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On Question about Extension of Maximin Problem with Phase Constraints

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Abstract

We study the asymptotic behavior of maximin values of a payoff function, when relaxed constraints are tightened. The payoff function depends on the trajectories of controlled systems of the first and second player. An extension in the class of the Radon measures is used. The asymptotic equivalence between two types of the constraints relaxations is shown.

Mathematics Subject Classification: 46N10, 49J35, 49J45, 49N99

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We study the asymptotic behavior of maximin values of a payoff function, when relaxed constraints are tightened. The payoff function depends on the trajectories of controlled systems of the first and second player. These systems can be non-linear; this is the important distinction from earlier papers [1, 2], where various asymptotic effects for linear systems with impulse constraints were considered and an extension in the class of finitely additive measures was used (see also [3, 4]). In the present paper a different approach is used (see [6]). We implement an extension in the class of the Radon measures. This approach is similar to the traditional one proposed in [5, 7, 8] and developed by N.N. Krasovskii and by his followers [9, 10].

We consider controlled systems of the first and second player:

$$\dot{x} = f(t, x, u), \quad u \in P, x \in \mathbb{R}^n, n \in \mathbb{N}, t \in I \triangleq [t_0, \vartheta_0], \tag{1}$$

$$\dot{y} = g(t, y, v), \quad v \in Q, y \in \mathbb{R}^m, m \in \mathbb{N}, t \in I. \tag{2}$$

Here $P, Q, P \subset \mathbb{R}^p, Q \subset \mathbb{R}^q, (p, q \in \mathbb{N})$ are compacta; $\mathbb{N} \triangleq \{1, 2, ...\}$. The sets of all admissible (open-loop) controls of players are defined as follows:

$$\mathcal{U} \triangleq \{u \in P^I \mid u \text{ is piecewise constant and right-continuous}\},$$
$$\mathcal{V} \triangleq \{v \in Q^I \mid v \text{ is piecewise constant and right-continuous}\}.$$

Phase constraints for the first and second player are set by families $(N_t)_{t \in I}$ and $(M_t)_{t \in I}$ respectively; where $(N_t \in \mathbb{R}^n) \& (M_t \in \mathbb{R}^m) \;\; \forall t \in I$. Let $\Upsilon(x(\cdot), y(\cdot))$ be a continuous functional defined on trajectories of systems (1) and (2). Players choose initial conditions for their systems from sets H, S; here $H \subset I \times \mathbb{R}^n, S \subset I \times \mathbb{R}^m$. The goal of the first player is to minimize value of Υ by choosing a control $u \in \mathcal{U}$ and an initial condition $h \in H$, the goal of the second player is to maximize value of Υ by choosing $v \in \mathcal{V}$ and $s \in S$. The players must comply with the phase constraints in terms of $(N_t)_{t \in]pr_1(h), \vartheta_0]}$ and $(M_t)_{t \in]pr_1(s), \vartheta_0]}$, where $pr_1(z)$ and $pr_2(z)$ are the first and second component of an ordered pair z. Thus we consider the following problem:

$$\Upsilon\Big(\phi(\cdot; h, u), \xi(\cdot; s, v)\Big) \to \sup_{(s,v) \in S \times \mathcal{V}} \inf_{(h,u) \in H \times \mathcal{U}} \tag{3}$$

$$w.r.t. \quad \Big(\phi(t; h, u) \in N_t \; \forall t \in]pr_1(h), \vartheta_0]\Big) \& \Big(\xi(t; s, v) \in M_t \; \forall t \in]pr_1(s), \vartheta_0]\Big);$$

where for all $(h, u) \in H \times \mathcal{U}, (s, v) \in S \times \mathcal{V}$ by functions $\phi(\cdot; h, u)$ and $\xi(\cdot; s, v)$ we denote trajectories of (1) and (2) respectively. Note that in this problem it is possible to observe the following instability effect: for an arbitrarily small relaxation of constraints in terms of the sets corresponding to the allowed phase states and initial positions, there is an jump-like change of a problem result (8). In this context, we construct a regularized version of the problem that does not cause the mentioned effect. For this purpose, we will use the approach proposed in [6, Chapter IV], [13, §7.7] and based on the using of generalized controls (the Radon measures).

Let $\mathcal{L}, \mathcal{L}_P, \mathcal{L}_Q$ be σ-algebras of compacta $I, I \times P, I \times Q$ respectively and let λ be the trace of the Lebesgue measure on $\mathcal{L}$. By $\mathcal{P}$ we denote the set of all nonnegative real-valued countably additive measures μ on $\mathcal{L}_P$ such that $\forall \Gamma \in \mathcal{L} : \mu(\Gamma \times P) = \lambda(\Gamma)$. Let $\mathcal{Q}$ be the set of all nonnegative real-valued countably additive measures ν on $\mathcal{L}_Q$ such that $\forall \Gamma \in \mathcal{L} : \nu(\Gamma \times Q) = \lambda(\Gamma)$.

Sets $\mathcal{P}$ and $\mathcal{Q}$ are compacta with respect to subspace $*$-weak topologies τ_*^P and τ_*^Q respectively (see [6,11]). Elements of these sets will be used as generalized controls. If (X,τ) is a topological space, then by $C(X)$ (by $C^k(X)$) we denote the set of all continuous functions from X into $\mathbb{R}$ (from X into $\mathbb{R}^k, k \in \mathbb{N}$). From the Riesz theorem it follows that for any $u \in \mathcal{U}$ there exists a unique measure (generalized control) μ_u^0 such that $\forall f_* \in C(I \times P)$

$$\int_{t_0}^{\vartheta_0} f_*(t,u(t))dt = \int_{I\times P} f_*(t,p)\mu_u^0(d(t,p)).$$

Then (for the first player) we define a mapping $\mathbf{s}_U$ as follows:

$$(h,u) \longmapsto (h,\mu_u^0) : H \times \mathcal{U} \to H \times \mathcal{P}. \tag{4}$$

In the same way for the second player we introduce the definition of $\nu_v^0 \in \mathcal{Q}$ for any $v \in \mathcal{V}$ and $\mathbf{s}_V$ by the rule $(s,v) \longmapsto (s,\nu_v^0) : S \times \mathcal{V} \to S \times \mathcal{Q}$.

If A is a subset of an Euclidean space, then $A^\kappa, \kappa > 0$, is the closed κ-neighborhood of A with respect to the Chebyshev distance. We introduce sets of all ordinary controls of the players that are admissible for the first type of the constraints relaxation:

$$\mathbb{P}[\varepsilon] \stackrel{\triangle}{=} \big\{(h,u) \in H \times \mathcal{U} \mid \phi(t;h,u) \in N_t^\varepsilon \ \forall t \in]pr_1(h)+\varepsilon, \vartheta_0]\big\};$$
$$\mathbb{Q}[\delta] \stackrel{\triangle}{=} \big\{(s,v) \in S \times \mathcal{V} \mid \xi(t;s,v) \in M_t^\delta \ \forall t \in]pr_1(s)+\delta, \vartheta_0]\big\}.$$

We now introduce the following key assumption of the generalized controls existence (see [11, Theorem 4.1]):

A1. $(\forall \varepsilon > 0 \ \ \mathbb{P}[\varepsilon] \neq \emptyset) \& (\forall \delta > 0 \ \ \mathbb{Q}[\delta] \neq \emptyset)$.

The following values correspond to the first type of the constraints relaxation: $\forall \varepsilon, \delta > 0$

$$V[\varepsilon,\delta] \stackrel{\triangle}{=} \sup_{(s,v)\in\mathbb{Q}[\delta]} \inf_{(h,u)\in\mathbb{P}[\varepsilon]} \Upsilon\Big(\phi(\cdot;h,u), \xi(\cdot;s,v)\Big) \in \mathbb{R}. \tag{5}$$

We introduce sets of all ordinary controls of the players that are admissible for the second type of the constraints relaxation:

$$\mathbb{P}_N[\varepsilon] \stackrel{\triangle}{=} \{(h,u) \in H^\varepsilon \times \mathcal{U} \mid \phi(t;h,u) \in N_t^\varepsilon \ \forall t \in]pr_1(h)+\varepsilon, \vartheta_0]\},$$
$$\mathbb{Q}_M[\delta] \stackrel{\triangle}{=} \{(s,v) \in S^\delta \times \mathcal{V} \mid \xi(t;s,v) \in M_t^\delta \ \forall t \in]pr_1(s)+\delta, \vartheta_0]\}.$$

The following values correspond to the second type of the constraints relaxation: $\forall \varepsilon, \delta > 0$

$$\mathbb{V}[\varepsilon,\delta] \stackrel{\triangle}{=} \sup_{(s,v)\in\mathbb{Q}_M[\delta]} \inf_{(h,u)\in\mathbb{P}_N[\varepsilon]} \Upsilon\Big(\phi(\cdot;h,u), \xi(\cdot;s,v)\Big) \in \mathbb{R}. \tag{6}$$

We will study the asymptotic behavior of values (5) and (6) in the case of the tightening of the constraints that are relaxed by $\varepsilon, \delta > 0$.

By definition, put $\overline{H} \triangleq cl(H, \tau_{\mathbb{R}}^{(n+1)})$ and $\overline{S} \triangleq cl(S, \tau_{\mathbb{R}}^{(m+1)})$; where $\tau_{\mathbb{R}}^{(k)}$ is the topology of coordinate-wise convergence in $\mathbb{R}^k$ and $cl(G, \tau)$ is the closure of G with respect to a topology τ. We suppose that the following conditions hold for systems (1), (2) [11, conditions 3.1, 3.2]:

C1. The condition of the generalized uniqueness;

C2. The condition of local boundedness of solutions for the systems.

If **C1** holds, then for any generalized control there exists a unique generalized trajectory (see [11]). This allows us to introduce the following definitions of generalized trajectories $\hat{\phi}(\cdot; h, \mu)$ (for (1)) and $\hat{\xi}(\cdot; s, \nu)$ (for (2)):

$$\Big(\hat{\phi}(t; h, \mu) = pr_2(h) \ \ \forall t \in [t_0, pr_1(h)[\Big) \& \Big(\hat{\phi}(t; h, \mu) = pr_2(h) + \\ + \int_{[pr_1(h), t[\times P} f(t, \phi(\tau), u)\mu(d(\tau, u)) \ \ \forall t \in [pr_1(h), \vartheta_0]\Big) \ \ \forall (h, \mu) \in \overline{H} \times \mathcal{P};$$

$$\Big(\hat{\xi}(t; s, \nu) = pr_2(s) \ \ \forall t \in [t_0, pr_1(s)[\Big) \& \Big(\hat{\xi}(t; s, \nu) = pr_2(s) + \\ + \int_{[pr_1(s), t[\times Q} g(t, \xi(\tau), v)\nu(d(\tau, v)) \ \ \forall t \in [pr_1(s), \vartheta_0]\Big) \ \ \forall (s, \nu) \in \overline{S} \times \mathcal{Q}.$$

If **C1** and **C2** hold, then the following mappings are continuous (see [11]):

$$(h, \mu) \longmapsto \hat{\phi}(\cdot; h, \mu) : \overline{H} \times \mathcal{P} \to C^n(I), \ \ (s, \nu) \longmapsto \hat{\xi}(\cdot; s, \nu) : \overline{S} \times \mathcal{Q} \to C^m(I). \quad (7)$$

By definition, put $\mathbf{P} \triangleq \{(h, \mu) \in \overline{H} \times \mathcal{P} \mid \hat{\phi}(t; h, \mu) \in N_t \ \forall t \in]pr_1(h), \vartheta_0]\}$ and $\mathbf{Q} \triangleq \{(s, \nu) \in \overline{S} \times \mathcal{Q} \mid \hat{\xi}(t; s, \nu) \in M_t \ \forall t \in]pr_1(s), \vartheta_0]\}$. Then sets $\mathbf{P}, \mathbf{Q}$ are closed. Combining this with **A1** and [11, Theorem 4.1], we obtain that $\mathbf{P}, \mathbf{Q}$ are compacta. Consequently the following value is well defined:

$$\mathbf{V} \triangleq \max_{(s,v) \in \mathbf{Q}} \min_{(h,\mu) \in \mathbf{P}} \Upsilon\Big(\hat{\phi}(\cdot; h, \mu), \hat{\xi}(\cdot; s, \nu)\Big) \in \mathbb{R}.$$

In this paper we show that if **A1** holds, then the asymptotics (in the sense of the tightening relaxed constraints) of values (5) and (6) coincide and equal to value $\mathbf{V}$ of the generalized problem. To prove this, we use the main results from papers [11, 12].

Let $\beta_0[X]$ be the set of all filter bases on X [15]. From **A1** it follows that

$$\{\mathbb{P}[\varepsilon] \mid \varepsilon > 0\} \in \beta_0[\mathbb{R}^{n+1} \times \mathcal{U}], \{\mathbb{P}_N[\varepsilon] \mid \varepsilon > 0\} \in \beta_0[\mathbb{R}^{n+1} \times \mathcal{U}], \quad (8)$$

$$\{\mathbb{Q}[\delta] \mid \delta > 0\} \in \beta_0[\mathbb{R}^{m+1} \times \mathcal{V}], \{\mathbb{Q}_M[\delta] \mid \delta > 0\} \in \beta_0[\mathbb{R}^{m+1} \times \mathcal{V}]. \quad (9)$$

Using ideas from article [12] (that deals with an asymptotic behavior of abstract maximin problems) and from articles [13, 14] (that deal with a construction of attraction sets [14, §1 and p. 1053]), we now specify the following compactifiers [14, p. 1060] for the ordinary control sets:

$$\Big[\overline{H}\times\mathcal{P}, \tau_{\mathbb{R}}^{n+1}|_{\overline{H}}\otimes\tau_*^P, \mathbf{s}_U, \big(\hat{\phi}(\cdot;h,\mu)\big)_{(h,\mu)\in\overline{H}\times\mathcal{P}}\Big], \Big[\overline{S}\times\mathcal{Q}, \tau_{\mathbb{R}}^{m+1}|_{\overline{S}}\otimes\tau_*^Q, \mathbf{s}_V, \big(\hat{\xi}(\cdot;s,\nu)\big)_{(s,\nu)\in\overline{S}\times\mathcal{Q}}\Big];$$

where $\otimes$ stands for a topology product and $\tau|_K$ means a subspace topology on set K with respect to a topology τ (see [15]). Note that we have (see [11]) the following presentation of auxiliary attraction sets (see [12–14]) via the sets of generalized controls:

$$\bigcap_{\varepsilon>0} cl\Big(\big\{(h,\mathbf{s}_U(u)) \mid (h,u)\in\mathbb{P}[\varepsilon]\big\}, \tau_{\mathbb{R}}^{n+1}\otimes\tau_*^P\Big) =$$
$$= \bigcap_{\varepsilon>0} cl\Big(\big\{(h,\mathbf{s}_U(u)) \mid (h,u)\in\mathbb{P}_N[\varepsilon]\big\}, \tau_{\mathbb{R}}^{n+1}\otimes\tau_*^P\Big) = \mathbf{P};$$
$$\bigcap_{\delta>0} cl\Big(\big\{(s,\mathbf{s}_V(v)) \mid (s,v)\in\mathbb{Q}[\delta]\big\}, \tau_{\mathbb{R}}^{m+1}\otimes\tau_*^Q\Big) =$$
$$= \bigcap_{\delta>0} cl\Big(\big\{(s,\mathbf{s}_V(v)) \mid (s,v)\in\mathbb{Q}_M[\delta]\big\}, \tau_{\mathbb{R}}^{m+1}\otimes\tau_*^Q\Big) = \mathbf{Q}. \quad (10)$$

We see that the auxiliary attraction sets are universal with respect to "the asymptotic constraints" defined as the first and second type of the constraints relaxation. By τ_k^C we denote the topology of uniform convergence in $C^k(I), k\in\mathbb{N}$. The next theorem is the specific version of Proposition 5.2.1 [13].

Theorem 1. *The following equations hold:*

$$\bigcap_{\varepsilon>0} cl\Big(\big\{\phi(\cdot;h,u) \mid (h,u)\in\mathbb{P}[\varepsilon]\big\}, \tau_n^C\Big) = \bigcap_{\varepsilon>0} cl\Big(\big\{\phi(\cdot;h,u) \mid (h,u)\in\mathbb{P}_N[\varepsilon]\big\}, \tau_n^C\Big) =$$
$$= \{\hat{\phi}(\cdot;h,\mu)|(h,\mu)\in\mathbf{P}\}\ ; \qquad \bigcap_{\delta>0} cl\Big(\big\{\xi(\cdot;s,v) \mid (s,v)\in\mathbb{Q}[\delta]\big\}, \tau_m^C\Big) =$$
$$= \bigcap_{\delta>0} cl\Big(\big\{\xi(\cdot;s,v) \mid (s,v)\in\mathbb{Q}_M[\delta]\big\}, \tau_m^C\Big) = \{\hat{\xi}(\cdot;s,\nu)|(s,\nu)\in\mathbf{Q}\}.$$

To prove this, we used continuity of (7) and (8)–(10). Theorem 1 shows the presentations of attraction sets in spaces of trajectories of (1) and (2).

Combining Theorem 1, (8), (9), continuity of (7), and [12, Theorem 1, Proposition 5], we obtain the following theorem that characterizes $\mathbf{V}$ as the generalized limit of values $V[\varepsilon,\delta]$ and $\mathbb{V}[\varepsilon,\delta]$ under the tightening constraints that are relaxed by $\varepsilon,\delta>0$.

Theorem 2. *The following is true:* $\forall\kappa>0\ \exists\psi_\kappa>0:$

$$\Big(|\mathbf{V}-V[\varepsilon,\delta]|<\kappa\Big)\&\Big(|\mathbf{V}-\mathbb{V}[\varepsilon,\delta]|<\kappa\Big)\ \ \forall\varepsilon,\delta\in]0,\psi_\kappa[.$$

Theorem 2 shows that for problem (3) there exists the asymptotic equivalence between two types of the constraints relaxations. Moreover, setting appropriate limits for extreme levels of the relaxations, we can guarantee any proximity of values (5) and (6).

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