Applied Mathematical Sciences, Vol. 7, 2013, no. 122, 6095 - 6099
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.310564

Purely Finitely Additive Measures as Generalized Elements in a Maximin Problem

Artem Baklanov

Ural Federal University
620002, Russia, Ekaterinburg, ul. Mira, 19
artem.baklanov@gmail.com

Abstract

We study the asymptotic behavior of maximin values of a payoff function, when admissible controls tend to infinity. The payoff function is superposition of a continuos function and a function that is uniform limit of step functions. An extension in the class of finitely additive measures is used.

Mathematics Subject Classification: 46N10, 49J35, 49J45, 49N99

Keywords: purely finitely additive measures, extension, maximin.

It is natural in control problems to use extensions constructions due to the non-existence of optimal solutions [1]. Moreover, extensions are needed for a 'regularization' of practically interesting problems that are associated with asymptotic relaxations of constraints (see [2, 3]). Obviously, game problems require extensions [4]. These problems can be divided into two types: those for which it is possible to well define an extension only on the product of ordinary controls sets of players [4–6], and those for which it is sufficient to construct an extension of an ordinary controls set for each player separately [7–10]. If we deals with a maximin problem of the second type, then we can use the following representation of the maximin value V_{ext} after an extension:

$$V_{ext} = \max_{\nu \in B} \min_{\mu \in A} \tilde{\alpha}(\mu, \nu), \tag{1}$$

where A, B are some sets of generalized elements (controls) and $\tilde{\alpha}$ is the generalized payoff function. In [7–10] the extension in the class of finitely additive measures (FAM) was used. In these papers sets A and B were some compacta in $*$-weak topology. The using of FAM helps to deal with the case of discontinuous control coefficients in the right-hand part of a differential equation. Often in this case purely FAM are essential elements of sets A, B (see [7, 10]). In this paper we consider a maximin problems such that admissible controls of players in some sense tends to infinity. We study asymptotics of values of the problems. We show that in this case sets A, B are subsets of purely FAM. The present paper extends results of [11].

Let (X, ρ_X) and (Y, ρ_Y) be unbounded metric spaces; fix $x_0 \in X$ and $y_0 \in Y$. By S_X^ε we denote open ε-neighborhood of x_0 w.r.t. ρ_X and by S_Y^κ we denote open κ-neighborhood of y_0 w.r.t. ρ_Y; here $\varepsilon \in]0, \infty[, \kappa \in]0, \infty[$. Let $H_X^\varepsilon \triangleq X \setminus S_X^\varepsilon, H_Y^\kappa \triangleq Y \setminus S_Y^\kappa$. By definition, put $\mathcal{H}_X \triangleq \{H_X^\varepsilon : \varepsilon \in]0, \infty[\}, \mathcal{H}_Y \triangleq \{H_Y^\kappa : \kappa \in]0, \infty[\}$. Families $\mathcal{H}_X$ and $\mathcal{H}_Y$ are filter bases in X and Y respectively [12]. By $\mathcal{L}_X$ and $\mathcal{L}_Y$ we denote semialgebras of subsets of X and Y respectively such that: $(\forall x_0 \in X\ \forall \varepsilon \in]0, \infty[: S_X^\varepsilon \in \mathcal{L}_X) \& (\forall y_0 \in Y\ \forall \kappa \in]0, \infty[: S_Y^\kappa \in \mathcal{L}_Y)$.

By $\mathbb{R}^n$ we denote n-dimensional arithmetic space. Let $\mathbf{u} : X \to \mathbb{R}^k$ and $\mathbf{v} : Y \to \mathbb{R}^l$ be uniform limits of $\mathcal{L}_X$-step and $\mathcal{L}_Y$-step functions respectively. Let $\Upsilon, \Upsilon : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$, be a jointly continuous payoff function. Thus,

$$\Upsilon(\mathbf{u}(x), \mathbf{v}(y)) \in \mathbb{R}\ \forall \varepsilon \in]0, \infty[\ \forall x \in H_X^\varepsilon\ \forall \kappa \in]0, \infty[\ \forall y \in H_Y^\kappa.$$

Now we can consider the following maximin problem for some $\varepsilon \in]0, \infty[, \kappa \in]0, \infty[$. The first player minimizes value of Υ by choosing $x \in H_X^\varepsilon$, the second player maximizes value of Υ by choosing $y \in H_Y^\kappa$. Thus we deal with problems

$$\Upsilon(\mathbf{u}(x), \mathbf{v}(y)) \to \sup_{y \in H_Y^\kappa} \inf_{x \in H_X^\varepsilon}, \quad \varepsilon \in]0, \infty[, \kappa \in]0, \infty[. \tag{2}$$

We will investigate asymptotics of values (2) as $\varepsilon, \kappa \to \infty$. Note that this asymptotics do not depend on x_0 and y_0.

By definition, put $\mathrm{V}(\varepsilon, \delta) \triangleq \sup_{y \in H_Y^\kappa} \inf_{x \in H_X^\varepsilon} \Upsilon(\mathbf{u}(x), \mathbf{v}(y))$.

Using continuity of Υ, boundedness of $\mathbf{v}^1(H_Y^\kappa), \mathbf{u}^1(H_X^\varepsilon)$, and [13, (2.34)], we obtain that

$$\mathrm{V}(\varepsilon, \delta) = \max_{b \in cl(\mathbf{v}^1(H_Y^\kappa), \tau_{\mathbb{R}}^{(l)})} \min_{a \in cl(\mathbf{u}^1(H_X^\varepsilon), \tau_{\mathbb{R}}^{(k)})} \Upsilon(a, b) \in \mathbb{R}\ \forall \varepsilon, \kappa \in]0, \infty[,$$

where $\tau_{\mathbb{R}}^{(k)}$ and $\tau_{\mathbb{R}}^{(l)}$ are the topologies of coordinate-wise convergence in $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively. Moreover, the asymptotic maximin as well-defined:

$$\mathcal{V} \triangleq \max_{b \in \mathbb{G}_2} \min_{a \in \mathbb{G}_1} \Upsilon(a, b) \in \mathbb{R},$$

where the following attraction sets (see [2, (3.3.10)])

$$\mathbb{G}_1 \triangleq \bigcap_{\varepsilon \in]0,\infty[} cl(\mathbf{u}^1(H_X^{\varepsilon}), \tau_{\mathbb{R}}^{(k)}), \mathbb{G}_2 \triangleq \bigcap_{\kappa \in]0,\infty[} cl(\mathbf{v}^1(H_Y^{\kappa}), \tau_{\mathbb{R}}^{(l)})$$

are compacta. We now ready to state the specific version of [13, theorem 1].

Theorem 1. *The following approximation property of* $\mathcal{V}$ *holds:*

$$\forall \xi \in]0,\infty[\ \exists \theta_\xi \in]0,\infty[: |\mathcal{V} - \mathrm{V}(\varepsilon,\delta)| < \xi\ \forall \varepsilon \in]0,\theta_\xi[\ \forall \kappa \in]0,\theta_\xi[.$$

The extension of the original problem (2) is constructed in the following way: for each point of the sets X and Y we assign the Dirac measure supported at this point (see immersion operator Δ in [3, p 1090] and [14, (4.4)]). The closures of resulting sets w.r.t. $*$-weak topology coincide, respectively, with the set of all $\{0,1\}$-valued FAM on $\mathcal{L}_X$ and on $\mathcal{L}_Y$ (see [3, p. 1090] and [14, (4.5)]). We now introduce the specific version of sets A and B (see (1)):

$$D_X \triangleq \bigcap_{\varepsilon \in]0,\infty[} cl(\{\delta_x^X : x \in H_X^{\varepsilon}\}, \tau_{\mathcal{L}_X}), D_Y \triangleq \bigcap_{\kappa \in]0,\infty[} cl(\{\delta_y^Y : y \in H_Y^{\kappa}\}, \tau_{\mathcal{L}_Y}); \quad (3)$$

where $cl(Z,\tau)$ is the closure of Z w.r.t. topology τ, δ_x^X is the Dirac measure on $\mathcal{L}_X$ supported at x, δ_y^Y is the Dirac measure on $\mathcal{L}_Y$ supported at y, and $\tau_{\mathcal{L}_X}, \tau_{\mathcal{L}_Y}$ are $*$-weak topologies. From [11, proposition 2.2] it follows that sets D_X, D_Y contains only purely FAM. Now we define the generalized payoff function $\tilde{\Upsilon} : D_X \times D_Y \to \mathbb{R}$ by the rule

$$\tilde{\Upsilon} \triangleq \Upsilon\Big(\big(\int_X \mathbf{u}_i\ \mu(dx)\big)_{i\in\overline{1,k}}, \big(\int_Y \mathbf{v}_j\ \nu(dy)\big)_{j\in\overline{1,l}}\Big)\ \forall \mu \in D_X\ \forall \nu \in D_Y. \quad (4)$$

Using [13, proposition 5],(3), and (4), we obtain the next statement.

Theorem 2. *The following representation of asymptotics of values of problems* (2) *holds:* $\mathcal{V} = \max\limits_{\nu \in D_Y} \min\limits_{\mu \in D_X} \tilde{\Upsilon}(\mu,\nu)$.

It is shown that if we use the extension in the class of FAM, then it is possible to obtain representation of asymptotics of values of problems (2). Moreover, from Theorem 2 it follows that these asymptotics can be defined in terms of generalized elements. These elements are purely FAM.

Acknowledgements. The reported study was partially supported by RFBR (research projects No. 13-01-90414 Ukr_f_a, No. 13-01-00304 A).

References

[1] J. Warga, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.

[2] A.G. Chentsov, *Asymptotic Attainability*, Kluwer, Dordrecht, 1997.

[3] A.G. Chentsov, Finitely-additive measures and relaxations of abstract control problems, *Journal of Mathematical Sciences*, Vol. 133, **2** (2006).

[4] N.N. Krasovskii and A.I. Subbotin, *Game-theoretical Control Problems*, New York, Springer-Verlag, 1988.

[5] Y.V. Averboukh, Nash equilibrium for differential game and nonanticipative strategies technique [In Russian]. *Matematicheskaya Teoriya Igr i Ee Prilozheniya,* **3** (2012), 3-20.

[6] D.V. Khlopin and A.G. Chentsov, On a control problem with incomplete information: quasistrategies and control procedures with a model. *Differential Equations*, **12** (2005), 1727-1742.

[7] A.P. Baklanov, On the representation of maximin of an impulse control problem [in Russian], *Differential Equations and Control Processes*, **3** (2012), 49-69.

[8] A.P. Baklanov and A.G. Chentsov, On question about extension of a game problem in the class of two-valued finitely-additive measures [in Russian], *Tambov University Bulletin*, **1** (2011), 15-37.

[9] A.G. Chentsov and Yu.V. Shapar, A game problem with approximate observation of constraints, *Doklady Mathematics*, **1** (2009), 497-502.

[10] A.P. Baklanov, A game problem with asymptotic impulse control [in Russian], *Bulletin of Udmurt University. Mathematics.*, **3** (2011), 3-14.

[11] S.I. Morina and A.G. Chentsov, On a problem of asymptotic optimization [in Russian], *Bulletin of Chelyabinsk State University. Mathematics, Mechanics, Computer Science*, **2** (1994), 80-87.

[12] N. Bourbaki, *General Topology,* Hermann, Paris, 1940.

[13] A.G. Chentsov, About presentation of maximin in the game problem with constraints of asymptotic character [in Russian]. *Bulletin of Udmurt University. Mathematics*, **3** (2010), 104-119.

[14] A.P. Baklanov and A.G. Chentsov, On question about extension of abstract attainability problems admitting discontinuous dependences, *Functional Differential Equations*, **12** (2010), 21–50.

Received: October 3, 2013