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Inclusion Properties of p-Valent Functions Associated with Borel Distribution Functions

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Abstract: In this paper, we define a differential operator on an open unit disk Δ by using the novel Borel distribution (BD) operator and means of convolution. This operator is adopted to introduce new subclasses of p-valent functions through the principle of differential subordination, and we focus on some interesting inclusion relations of these classes. Additionally, some inclusion relations are derived by using the Bernardi integral operator. Moreover, relevant convolution results are established for a class of analytic functions on Δ, and other results of analytic univalent functions are derived in detail. This study provides a new perspective for developing p-univalent functions with BD and offers valuable understanding for further research in complex analysis.

Keywords: p-valent function; Borel distribution; inclusion relation; integral operator; convolution

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1. Introduction

Probability distributions are applied in a wide range of scientific areas, such as neural networks, economic forecasting, radiationless sources, and meteorology, and are used to describe several real-life phenomena. In mathematics, the concept is extensively used to study singular structures of Laplacian eigenfunctions, derivatives of distributions, orthogonal polynomials, transmission eigenfunctions, and impulse functions (see, for example, [1–5]).

The Borel distribution (BD) was introduced by Wanas et al. [6] as

$$P(X=\mu)=\frac{(\mu\lambda)^{\mu-1}e^{-\lambda\mu}}{\mu!},\quad 0<\lambda\leq 1,\ \ \mu=1,2,3,\dots$$

Furthermore, they introduced the series

$$M_\lambda(z)=z^p+\sum_{k=p+1}^{\infty}\frac{[\lambda(k-p)]^{k-p-1}e^{-\lambda(k-p)}}{(k-p)!}z^k,\quad 0<\lambda\leq 1, p\in\mathbb{N},$$

whose coefficients are probabilities of the BD.

By using the familiar Ratio Test, it can be shown that the above series is convergence on the unit disk Δ. Later, some researchers used the concept of BD of probabilities to obtain several properties of geometric functions theory in complex analysis. Alatawi [7] studied the Gegenbauer polynomials by using distributions of probabilities and the Mittag-Leffler operator. Amourah et al. [8] introduced a subclass of bi-univalent functions by using the

BD operator. Srivastava et al. [9] used the Mittag-Leffler-type BD series to address fuzzy differential subordinations.

Let us consider that $\mathcal{A}_p(a)$ is the class of functions analytic in $\Delta = \{z \in \mathbb{C}; |z| < 1\}$, given by

$$f(z) = a + z^p + \sum_{k=p+1}^{\infty} a_k z^k. \tag{1}$$

For $a = 0$, we get the class of analytic p-valent functions in Δ, and we denote $\mathcal{A}_p(0) = \mathcal{A}_p$. If $p = 1$ and $a = 0$, then we obtain the class of analytic functions, and we use $\mathcal{A}_1(0) = \mathcal{A}$.

Let $\mathfrak{P}_{p,n}(\nu)$ consist of analytic functions $\eta(z) \in A_p(p)$ such that

$$\int_0^{2\pi} \left| \frac{Re\{\eta(z)\} - \nu}{p - \nu} \right| d\theta < n\pi,$$

where $z = re^{i\theta}$, $n \geq 2$ and $0 \leq \nu < p$ (see [10]). Note that, for $n \geq 2$ and $0 \leq \nu < p$, we have $\mathfrak{P}_{1,n}(\nu) = \mathfrak{P}_n(\nu)$ and $\mathfrak{P}_{1,n}(0) = \mathfrak{P}_n$ [11,12]. Moreover, for $0 \leq \nu < p$ and $p \in \mathbb{N}$, we have $\mathfrak{P}_{p,2}(\nu) = \mathfrak{P}_{p,\nu}$, where $\mathfrak{P}_{p,\nu}$ is the class of analytic functions that have real positive part. Furthermore, $\mathfrak{P}_{p,2}(0) = \mathfrak{P}_p$ (see [10]).

Now, for $p \in \mathbb{N}$, $n \geq 2$ and $0 \leq \rho, \theta < \pi$, we define the subclasses $\mathtt{S}_p^n(\nu), \mathtt{C}_p^n(\nu)$ and $\mathtt{K}_p^n(\tau,\nu)$ of $\mathcal{A}_p$ as follows

$$\mathtt{S}_p^n(\nu) = \left\{ f : f \in \mathcal{A}_p \text{ and } \tfrac{zf'(z)}{f(z)} \in \mathfrak{P}_{p,n}(\nu), z \in \Delta \right\},$$
$$\mathtt{C}_p^n(\nu) = \left\{ f : f \in \mathcal{A}_p \text{ and } 1 + \tfrac{zf''(z)}{f'(z)} \in \mathfrak{P}_{p,n}(\nu), z \in \Delta \right\},$$
$$\mathtt{K}_p^n(\tau,\nu) = \left\{ f : f \in \mathcal{A}_p, g \in \mathtt{S}_p^2(\tau) \text{ and } \tfrac{zf'(z)}{g(z)} \in \mathfrak{P}_{p,n}(\nu), z \in \Delta \right\}.$$

We should note that $f(z) \in \mathtt{C}_p^n(\nu)$ if and only if $\frac{zf'(z)}{p} \in \mathtt{S}_p^n(\nu)$.

Let $\mathtt{S}$, $\mathtt{S}_1^2(\nu)$, $\mathcal{C}_1^2(\nu)$ and $\mathtt{K}_1^2(\tau,0)$ denote subclass families of $\mathcal{A}$, made of functions that are, univalent, starlike of order ν, convex of order ν, and close to starlike of order τ, respectively [13,14].

Let $f(z)$ be given by (1) and $g(z) \in A_p(b)$ be $g(z) = b + z^p + \sum_{k=p+1}^{\infty} b_k z^k$. The convolution is defined in [15] (see also [16]) and is expressed as

$$f(z) * g(z) = ab + z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, \quad z \in \Delta.$$

Lashin [17] investigated interesting criteria for convolution properties of analytic functions. Moreover, Patel et al. [18] studied convolution properties of multivalent functions by means of the Dziok–Srivastava operator. Other authors [19–21] studied various subclasses of analytic functions using the technique of criteria convolution.

The function $f(z)$ is said to be subordinate to $g(z)$, that is, $f(z) \prec g(z)$, if there exists an analytic Schwarz function $w(z)$ in Δ with $|w(z)| < 1$ and $w(0) = 0$, such that $g(z) = f(w(z))$. If $g(z)$ is univalent in Δ, then the subordination is equivalent to have $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$ (see [22]). Miller and Mocanu [23,24] investigated some applications of subordination to obtain several properties of first-order and second-order differential equations.

An analytic function $f(z) \in \mathcal{A}$ is in the class $\mathfrak{R}(\nu)$, consisting of prestarlike functions of order ν in Δ, if

$$\frac{z}{(1-z)^{2(1-\nu)}} * f(z) \in \mathtt{S}_1^2, \qquad \nu < 1.$$

A function $f(z) \in \mathcal{A}$ is in the class $\mathtt{C}_1^2$, consisting of univalent convex functions, if and only if it is in the class $\mathfrak{R}(0)$. Furthermore, $\mathfrak{R}(\frac{1}{2}) = \mathtt{S}_1^2$ (see [18]).

The research on inclusion relations of analytic functions in certain special sets is a subject that has its origin at the beginning of the study of geometric function theory. Ruscheweyh in [25] studied neighborhood and inclusion relations of univalent functions. Srivastava et al. [26] investigated the inclusion properties of multivalent functions. The authors in [27] derived inclusion symmetric relations for (q,δ)-neighborhoods of analytic univalent functions. For further results, please see [28–30] and works cited therein. Recently, various subclasses of univalent functions in geometric function theory have been investigated (for details, see [27,31–34]).

Let us consider the linear operator $\mathfrak{B}_\lambda f(z) : \mathcal{A}_p \to \mathcal{A}_p$ as

$$\mathfrak{B}_\lambda f(z) = M_\lambda(z) * f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} a_k z^k, \quad 0 < \lambda \leq 1, p \in \mathbb{N}.$$

For $\delta \geq 0$, we define the operator $\mathfrak{B}^m_{\lambda,\delta} f(z) : \mathcal{A}_p \to \mathcal{A}_p$ as

$$\begin{aligned}
\mathfrak{B}^0_{\lambda,\delta} f(z) &= z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} a_k z^k, \\
\mathfrak{B}^1_{\lambda,\delta} f(z) &= (1-\delta)\mathfrak{B}^0_{\lambda,\delta} f(z) + \frac{\delta}{p} z \left(\mathfrak{B}^0_{\lambda,\delta} f(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} \left[1 + \delta\left(\frac{k}{p} - 1\right)\right] a_k z^k, \\
\mathfrak{B}^2_{\lambda,\delta} f(z) &= (1-\delta)\mathfrak{B}^1_{\lambda,\delta} f(z) + \frac{\delta}{p} z \left(\mathfrak{B}^1_{\lambda,\delta} f(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} \left[1 + \delta\left(\frac{k}{p} - 1\right)\right]^2 a_k z^k, \\
&\vdots \\
\mathfrak{B}^m_{\lambda,\delta} f(z) &= (1-\delta)\mathfrak{B}^{m-1}_{\lambda,\delta} f(z) + \frac{\delta}{p} z \left(\mathfrak{B}^{m-1}_{\lambda,\delta} f(z)\right)' \\
&= z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} \left[1 + \delta\left(\frac{k}{p} - 1\right)\right]^m a_k z^k, \qquad (2) \\
& \qquad m \in \mathbb{N} \bigcup \{0\}, \delta \geq 0, 0 < \lambda \leq 1.
\end{aligned}$$

From $\mathfrak{B}^m_{\lambda,\delta} f(z)$ defined in (2), we can show that the following differential relation holds for all $f \in \mathcal{A}_p$:

$$\delta z \left(\mathfrak{B}^m_{\lambda,\delta} f(z)\right)' = p \mathfrak{B}^{m+1}_{\lambda,\delta} f(z) - p(1-\delta)\mathfrak{B}^m_{\lambda,\delta} f(z), \quad z \in \Delta. \qquad (3)$$

Now, we define the class $\Omega^m_{\lambda,\delta}(\alpha,\eta)$ of functions $f(z) \in \mathcal{A}_p$ such that the next subordination condition is satisfied

$$(1-\alpha) z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z) + \frac{\alpha}{p} z^{-p+1} \left(\mathfrak{B}^m_{\lambda,\delta} f(z)\right)' \prec \eta(z),$$

where $\alpha \in \mathbb{C}$ and $\eta(z) \in \mathfrak{P}$.

Example 1. *If we define*

$$\Lambda^{\alpha,p}_k = \left(1 - \frac{\alpha(k-p)}{p}\right) \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} \left[1 + \delta\frac{k-p}{p}\right]^m,$$
$$\alpha \in \mathbb{C},\ p \in \mathbb{N},\ 0 < \lambda \leq 1,\ m \in \mathbb{N} \bigcup \{0\},\ \delta \geq 0,\ k = p+1, p+2, \ldots,$$

then

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{2}{\Lambda_k^{\alpha,p}}z^k\in\Omega_{\lambda,\delta}^{m}(\alpha,\frac{1+z}{1-z}).$$

Example 2. *If we define*

$$_1\Lambda_k^{\alpha,p}=\left(1-\frac{\alpha(k-p)}{p}\right)\frac{[(k-p)]^{k-p-1}e^{-\lambda(k-p)}}{\lambda}\left[1+\delta\frac{k-p}{p}\right]^m,$$
$$\alpha\in\mathbb{C},\ p\in\mathbb{N},\ 0<\lambda\le 1,\ m\in\mathbb{N}\bigcup\{0\},\ \delta\ge 0,\ k=p+1,p+2,\dots,$$

then

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{1}{{}_1\Lambda_k^{\alpha,p}}z^k\in\Omega_{\lambda,\delta}^{m}(\alpha,e^{\lambda z}),\quad 0<\lambda\le 1.$$

Furthermore, we define the subclass of the class $\mathcal{A}_p$ as follows

$$\mathtt{S}_{\delta,n}^{\lambda,m}(\nu)=\left\{f:f\in\mathcal{A}_p\text{ and }\mathfrak{B}_{\lambda,\delta}^{m}f(z)\in\mathtt{S}_p^n(\nu),z\in\Delta\right\},$$
$$\mathtt{C}_{\delta,n}^{\lambda,m}(\nu)=\left\{f:f\in\mathcal{A}_p\text{ and }\mathfrak{B}_{\lambda,\delta}^{m}f(z)\in\mathtt{C}_p^n(\nu),z\in\Delta\right\},$$
$$\mathtt{K}_{\delta,n}^{\lambda,m}(\tau,\nu)=\left\{f:f\in\mathcal{A}_p\text{ and }\mathfrak{B}_{\lambda,\delta}^{m}f(z)\in\mathtt{K}_p^n(\tau,\nu),z\in\Delta\right\}.$$

We can easily show that $f(z)\in\mathtt{C}_{\delta,n}^{\lambda,m}(\nu)$ if and only if $\frac{zf'(z)}{p}\in\mathtt{S}_{\delta,n}^{\lambda,m}(\nu)$.

Inspired by the previous result using the BD functions, in this paper, we study several inclusion relations. Additionally, we obtain convolution results of these functions. Therefore, we study properties of the subclasses of univalent functions by means of the operator $\mathfrak{B}_{\lambda,\delta}^{m}$. The paper is organized into five Sections. In Section 2 we introduce useful lemmas, which are used to simplify the derived results. In Section 3, we present inclusion results and applications. In Section 4, we obtain some interesting convolution results, and we derive several theorems and remarks in some detail. In Section 5, we draw some conclusions.

2. Some Useful Lemmas

Some useful lemmas for the analysis carried out in the follow-up are introduced.

Lemma 1 (Theorem 6, [23])**.** *Let the function $\gamma(z)$ be analytic in Δ, and $\eta(z)$ be analytic and convex univalent in Δ with $\gamma(0)=\eta(0)$. If*

$$\gamma(z)+\frac{1}{\rho}z\gamma'(z)\prec\eta(z),\quad z\in\Delta,\tag{4}$$

with $Re(\rho)\ge 0$ and $\rho\ne 0$, then

$$\gamma(z)\prec\tilde{\eta}(z)=\rho z^{-\rho}\int_0^z\tau^{\rho-1}\eta(\tau)d\tau\prec\eta(z),\quad z\in\Delta,$$

where $\tilde{\eta}(z)$ is the best dominant of the differential subordination (4).

Lemma 2 (Theorem 13, [24])**.** *Let us consider $v=v_1+iv_2$ and $w=w_1+iw_2$, and let $\Psi(v,w)$, $\Psi:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$, be a complex-valued function. Suppose that $\Psi(v,w)$ meets:*

(i) $\Psi(v,w)$ *is continuous in domain $\mathfrak{D}$;*
(ii) $Re\{\Psi(1,0)\}>0$ *and $(1,0)\in\mathfrak{D}$;*
(iii) $Re\{\Psi(iw_2,v_1)\}\le 0$ *for all $(iw_2,v_1)\in\mathfrak{D}$, and such that $v_1\le-\frac{1}{2}(1+w_1^2)$.*

Let $\eta(z) \in \mathfrak{P}$ such that $(\eta(z), z\eta'(z)) \in \mathfrak{D}$ for all $z \in \Delta$. If

$$Re\{\Psi(\eta(z), z\eta'(z))\} > 0, \quad z \in \Delta,$$

then

$$Re\{\eta(z)\} > 0 \quad z \in \Delta.$$

Lemma 3 (Theorem 2.4, [15])**.** *Let $\nu < 1$, $f(z) \in S_2^1(\nu)$ and $g(z) \in R(\nu)$. Then,*

$$\frac{g(z) * \psi(z) f(z)}{g(z) * f(z)} \subseteq \overline{co}(\psi(\Delta)),$$

for any $\psi(z)$ analytic function in Δ, where $\overline{co}(\psi(\Delta)$ represents the $\psi(\Delta)$ closed convex hull.

Lemma 4 (Theorem 6, [35])**.** *Let $f(z) \in A_p(1)$. If*

$$Re\{f(z)\} > 0, \quad z \in \Delta,$$

then

$$Re\{f(z)\} \geq \frac{1 - |z|^p}{1 + |z|^p}, \qquad z \in \Delta.$$

3. Inclusion Relations

We present some inclusion relations on the newly defined subclasses of analytic functions.

Theorem 1. *Let $f(z) \in \mathcal{A}_p$ and $0 \leq \alpha_1 < \alpha_2$. Then*

$$\Omega^m_{\lambda,\delta}(\alpha_2, \eta) \subseteq \Omega^m_{\lambda,\delta}(\alpha_1, \eta).$$

Proof. Suppose that $f(z) \in \Omega^m_{\lambda,\delta}(\alpha_2, \eta)$. We define

$$\gamma(z) = z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z).$$

Hence, the function $\gamma(z)$ is analytic in Δ with $\gamma(0) = 1$. Thus,

$$(1 - \alpha_2) z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z) + \frac{\alpha_2}{p} z^{-p+1} \left(\mathfrak{B}^m_{\lambda,\delta} f(z)\right)' = \gamma(z) + \frac{\alpha_2}{p} z\gamma'(z) \prec \eta(z).$$

Thus, by using Lemma 1, we obtain

$$\gamma(z) \prec \eta(z), \quad z \in \Delta.$$

Since $0 \leq \alpha_1 < \alpha_2$, we obtain that

$$0 \leq \frac{\alpha_1}{\alpha_2} < 1.$$

Furthermore, as $\eta(z)$ is analytic convex in Δ, we have

$$\begin{aligned} & (1 - \alpha_1) z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z) + \frac{\alpha_1}{p} z^{-p+1} \left(\mathfrak{B}^m_{\lambda,\delta} f(z)\right)' \\ = \; & \frac{\alpha_1}{\alpha_2} \left((1 - \alpha_2) z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z) + \frac{\alpha_2}{p} z^{-p+1} \left(\mathfrak{B}^m_{\lambda,\delta} f(z)\right)' \right) + \left(1 - \frac{\alpha_1}{\alpha_2}\right) z^{-p} \mathfrak{B}^m_{\lambda,\delta} f(z) \prec \eta(z). \end{aligned}$$

Thus, $f(z) \in \Omega^m_{\lambda,\delta}(\alpha_1, \eta)$, and we get the desired result. □

Theorem 2. *Let $f \in \mathcal{A}_p$ and $\alpha \in \mathbb{C}$. Then*

$$\Omega^{m+1}_{\lambda,\delta}(\alpha,\eta) \subseteq \Omega^{m}_{\lambda,\delta}(\alpha,\tilde{\eta}), \tag{5}$$

where

$$\tilde{\eta}(z) = \frac{p}{\delta} z^{-\frac{p}{\delta}} \int_0^z \tau^{\rho-1}\eta(\tau)d\tau, \quad z \in \Delta.$$

Proof. Suppose that $f(z) \in \Omega^{m+1}_{\lambda,\delta}(\alpha,\eta)$. We define

$$\gamma(z) = (1-\alpha)z^{-p}\mathfrak{B}^m_{\lambda,\delta}f(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^m_{\lambda,\delta}f(z)\right)'. \tag{6}$$

From Equations (3) and (6),

$$pz^p\gamma(z) = p\left(1-\alpha-\alpha\frac{1-\delta}{\delta}\right)\mathfrak{B}^m_{\lambda,\delta}f(z) + \frac{\alpha p}{\delta}\mathfrak{B}^{m+1}_{\lambda,\delta}f(z). \tag{7}$$

By computing the derivative of the relation (7) and using (3) , we get

$$\begin{aligned} pz^p\left(z\gamma'(z)+p\gamma(z)\right) &= \frac{\alpha p}{\delta}z\left(\mathfrak{B}^{m+1}_{\lambda,\delta}f(z)\right)' + \frac{p^2}{\delta}\left(1-\alpha-\alpha\frac{1-\delta}{\delta}\right)\mathfrak{B}^{m+1}_{\lambda,\delta}f(z) \\ &- \frac{p^2(1-\delta)}{\delta}\left(1-\alpha-\alpha\frac{1-\delta}{\delta}\right)\mathfrak{B}^m_{\lambda,\delta}f(z). \end{aligned} \tag{8}$$

By using (7) and (8), we have

$$pz^p\left(z\gamma'(z)+\frac{p}{\delta}\gamma(z)\right) = \frac{p^2}{\delta}(1-\alpha)\mathfrak{B}^{m+1}_{\lambda,\delta}f(z) + \frac{\alpha p}{\delta}z\left(\mathfrak{B}^{m+1}_{\lambda,\delta}f(z)\right)',$$

which implies that

$$\gamma(z) + \frac{z\gamma'(z)}{\frac{p}{\delta}} = (1-\alpha)z^{-p}\mathfrak{B}^{m+1}_{\lambda,\delta}f(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m+1}_{\lambda,\delta}f(z)\right)'.$$

Since $f(z) \in \Omega^{m+1}_{\lambda,\delta}(\alpha,\eta)$, $Re\{\frac{p}{\delta}\} > 0$ and $\frac{p}{\delta} > 0$, we obtain

$$\gamma(z) + \frac{z\gamma'(z)}{\frac{p}{\delta}} \prec \eta(z), \quad z \in \Delta.$$

Therefore, with Lemma 1, we obtain

$$\gamma(z) \prec \tilde{\eta}(z) = \frac{p}{\delta}z^{-\frac{p}{\delta}}\int_0^z \tau^{\rho-1}\eta(\tau)d\tau \prec \eta(z).$$

Hence, this establishes relation (10) and demonstrates Theorem 2. □

Putting $\eta(z) = \frac{1+z}{1-z}$ into Theorem 2 leads to corollary 1.

Corollary 1. *Let $f \in \mathcal{A}_p$ and $\alpha \in \mathbb{C}$. Then*

$$\Omega^{m+1}_{\lambda,\delta}\left(\alpha, \frac{1+z}{1+z}\right) \subseteq \Omega^{m}_{\lambda,\delta}(\alpha,\tilde{\eta}), \tag{9}$$

where

$$\tilde{\eta}(z) = \frac{1}{\delta}z^{p(1-\frac{1}{\delta})} + \frac{p}{\delta}z^{-\frac{p}{\delta}}\int_0^z \frac{\tau^{p-1}}{\tau-1}d\tau, \quad z \in \Delta.$$

Putting $\eta(z) = e^{\lambda z}$, $0 < \lambda \leq 1$ and $p = 2$ into Theorem 2 yields Corollary 2.

Corollary 2. *Let $f \in \mathcal{A}_2$ and $\alpha \in \mathbb{C}$. Then*

$$\Omega_{\lambda,\delta}^{m+1}\left(\alpha, e^{\lambda z}\right) \subseteq \Omega_{\lambda,\delta}^{m}\left(\alpha, \frac{2}{\delta} z^{\frac{-2}{\delta}}\left[\frac{z}{\lambda}e^{\lambda z} - \frac{1}{\lambda^2}e^{\lambda z} + \frac{1}{\lambda^2}\right]\right). \tag{10}$$

Theorem 3. *Let $0 \leq \theta < \nu < p$. Then,*

$$\mathrm{S}_{\delta,n}^{\lambda,m+1}(\nu) \subseteq \mathrm{S}_{\delta,n}^{\lambda,m}(\theta),$$

where θ is given by

$$\theta = \frac{2p\left(1 - 2\nu\frac{1-\delta}{\delta}\right)}{\sqrt{\left(1 - 2\nu - 2\frac{p(1-\delta)}{\delta}\right)^2 + 8p\left(1 - 2\nu\frac{1-\delta}{\delta}\right) + \left(1 - 2\nu - 2\frac{p(1-\delta)}{\delta}\right)}}. \tag{11}$$

Proof. Assume that $f \in \mathrm{S}_{\delta,n}^{\lambda,m+1}(\nu)$. We suppose that

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m} f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m} f(z)} = (p-\theta)\eta(z) + \theta, \tag{12}$$

where

$$\eta(z) = \left(\frac{n}{4} + \frac{1}{2}\right)\eta_1(z) - \left(\frac{n}{4} - \frac{1}{2}\right)\eta_2(z), \tag{13}$$

and $\eta_i(z) \in \mathfrak{P}$ $(i = 1, 2)$.

Using (3) and (12), we obtain

$$\frac{p}{\delta}\frac{\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)}{\mathfrak{B}_{\lambda,\delta}^{m} f(z)} = (p-\theta)\eta(z) + \theta + \frac{p(1-\delta)}{\delta}. \tag{14}$$

By computing the logarithmical derivative of relation (14), we get

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)} - \nu = \theta - \nu + (p-\theta)\eta(z) + \frac{(p-\theta)z\eta'(z)}{(p-\theta)\eta(z) + \theta + \frac{p(1-\delta)}{\delta}}. \tag{15}$$

Since $\eta_i(z) \in \mathfrak{P}$ $(i = 1, 2)$ and $(p-\theta)\eta(z) + \theta \in \mathfrak{P}_{p,n}(\theta)$, then from (13) and (15) we have

$$\begin{aligned}
\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)} - \nu \quad &= \quad \left(\frac{n}{4} + \frac{1}{2}\right)\left(\theta - \nu + (p-\theta)\eta_1(z) + \frac{(p-\theta)z\eta_1'(z)}{(p-\theta)\eta(z) + \theta + \frac{p(1-\delta)}{\delta}}\right) \\
&- \quad \left(\frac{n}{4} - \frac{1}{2}\right)\left(\theta - \nu + (p-\theta)\eta_2(z) + \frac{(p-\theta)z\eta_2'(z)}{(p-\theta)\eta(z) + \theta + \frac{p(1-\delta)}{\delta}}\right).
\end{aligned}$$

Furthermore, we have

$$Re\left\{\theta - \nu + (p-\theta)\eta_i(z) + \frac{(p-\theta)z\eta_i'(z)}{(p-\theta)\eta(z) + \theta + \frac{p(1-\delta)}{\delta}}\right\} > 0, \quad i = 1, 2, \quad z \in \Delta.$$

We define $\Psi(v,w)$ by choosing $v = \eta_i(z)$ and $w = z\eta_i'(z)$ as follows:

$$\Psi(v,w) = \theta - \nu + (p-\theta)v + \frac{(p-\theta)w}{(p-\theta)v + \theta + \frac{p(1-\delta)}{\delta}}.$$

Then, the next conditions are obtained:

(i) $\Psi(v,w)$ is continuous in $\mathfrak{D} = \left(\mathbb{C} - \left\{\frac{\theta + \frac{p(1-\delta)}{\delta}}{\theta - p}\right\}\right) \times \mathbb{C}$;

(ii) $Re\{\Psi(1,0)\} = p - \nu > 0$ and $(1,0) \in \mathfrak{D}$;

(iii) for all $(iw_2, v_1) \in \mathfrak{D}$ such that $v_1 \leq -\frac{1}{2}(1+w_2^2)$

$$\begin{aligned} Re\{\Psi(iw_2,v_1)\} &= Re\left\{\theta - \nu + (p-\theta)iw_2 + \frac{(p-\theta)v_1}{(p-\theta)iw_2 + \theta + \frac{p(1-\delta)}{\delta}}\right\} \\ &= \theta - \nu + \frac{(p-\theta)\left(\theta + \frac{p(1-\delta)}{\delta}\right)v_1}{(p-\theta)^2 w_2^2 + \left(\theta + \frac{p(1-\delta)}{\delta}\right)^2} \\ &\leq \theta - \nu - \frac{(p-\theta)\left(\theta + \frac{p(1-\delta)}{\delta}\right)(1+w_2^2)}{2\left[(p-\theta)^2 w_2^2 + \left(\theta + \frac{p(1-\delta)}{\delta}\right)^2\right]} = \frac{\Lambda_1 + \Lambda_2 w_2^2}{2\Lambda_3}, \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 &= 2(\theta-\nu)\left(\theta + \frac{p(1-\delta)}{\delta}\right)^2 - (p-\theta)\left(\theta + \frac{p(1-\delta)}{\delta}\right), \\ \Lambda_2 &= 2(\theta-\nu)(p-\theta)^2 - (p-\theta)\left(\theta + \frac{p(1-\delta)}{\delta}\right), \\ \Lambda_3 &= (p-\theta)^2 w_2^2 + \left(\theta + \frac{p(1-\delta)}{\delta}\right)^2. \end{aligned}$$

It is clear that $\Lambda_3 > 0$. Since $0 \leq \theta \leq \nu < p$, we obtain that $\Lambda_2 < 0$. From the assertion (11) we obtain that $\Lambda_1 < 0$. Thus, $Re\{\Psi(iw_2, v_1)\} < 0$.

Therefore, from $\eta_1(z), \eta_2(z) \in \mathfrak{P}$, applying Lemma 2 establishes that $(p-\theta)\eta(z) + \theta \in P_{p,n}(\theta)$ for all $z \in \Delta$. Thus, this completes the proof of the theorem. □

Theorem 4. *Let $0 \leq \theta < \nu < p$. Then*

$$\mathrm{C}_{\delta,n}^{\lambda,m+1}(\nu) \subseteq \mathrm{C}_{\delta,n}^{\lambda,m}(\theta),$$

where θ is given by (11).

Proof. Let

$$\begin{aligned} f \in \mathrm{C}_{\delta,n}^{\lambda,m+1}(\nu) &\Rightarrow \mathfrak{B}_{\lambda,\delta}^{m+1} f(z) \in \mathrm{C}_p^k(\nu) \Rightarrow \frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1} f(z)\right)'}{p} \in \mathrm{S}_p^k(\nu) \\ &\Rightarrow \mathfrak{B}_{\lambda,\delta}^{m+1}\left(\frac{zf'(z)}{p}\right) \in \mathrm{S}_p^k(\nu) \Rightarrow \frac{zf'(z)}{p} \in \mathrm{S}_{\delta,n}^{\lambda,m+1}(\nu) \subseteq \mathrm{S}_{\delta,n}^{\lambda,m}(\nu) \\ &\Rightarrow \mathfrak{B}_{\lambda,\delta}^{m}\left(\frac{zf'(z)}{p}\right) \in \mathrm{S}_p^k(\theta) \Rightarrow \mathfrak{B}_{\lambda,\delta}^{m} f(z) \in C_p^k(\theta) \Rightarrow f(z) \in \mathrm{C}_{\delta,n}^{\lambda,m}(\theta). \end{aligned}$$

Thus, this ends the proof of the theorem. □

Theorem 5. *Let $0 \leq \nu < \tau < p$ and $m > p$. Then, we have*

$$\mathrm{K}_{\delta,n}^{\lambda,m+1}(\tau,\nu) \subset \mathrm{K}_{\delta,n}^{\lambda,m}(\tau,\nu).$$

Proof. Assume that $f \in \mathrm{K}_{\delta,n}^{\lambda,m+1}(\tau,\nu)$. Thus, there exists $g \in \mathcal{A}_p$ such that $\mathfrak{B}_{\lambda,\delta}^{m+1}g(z) \in S_p^2(\tau)$ and

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1}f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1}g(z)} \in \mathfrak{P}_{p,n}(\nu), \tag{16}$$

where $g(z) \in \mathrm{S}_{\delta,2}^{\lambda,m+1}(\tau)$. Now, we set

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}g(z)} = \beta(z) = (p-\nu)\eta(z) + \nu, \tag{17}$$

where $\eta(z)$ is given by (13).

By using (3) in (16), we obtain that

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1}f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1}g(z)} = \frac{\frac{\delta}{p}z\left(\mathfrak{B}_{\lambda,\delta}^{m}zf'(z)\right)' + (1-\delta)\mathfrak{B}_{\lambda,\delta}^{m}(zf'(z))}{\frac{\delta}{p}z\left(\mathfrak{B}_{\lambda,\delta}^{m}g(z)\right)' + (1-\delta)\mathfrak{B}_{\lambda,\delta}^{m}g(z)}. \tag{18}$$

Because of $\mathfrak{B}_{\lambda,\delta}^{m+1}g(z) \in \mathrm{S}_{\delta,2}^{\lambda,m+1}(\tau)$ and by using Theorem 3, we deduce that $\mathfrak{B}_{\lambda,\delta}^{m}g(z) \in \mathrm{S}_{\delta,2}^{\lambda,m}(\tau)$.

Therefore, we have

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}g(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}g(z)} = \beta_0(z) = (p-\tau)\eta_0(z) + \tau, \tag{19}$$

where $\eta_0(z)$ is analytic in Δ with $\eta_0(0) = 1$.

Using the derivative of Equation (17) in order to z, we have

$$z\left(\mathfrak{B}_{\lambda,\delta}^{m}zf'(z)\right)' = z\beta'(z)\mathfrak{B}_{\lambda,\delta}^{m}g(z) + \beta(z)z\left(\mathfrak{B}_{\lambda,\delta}^{m}g(z)\right)'. \tag{20}$$

By using assertion (19) and (20), we obtain

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}zf'(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}g(z)} = z\beta'(z) + \beta(z)\beta_0(z). \tag{21}$$

From (18) and (21), we obtain

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1}f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1}g(z)} = \frac{\delta(z\beta'(z) + \beta(z)\beta_0(z)) + p(1-\delta)\beta(z)}{\delta\beta_0(z) + p(1-\delta)},$$

or equivalently

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m+1}f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m+1}g(z)} = \frac{z\beta'(z)}{\beta_0(z) + \frac{p}{\delta}(1-\delta)} + \beta(z). \tag{22}$$

Let

$$\beta(z) = \left(\frac{n}{4} + \frac{1}{2}\right)[(p-\nu)\eta_1(z) + \nu] - \left(\frac{n}{4} - \frac{1}{2}\right)[(p-\nu)\eta_2(z) + \nu],$$

and

$$\beta_0(z) + \frac{p}{\delta}(1-\delta) = (p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta).$$

From (17) and (22), we get

$$\begin{aligned} \frac{z\left(\mathfrak{B}^{m+1}_{\lambda,\delta} f(z)\right)'}{\mathfrak{B}^{m+1}_{\lambda,\delta} g(z)} - \nu \quad &= \left(\frac{n}{4} + \frac{1}{2}\right)\left((p-\nu)\eta_1(z) + \frac{(p-\nu)z\eta_1'(z)}{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)}\right) \\ &- \left(\frac{n}{4} - \frac{1}{2}\right)\left((p-\nu)\eta_2(z) + \frac{(p-\nu)z\eta_2'(z)}{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)}\right). \end{aligned}$$

Furthermore, we have

$$Re\left\{(p-\nu)\eta_i(z) + \frac{(p-\nu)z\eta_i'(z)}{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)}\right\} > 0, \quad i = 1,2, \quad z \in \Delta.$$

Now, we define $\Psi(v,w)$ by choosing $v = \eta_i(z)$ and $w = z\eta_i'(z)$ as

$$\Psi(v,w) = (p-\nu)v + \frac{(p-\nu)w}{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)}.$$

Then, we have the next conditions:

(i) $\Psi(v,w)$ is continuous in $\mathfrak{D} = \mathbb{C} \times \mathbb{C}$;
(ii) $Re\{\Psi(1,0)\} = p - \nu > 0$ and $(1,0) \in \mathbb{C} \times \mathbb{C}$;
(iii) for all $(iw_2, v_1) \in \mathbb{C} \times \mathbb{C}$ such that $v_1 \le -\frac{1}{2}(1 + w_2^2)$

$$\begin{aligned} Re\{\Psi(iw_2, v_1)\} &= Re\left\{(p-\nu)iw_2 + \frac{(p-\nu)v_1}{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)}\right\} \\ &= Re\left\{(p-\nu)iw_2 + \frac{(p-\nu)v_1}{(p-\tau)(\eta_{01} + i\eta_{02}) + \tau + \frac{p}{\delta}(1-\delta)}\right\} \\ &= \frac{(p-\nu)[(p-\tau)\eta_{01} + \tau + \frac{p}{\delta}(1-\delta)]v_1}{\left((p-\tau)\eta_{01} + \tau + \frac{p}{\delta}(1-\delta)\right)^2 + \left((p-\tau)\eta_{02}\right)^2} < 0. \end{aligned}$$

Therefore, from $\eta_1(z), \eta_2(z) \in \mathfrak{P}$, applying Lemma 2, is established that $(p-\nu)\eta(z) + \nu \in P_{p,n}(\nu)$ for all $z \in \Delta$. Hence, $Re\{(p-\tau)\eta_0(z) + \tau + \frac{p}{\delta}(1-\delta)\} > 0$ for all $z \in \Delta$. Thus, the proof of Theorem 5 is completed. □

4. Convolution

We discuss various results involving the new subclass of univalent function with BD. In addition, we obtain some convolution properties.

Theorem 6. *Let $f(z) \in \Omega^m_{\lambda,\delta}(\alpha,\eta)$, $g(z) \in A_p$ and*

$$Re(z^{-p}g(z)) > \frac{1}{2}, \quad z \in \Delta. \tag{23}$$

Then,

$$(f * g)(z) \in \Omega^{m}_{\lambda,\delta}(\alpha,\eta).$$

Proof. Assume that $f(z) \in \Omega^{m}_{\lambda,\delta}(\alpha,\eta)$ and $g(z) \in A_p$, we obtain that

$$(1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z)) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z))\right)' \prec \eta(z).$$

Now,

$$\begin{aligned} & (1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z)) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z))\right)' \\ = & \left[(1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}f(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}f(z)\right)'\right] * \left(z^{-p}g(z)\right) \\ = & \varphi(z) * \left(z^{-p}g(z)\right), \end{aligned} \tag{24}$$

where

$$\varphi(z) = (1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}f(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}f(z)\right)'. \tag{25}$$

From the inequality (23), we can use the Herglotz representation (see [14], p. 96) as follows

$$z^{-p}g(z) = \int \frac{d\mu(\tau)}{1-z\tau}, \quad z \in \Delta, \tag{26}$$

where $\mu(\tau)$ is the probability measure

$$\int_{|\tau|=1} d\mu(\tau) = 1.$$

Now, from (24) and (26), we get

$$(1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z)) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z))\right)' = \int_{|x|=1} \varphi(xz)d\mu(x) \prec \eta(z).$$

Hence, the desired result is obtained. □

Theorem 7. *Let $f(z) \in \Omega^{m}_{\lambda,\delta}(\alpha,\eta)$, $g(z) \in A_p$ and $z^{-p+1}g(z) \in \mathfrak{R}(\nu)$, $(\nu < 1)$. Then,*

$$f(z) * g(z) \in \Omega^{m}_{\lambda,\delta}(\alpha,\eta), \qquad (z \in \Delta). \tag{27}$$

Proof. Assume that $f(z) \in \Omega^{m}_{\lambda,\delta}(\alpha,\eta)$ and $g(z) \in A_p$, we have

$$\begin{aligned} & (1-\alpha)z^{-p}\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z)) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^{m}_{\lambda,\delta}(f(z) * g(z))\right)' \\ = & \frac{z^{-p+1}g(z) * z\varphi(z)}{z} = \frac{z^{-p+1}g(z) * z\varphi(z)}{z^{-p+1}g(z) * z}, \end{aligned}$$

where $\varphi(z)$ is given in (25). By Lemma 3, we obtain

$$\frac{z^{-p+1}g(z) * z\varphi(z)}{z^{-p+1}g(z) * z} \subseteq \overline{co}(\varphi(\Delta)).$$

Since $\eta(z)$ is convex in Δ and $\varphi(z) \prec \eta(z)$, then this establishes the assertion (27). This demonstrates Theorem 7. □

Theorem 8. *Let $f \in \mathtt{S}_{\delta,2}^{\lambda,m}(\nu)$ and $g \in \mathfrak{R}(\nu)$. Then,*

$$f * g \in \mathtt{S}_{\delta,2}^{\lambda,m}(\nu).$$

Proof. Assume that

$$\psi(z) = \frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m} f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m} f(z)}, \qquad z \in \Delta.$$

Now, we have

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))} = \frac{g(z) * z\left(\mathfrak{B}_{\lambda,\delta}^{m} f(z)\right)'}{g(z) * \mathfrak{B}_{\lambda,\delta}^{m} f(z)} = \frac{g(z) * \psi(z)\mathfrak{B}_{\lambda,\delta}^{m} f(z)}{g(z) * \mathfrak{B}_{\lambda,\delta}^{m} f(z)}.$$

Using Lemma 3, we obtain

$$\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))} \subseteq \overline{co}(\psi(\Delta)).$$

Since $\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))\right)'}{\mathfrak{B}_{\lambda,\delta}^{m}(f(z) * g(z))}$ is analytic in Δ and

$$\psi(\Delta) \subseteq \Omega = \left\{ w : \frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m} w(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m} w(z)} \in \mathfrak{P}_{p,2}(0) \right\},$$

then $\frac{z\left(\mathfrak{B}_{\lambda,\delta}^{m} f(z)\right)'}{\mathfrak{B}_{\lambda,\delta}^{m} f(z)}$ lies in Ω. This established that $f * g \in \mathtt{S}_{\delta,2}^{\lambda,m}(\nu)$. This ends the proof of Theorem 8. □

Remark 1. *Let $f \in \mathtt{C}_{\delta,2}^{\lambda,m}(\nu)$ and $g \in \mathfrak{R}(\nu)$. Then,*

$$f * g \in \mathtt{C}_{\delta,2}^{\lambda,m}(\nu).$$

Theorem 9. *Let $\alpha \geq 0$, $h(z) \in \mathcal{A}_p(1)$ with $Re\{\eta(z)\} > \frac{1}{2}$ and*

$$f_j(z) = z^p + \sum_{k=p}^{\infty} a_{k,j} z^{k+p} \in \Omega_{\lambda,\delta}^{m}(\alpha, \eta_j), \quad j = 1, 2, \tag{28}$$

such that

$$\eta_j(z) = \vartheta_j + (1 - \vartheta_j)h(z), \quad \text{and} \quad 0 < \vartheta_j < 1, \quad j = 1, 2.$$

If $f(z) \in \mathcal{A}_p$ is given by

$$\mathfrak{B}_{\lambda,\delta}^{m} f(z) = \left(\mathfrak{B}_{\lambda,\delta}^{m} f_1(z)\right) * \left(\mathfrak{B}_{\lambda,\delta}^{m} f_2(z)\right), \tag{29}$$

then $f(z) \in \Omega_{\lambda,\delta}^{m}(\alpha, \eta)$, where

$$\eta(z) = \vartheta + (1 - \vartheta)h(z) \quad \text{and} \quad 0 < \vartheta < 1 \quad j = 1, 2, \tag{30}$$

and the parameter ϑ is given by

$$\vartheta = \begin{cases} 1-2(1-\vartheta_1)(1-\vartheta_2)\left(1-\frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\frac{1-t^p}{1+t^p}dt\right), & \alpha>0, \\ 1-2(1-\vartheta_1)(1-\vartheta_2), & \alpha=0. \end{cases}$$

Proof. Assume that

$$\mathsf{F}_j(z) = (1-\alpha)z^{-p}\mathfrak{B}^m_{\lambda,\delta}f_j(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^m_{\lambda,\delta}f_j(z)\right)', \qquad j=1,2, \tag{31}$$

where $f_j(z)$, $(j=1,2)$ is given by (28). Thus, we obtain

$$\mathsf{F}_j(z) = 1+\sum_{k=0}^{\infty} b_{k,j}z^{k+p} \prec \vartheta_j + (1-\vartheta_j)\eta(z), (z\in\Delta, j=1,2), \tag{32}$$

and

$$\mathfrak{B}^m_{\lambda,\delta}f_j(z) = \frac{p}{\alpha}z^{\frac{-p(1-\alpha)}{\alpha}}\int_0^z \tau^{\frac{p}{\alpha}-1}\mathsf{F}_j(\tau)d\tau, \quad j=1,2.$$

From (29) and (31), we obtain

$$\begin{aligned} \mathfrak{B}^m_{\lambda,\delta}f(z) &= \left(\mathfrak{B}^m_{\lambda,\delta}f_1(z)\right) * \left(\mathfrak{B}^m_{\lambda,\delta}f_2(z)\right) \\ &= \left(\frac{p}{\alpha}z^p\int_0^1 t^{\frac{p}{\alpha}-1}\mathsf{F}_1(tz)dt\right) * \left(\frac{p}{\alpha}z^p\int_0^1 t^{\frac{p}{\alpha}-1}\mathsf{F}_2(tz)dt\right) \\ &= \frac{p}{\alpha}z^p\int_0^1 t^{\frac{p}{\alpha}-1}\mathsf{F}(tz)du, \end{aligned} \tag{33}$$

where

$$\mathsf{F}(z) = \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}(\mathsf{F}_1*\mathsf{F}_1)(tz)dt.$$

Now, by applying the Herglotz theorem on Equation (32), we get

$$Re\left\{\left(\frac{\mathsf{F}_1(z)-\vartheta_1}{1-\vartheta_1}\right) * \left(\frac{1}{2}+\frac{\mathsf{F}_2(z)-\vartheta_2}{2(1-\vartheta_2)}\right)\right\} > 0.$$

This implies that

$$Re\{(\mathsf{F}_1*\mathsf{F}_2)(z)\} \ge \vartheta_0 = 1-2(1-\vartheta_1)(1-\vartheta_2), \qquad z\in\Delta.$$

Furthermore, by applying Lemma 4, we obtain

$$Re\{(\mathsf{F}_1*\mathsf{F}_2)(z)\} \ge \vartheta_0 + (1-\vartheta_0)\frac{1-|z|^p}{1+|z|^p}, \qquad z\in\Delta. \tag{34}$$

Thus, from the Equations (33) and (34), we have

$$\begin{aligned} & Re\left\{(1-\alpha)z^{-p}\mathfrak{B}^m_{\lambda,\delta}f(z) + \frac{\alpha}{p}z^{-p+1}\left(\mathfrak{B}^m_{\lambda,\delta}f(z)\right)'\right\} = Re\{\mathsf{F}(z)\} \\ = \; & Re\left\{\frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}(\mathsf{F}_1*\mathsf{F}_2)(tz)dt\right\} = \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}Re\{(\mathsf{F}_1*\mathsf{F}_2)(tz)\}dt \\ \ge \; & \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\left(\vartheta_0+(1-\vartheta_0)\frac{1-|z|^pt^p}{1+|z|^pt^p}\right)dt = \vartheta_0 + \frac{p(1-\vartheta_0}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\frac{1-|z|^pt^p}{1+|z|^pt^p}dt \\ > \; & \vartheta_0 + \frac{p(1-\vartheta_0)}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\frac{1-t^p}{1+t^p}dt = 1-2(1-\vartheta_1)(1-\vartheta_2)\left(1-\frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\frac{1-t^p}{1+t^p}dt\right) := \vartheta. \end{aligned}$$

Thus, this establishes that $f(z) \in \Omega^m_{\lambda,\delta}(\alpha,\eta)$ for the function $\eta(z)$ that is given by (30). Hence, the proof of the theorem is completed. $\square$

5. Conclusions

For univalent functions in a unit disk Δ, the new differential operator $\mathfrak{B}^m_{\lambda,\delta}$ was defined by using probability mass functions. This operator is a generalization of the BD. Moreover, several subclasses of multivalent functions were defined by means of the operator $\mathfrak{B}^m_{\lambda,\delta}$. Several inclusion relations for the new subclasses of analytic functions were obtained. Finally, convolution properties for functions in the defined subclasses were studied.

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