On the product of almost discrete Grothendieck spaces

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Abstract

A topological space X is called almost discrete, if it has precisely one nonisolated point. In this paper, we get that for a countable product $X = \prod X_i$ of almost discrete spaces X_i the space $C_p(X)$ of continuous real-valued functions with the topology of pointwise convergence is a μ-space if, and only if, X is a weak q-space if, and only if, $t(X) = \omega$ if, and only if, X is functionally generated by the family of all its countable subspaces.

This result makes it possible to solve Archangel'skii's problem on the product of Grothendieck spaces. It is proved that in the model of ZFC, obtained by adding one Cohen real, there are Grothendieck spaces X and Y such that $X \times Y$ is not weakly Grothendieck space. In (PFA): the product of any countable family almost discrete Grothendieck spaces is a Grothendieck space.

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1. Introduction

In 1952 Grothendieck [14] proved the following result.

Theorem 1.1. *(Grothendieck) Let X be a compact space and let Y be a metrizable space. Then each relatively countably compact subspace of $C_p(X, Y)$ is relatively compact.*

This theorem has played an important role in topology, mathematical and functional analysis. Grothendieck's theorem has been generalized many times [1, 2, 9, 18, 19].

A topological space X is called *almost discrete*, if it has precisely one nonisolated point.

A space X is called *g-space*, if for every subset A of X such that A is countably compact in X, the closure of A in X is compact. A space X is called *a Grothendieck space (a weakly Grothendieck space)*, if $C_p(X)$ is a hereditary g-space (a g-space).

The product of two weakly Grothendieck spaces need not be a weakly Grothendieck space (see Proposition 4.1).

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In 1998, A.V. Archangel'skii posed the following question (Question 5.13 in [4], see also Problem 4.8.3 in [11]):

Is the product of two Grothendieck spaces a weakly Grothendieck? A Grothendieck space?

In this paper we get the following results:

• Countable product X of almost discrete spaces is weakly Grothendieck if, and only if, the tightness of X is countable (Th. 3.3).

• Countable product X of almost discrete spaces is Grothendieck if, and only if, the tightness of X is countable and X is Lindelöf (Th. 3.6).

• In the model of ZFC, obtained by adding one Cohen real, there are almost discrete Grothendieck spaces Y_0 and Y_1 such that $Y_0 \times Y_1$ is not weakly Grothendieck (Th. 5.2).

• (PFA) Countable product of almost discrete Grothendieck spaces is a Grothendieck space. (Th. 5.5).

2. Notation and terminology

The set of positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the real line. We denote by $\overline{A}$ the closure of A (in X).

Recall that a subset A of a topological space X is called

• *relatively compact* if A has a compact closure in X;

• *bounded* if every continuous function on X is bounded on A;

• *countably compact* if each sequence of A has a limit point in A;

• *relatively countably compact* if each sequence of A has a limit point in X;

• *pseudocompact* if any continuous real-valued function on A is bounded;

• *countably pracompact* subspace of X, if there is a subspace $Y \subseteq A$ which is dense in A and countably compact in A in the following sense: every infinite set $B \subseteq Y$ has a limit point in A.

A topological space X is called *μ-space*, if each bounded subset of X is relatively compact.

Let X be a Tychonoff topological space, $C(X, \mathbb{R})$ be the space of all continuous functions on X with values in $\mathbb{R}$ and τ_p be the pointwise convergence topology. Denote by $C_p(X)$ the topological space $(C(X, \mathbb{R}), \tau_p)$.

It will be convenient for us to use (for brevity) the notation proposed by E. Reznichenko for the following general property.

Let $\mathcal{P}$ be a family subsets of $C_p(X)$. If A is relatively compact in $C_p(X)$ for any $A \in \mathcal{P}$, then we will write that X is a *$\mu^{\sharp}_{\mathcal{P}}$-space*.

In this paper we consider the following families $\mathcal{P}$ of subsets of $C_p(X)$:

rc — family of all relatively compact subsets;

b — family of all bounded subsets;

cc — family of all countably compact subsets;
rcc — family of all relatively countably compact subsets;
pcc — family of all pseudocompact subsets;
cp — family of all countably pracompact subsets.

Note that $C_p(X)$ is a μ-space equivalent to X is a $\mu_b^\sharp$-space.
Also note that weakly Grothendieck spaces in this terminology are $\mu_{cc}^\sharp$-spaces.

Let $\mathcal{M}$ be a family of subspaces of a space X. We say that X is *functionally generated by* $\mathcal{M}$ if for any discontinuous function $f : X \to \mathbb{R}$ there is an $A \in \mathcal{M}$ such that $f|A$ cannot be extended to a continuous real-valued function on the whole of X (Definition 2.11 in [6]).

For example, X is functionally generated by the family of all its countable subspaces if and only if its weak functional tightness $t_{\mathbf{R}}$ is countable: $t_{\mathbf{R}}(X) = \omega$; this follows directly from the definition of weak functional tightness $t_{\mathbf{R}}$ in [5].

The *tightness* $t(X)$ of X is the smallest infinite cardinal τ such that for any set $A \subset X$ and any point $x \in X$ there is a set $B \subset X$ for which $|B| \leq \tau$ and $x \in \overline{B}$.

The notation of a weak q-space defined in [8]. Recall that a space is called *realcomplete* if it is homeomorphic to a closed subspace of the space $\mathbb{R}^\tau$ for a certain τ.

For other notation and terminology almost without exceptions we follow the Engelking's book [10].

3. Countable product of almost discrete spaces

Lemma 3.1. *(Corollary II.4.17 in [5]) If $t(X) = \omega$, then $C_p(X)$ is realcomplete and, hence, $C_p(X)$ is a μ-space.*

Let $f : X \to Y$ be a map (between *sets* X and Y). Define the map $f^\sharp : \mathbb{R}^Y \to \mathbb{R}^X$ (between *topological spaces*) dual to f as follows: if $\phi \in \mathbb{R}^Y$, then $f^\sharp(\phi)(x) = \phi(f(x))$ for all $x \in X$, i.e. $f^\sharp(\phi) = \phi \circ f$.

Lemma 3.2. *Let X and Y be topological spaces, $f : X \to Y$ be a quotient map from X onto Y. Then, Y is a weakly Grothendieck space provided X is a weakly Grothendieck space.*

Proof. By Proposition 0.4.8 (2) in [5], $f^\sharp(C_p(Y))$ is a closed subset of $C_p(X)$. Hence, $f^\sharp(\overline{A}^{C_p(Y)}) = \overline{f^\sharp(A)}^{C_p(X)}$ for any $A \subset C_p(Y)$. Note that $\overline{f^\sharp(A)}^{C_p(X)}$ is compact for any countably compact subset A of $C_p(Y)$.

Since f is surjection, by Proposition 0.4.6 in [5], $f^\sharp$ is a homeomorphism. Hence, $\overline{A}^{C_p(Y)}$ is compact.

□

Theorem 3.3. *Let X be a countable product of almost discrete spaces. Then the following statements are equivalent:*

1. $C_p(X)$ *is a* μ*-space;*
2. $C_p(X)$ *is realcomplete;*
3. X *is a* $\mu^{\sharp}_{pc}$*-space;*
4. X *is a* $\mu^{\sharp}_{rcc}$*-space;*
5. X *is a* $\mu^{\sharp}_{cp}$*-space;*
6. X *is a weakly Grothendieck space;*
7. $t(X) = \omega$*;*
8. $t_{\mathbf{R}}(X) = \omega$*;*
9. X *is a weak* q*-space.*

Proof. The following implications are trivial: (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (6); (1) $\Rightarrow$ (4) $\Rightarrow$ (5); (7) $\Rightarrow$ (8); (7) $\Rightarrow$ (9).

By Theorem 2.16 in [6], (8) $\Rightarrow$ (1). By Theorem 1 in [8], (9) $\Rightarrow$ (1). By Corollary II.4.17 in [5], (2) $\Leftrightarrow$ (8).

(6) $\Rightarrow$ (7). (a) Let $X = \prod\limits_{i=1}^{n} X_i$ be a finite product of almost discrete spaces X_i ($i = 1, ..., n$) and X is a weakly Grothendieck space. Claim that $t(X) = \omega$.

• Let $n = 1$. Assume that $t(X) > \omega$.

Let x_0 be a nonisolated point of X. Then there is $A \subset X$ such that $|A| > \omega$, $x_0 \in \overline{A} \setminus A$ and any countable subset of A is closed in X.

Let $F = \{f \in C_p(X) : f(X \setminus S) = 0,\ f(S) = 1,\ S \in [A]^{\omega}\}$. Note that any $S \in [A]^{\omega}\}$ is clopen subset of X.

Claim that F is countably compact and, hence, is bounded in $C_p(X)$. Indeed, if $\{f_n : n \in \mathbb{N}\} \subseteq F$, and $B = \bigcup\{f_n^{-1}(\{1\}) : n \in \mathbb{N}\}$ then the function

$$f_0(x) = \begin{cases} 0 \ \ for \ \ x \notin B; \\ s(x) \ \ for \ \ x \in B \end{cases}$$

where $s(x)$ is a limit point the sequence $\{f_n|B : n \in \mathbb{N}\} \subset 2^B$ belongs to F and is an accumulation point for the set $\{f_n : n \in \mathbb{N}\}$.

On the other hand, $\overline{F}^{C_p(X)}$ is not compact because the function

$$h(x) = \begin{cases} 0 \ \ for \ \ x \notin A; \\ 1 \ \ for \ \ x \in A \end{cases}$$

is discontinuous and $h \in \overline{F}^{\mathbb{R}^X}$.

• Let $n > 1$ and the theorem is true for any $k < n$.

Let $X = X_1 \times ... \times X_n$ and X is a weakly Grothendieck space. Denote by x_i is one nonisolated point in X_i for every $i = 1, ..., n$. Consider the subspace $S = \bigcup\{(X_1 \times ... \times X_{i-1} \times \{x_i\} \times X_{i+1} \times ... \times X_n) : i = 1, ..., n\}$. The set S is closed in X and any point in $X \setminus S$ is a isolated point in X. Note that S is a quotient image of the space $Q = \oplus\{(X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n) : i = 1, ..., n\}$. By induction, $t(Q) = \omega$ and $t(S) = \omega$ (Proposition 3 in [7]).

Let Z be a quotient space of X with S identified to a point. Let $\pi : X \to Z$ be the quotient map. Note that π is a closed map and $\{S\}$ is one nonisolated point in Z. By Lemma 3.2, Z is a weakly Grothendieck space and, hence, $t(Z) = \omega$. Since π is a continuous closed surjective map, $t(X) = \sup\{t(Z), t(\pi^{-1}(x)) : x \in Z\}$ (Theorem 4.5 in [3]). It follows that $t(X) = \omega$.

(b) Let $X = \prod X_i$ be a countable product of almost discrete spaces X_i and X is a weakly Grothendieck space. Let $\pi_{i_1,...,i_k} : X \to X_{i_1} \times ... \times X_{i_k}$ be a projection function from X onto $X_{i_1} \times ... \times X_{i_k}$. Then $\pi_{i_1,...,i_k}$ is a quotient map and, by Lemma 3.2, $X_{i_1} \times ... \times X_{i_k}$ is a weakly Grothendieck space. By (a), $t(X_{i_1} \times ... \times X_{i_k}) = \omega$. By Remark 3 in [17], $t(X) = \omega$.

□

Corollary 3.4. An almost discrete space X is weakly Grothendieck if and only if $t(X) = \omega$.

Theorem 3.5. *(Theorem 5.6 in [4]) A weakly Grotendieck space X is a Grothendieck space if and only if every compact subspace of $C_p(X)$ is Fréchet-Urysohn.*

Let τ be an infinite cardinal, $D(\tau)$ the discrete space of cardinality τ, $\xi \notin D(\tau)$, and $L(\tau) = D(\tau) \cup \{\xi\}$ a space in which only the point ξ is not isolated, and where the neighborhoods of ξ are all sets $V \subset L(\tau)$ such that $\xi \in V$ and $L(\tau) \setminus V$ is countable. Clearly, $L(\tau)$ is a Lindelöf P-space.

Recall that a space Y is called *a primary Lindelöf space*, if Y is a continuous image of closed subspace of the space $(L(\tau))^\omega$.

Theorem 3.6. *A countable product $X = \prod X_i$ of almost discrete spaces X_i is Grothendieck if, and only if, $t(X) = \omega$ and X is Lindelöf.*

Proof. ($\Rightarrow$). By Theorem 3.3, $t(X) = \omega$. Every continuous image of a Grothendieck space is Grothendieck (Theorem 5.4 in [4]). Then, X_i is Grothendieck. It remains to note that if an almost discrete space is Grothendieck, then it is a continuous bijective image of one-point Lindelöfication $L(\tau)$ of a discrete space. Note that the space $(L(\tau))^\omega$ is Lindelöf (Proposition IV.2.19 in [5]). It follows that X is Lindelöf.

($\Leftarrow$). By Theorem 3.3, X is a weakly Grothendieck space. Note that an almost discrete Lindelöf space is a continuous bijective image of one-point Lindelöfication of discrete space. Thus, for every $i \in \mathbb{N}$ there is a one-point Lindelöfication Y_i of discrete space of cardinality $|X_i|$ such that X_i is a bijective continuous image of Y_i.

The product $Y = \prod Y_i$ is a primary Lindelöf space. Then, $C_p(Y)$ can be mapped by a one-to-one continuous linear map into the Σ-product of a certain number of copies of the real line (Proposition IV.3.10 in [5]). Since $C_p(X)$ is homeomorphic to a subset of $C_p(Y)$, every compact subset of $C_p(X)$ is a Corson compactum and, hence, it is Fréchet-Urysohn. By Theorem 3.5, X is Grothendieck. □

Corollary 3.7. An almost discrete space X is Grothendieck if, and only if, X is Lindelöf and $t(X) = \omega$.

4. Examples

In [4], A.V. Archangel'skii noted the following result.

Proposition 4.1. *There are almost discrete Fréchet-Urysohn spaces X and Y such that $X \times Y$ is not weakly Grothendieck space.*

Proof. Consider X is the countable sequential fan S_ω and Y is the sequential fan $S_\mathfrak{c}$ of cardinality 2^ω. Then, $t(X \times Y) > \omega$. By Theorem 3.3, $X \times Y$ is not weakly Grothendieck space. □

Note that Archangel'skii's example such that each of the factors is a closed image of the metric space. We can provide an alternative proof of Proposition 4.1. Consider the spaces X and Y in Example 1 in [13]. Then, X and Y are almost discrete Fréchet-Urysohn spaces and $t(X \times Y) > \omega$. By Theorem 3.3, $X \times Y$ is not weakly Grothendieck space. Note that in our example the closure of a countable set is a subspace with the first axiom of countability.

Proposition 4.2. *There is an almost discrete Fréchet-Urysohn space Z such that $Z \times Z$ is not weakly Grothendieck space.*

Proof. Let X and Y be spaces from Proposition 4.1 where x and y are non-isolated points of X and Y, respectively. Let Z be a quotient space of $X \bigcup Y$ with $\{x, y\}$ identified to a point. Then Z is an almost discrete Fréchet-Urysohn space. Since $Z \times Z$ contains $X \times Y$, $t(Z \times Z) > \omega$. By Theorem 3.3, $Z \times Z$ is not weakly Grothendieck space. □

5. Archangel'skii's problem on the product of Grothendieck spaces

Definition 5.1. ([16]) A pair (Y_0, Y_1) of spaces Y_0 and Y_1 is called *a Leiderman pair*, if

(1) Y_0 and Y_1 are almost discrete spaces of cardinality ω_1;
(2) Y_0 and Y_1 are Lindelöf;
(3) $t(Y_0) = t(Y_0^\omega) = t(Y_1) = t(Y_1^\omega) = \omega$;
(4) $t(Y_0 \times Y_1) > \omega$.

The following theorem answers the question posed.

Theorem 5.2. *In the model of ZFC, obtained by adding one Cohen real, there are almost discrete Grothendieck spaces Y_0 and Y_1 such that $Y_0 \times Y_1$ is not weakly Grothendieck space.*

Proof. By Theorem 2 in [15], there is a Leiderman pair (Y_0, Y_1) of almost discrete spaces Y_0 and Y_1. By Corollary 3.7, Y_0 and Y_1 are Grothendieck spaces. Since $t(Y_0 \times Y_1) > \omega$, by Theorem 3.3, $Y_0 \times Y_1$ is not weakly Grothendieck space. □

Similarly Theorem 2 in [15], we note the following result.

Proposition 5.3. *The assertion of the existence of two almost discrete Grothendieck spaces (a Leiderman pair) for which their product is not weakly Grothendieck space, is consistent with any cardinal arithmetic (including with CH and also those with the negation of CH).*

We shall say that a (completely regular) space X is *condensed*, if there is a single point $x \in X$ such that every uncountable subset of X accumulates to x.

Theorem 5.4. *(Theorem 9 in [12]) (PFA) If X is a countably tight condensed space, then its countable power X^ω is also countably tight.*

Theorem 5.5. *(PFA) The product of any countable family almost discrete Grothendieck spaces is a Grothendieck space.*

Proof. Let $Y = \prod Y_i$ be a countable product of almost discrete Grothendieck spaces Y_i. By Corollary 3.7, Y_i is Lindelöf and $t(Y_i) = \omega$ for every $i \in \mathbb{N}$. By Theorem 3.6, we just need to check that $t(Y) = \omega$. Let y_i be one non-isolated point of Y_i for every $i \in \mathbb{N}$. Let Z be a quotient space of $\bigcup\{Y_i : i \in \mathbb{N}\}$ with $\{y_i : i \in \mathbb{N}\}$ identified to a point. Then Z is an almost discrete Lindelöf space and $t(Z) = \omega$. By Theorem 5.4, $t(Z^\omega) = \omega$. Since Y is homeomorphic to a subset of Z^ω, $t(Y) = \omega$.

□

A.V. Archangel'skii and V.V. Tkachuk posed the following question (Question 5.16 in [4] and Problem 4.8.5 in [11]):

Let X and Y be Grothendieck spaces. Is then the free topological sum of X and Y a Grothendieck space?

Note, that the free topological sum of any family of weakly Grothendieck spaces is a weakly Grothendieck space.

The following theorem answers the question in the class of almost discrete spaces.

Theorem 5.6. *Let $X = \bigoplus\{X_i : i \in \mathbb{N}\}$ be a free topological sum of almost discrete Grothendieck spaces X_i. Then X is a Grothendieck space.*

Proof. Note that X_i is a biijective continuous image of one-point Lindelöfication $Y_i = L(\tau)$ of a discrete space of size $\tau_i = |X_i|$ for every i. Then $Y = \bigoplus\{Y_i : i \in \mathbb{N}\}$ is a Lindelöf P-space. Since $C_p(X) \subset C_p(Y)$, every compact subset of $C_p(X)$ is Fréchet-Urysohn. By Theorem 3.5, X is Grothendieck. □

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