Superfluid – Bose glass transition in two dimensions at $T = 0$: analytic solution of a mode-coupling toy model.

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Analytic expression for the memory function and the optical conductivity of the two-dimensional Bose gas with logarithmic interaction at $T = 0$ in presence of point-like impurities is obtained within the mode-coupling approximation. Depending on the value of a dimensionless combination of the model parameters proportional to the strength of the impurity potential, two different phases are distinguished, viz. the disordered superfluid and insulator (Bose glass), separated by an intermediate (quasi)metal state.

Keywords: glass transition, memory function, mode-coupling theory, Bose condensate

I. INTRODUCTION

The behaviour of many-particle classical and quantum systems in presence of a random perturbation is a long-standing problem of great scientific interest.

Various methods have been developed to describe the behaviour of a system subject to the random external force, such as the coherent potential and other effective medium types of approximations[1,2], the self-consistent hydrodynamic scheme for the description of the over-damped charge dynamics in presence of a quenched disorder, elaborated by W. Götze and coworkers[3], the replica theories of flux lattices and Wigner crystals[4].

It is interesting to observe, that for very different model systems in the localization regime, many quantities of physical interest, such as the dynamic conductivity, were found to manifest an intimate resemblance. E.g., in the three-dimensional (3D) electron system, the Drude tail of the optical conductivity flattens out and gradually evolves into a finite energy peak as the system approaches the Anderson transition[5] from the metallic side of the phase diagram. Similar results are obtained for the two-dimensional (2D) electron gas and also for 2D and 3D Bose gas at zero temperature[6,7], where the effects of the disorder lead to the decay of the coherent response of the condensate σ_{coh} and the transfer of the spectral weight to a finite-energy broad feature. The same general picture holds also for pinned density waves[8] and Wigner crystals[4], where the conductivity displays the inhomogeneously broadened feature at the pinning energy.

However, the understanding of the disorder effects in complex quantum systems is still an issue, that have been given the new impetus since the discovery of the strongly correlated oxides. An extreme complexity of these problems usually rules out the possibility of a tractable analytic treatment. In this article we present a simple exactly solvable model of the superfluid-insulator transition in the 2D Bose gas with logarithmic interaction in presence of the random short-range potential of the point defects at zero temperature.

The key result of our analysis is the analytic expression for the memory function, obtained within the framework of the mode-coupling theory[6]. This enables us to explore the evolution of the optical conductivity in the full range of the relevant physical parameters and to trace the superfluid-insulator transition, that takes place, when a dimensionless parameter of the disorder exceeds a critical value. Despite a number of simplifying assumptions, we believe, that the theory, developed in the next section, is of a certain interest, being an elementary but instructive exactly solvable model of the disorder-induced glass transition.

II. THE MODEL

The central role in our analysis plays the memory function $M(z) = M' + iM''$, that measures the relaxation rate of the current excitations and incorporates the effects of an external random perturbation due to the impurities.

Within the mode coupling theory[3] the memory function obeys the self-consistency relation:

$$M(z) = i\gamma + \frac{1}{2nm} \int \frac{\Phi_0(k, z + M(z))}{1 + M(z)\Phi_0(k, z + M(z))/\chi(k, z = 0)} k^2 \langle |U(k)|^2 \rangle d\mathbf{k}, \tag{1}$$

where γ is a bare relaxation constant, corresponding to some internal current decay channels, $U(k)$ is the Fourier transform of the random potential, n and m are the con-centration and the mass of the particles, respectively. The angular brackets stand for the averaging over the disorder. The density response function Φ_0 is related to

the generalized susceptibility $\chi_0(k,z)$ of the system without disorder:

$$\Phi_0(k,z) = \frac{\chi_0(k,z) - \chi_0(k,0)}{z}. \qquad (2)$$

Thus, the free model being specified by its correlation functions, eq. (1) offers a direct way to explore the effects of the disorder on its electric and optical properties.

We consider the 2D Bose gas at zero temperature, where all particles are condensed into the superfluid phase. Hence, no bare mechanisms of relaxation are presumed[20] and $\gamma = 0$ in (1). The susceptibility of noninteracting 2D bosons is readily obtained[11]:

$$\chi_{nonint.}(k,z) = \frac{2n\varepsilon_k}{\varepsilon_k^2 - z^2}, \qquad (3)$$

where $\varepsilon_k = k^2/2m$ is the dispersion of free particles. The interparticle interaction V modifies the susceptibility. A qualitative estimate of this effect may be done within the random phase approximation:

$$\chi_0(k,z) = \frac{\chi_{nonint.}(k,z)}{1 + V(k)\chi_{nonint.}(k,z)}. \qquad (4)$$

Having in mind to work out a simple tractable model, we neglect here a certain limitation of this mean-field type of approximation in low dimensions, as well as the local-field corrections[12]. On the other hand, the mean-field concept retains its significance as a highly intuitive approach, that makes it possible to get insight even into the problems, where its formal substantiation is problematic, e.g. in the physics of the superfluid films[13] and the plane vortices in quasi-two-dimensional magnets[14].

In purely two-dimensional systems, such as the planar vortices, the axially symmetric solution of the 2D Laplace equation provides the following form of the interparticle interaction:

$$V(r) = Q^2 \ln\left(\frac{r_0}{r}\right), \qquad (5)$$

(the two-dimensional analog of the Coulomb potential), where Q is the boson charge and r_0 is an appropriate length scale, that may be considered as a mean interparticle distance, $r_0 \sim n^{1/2}$, where n is the two-dimensional concentration of particles. Thus, the potential $V(r)$ is repulsive for $r < r_0$ and attractive for $r > r_0$, favouring the homogenisation of the system. The ground-state properties of the free 2D Bose gas with the interaction (5) have been investigated in[15]. Correspondingly, the potential in momentum space is:

$$V(k) = \frac{2\pi Q^2}{k^2}, \qquad (6)$$

that may be obtained as the 2D Fourier transform of $V(r)\exp(-\alpha\,k\,r)$ in the limit $\alpha \to 0$.

Hereafter it is convenient to introduce the dimensionless variables $\widetilde{p} = p/p_0$, $\widetilde{\omega} = \omega/\varepsilon_0$, $\widetilde{M} = M/\varepsilon_0$,

defining the units of momentum and energy as $p_0 = (8\pi Q^2\hbar^2 mn)^{1/4}$ and $\varepsilon_0 = p_0^2/2m$, respectively[21]. In what follows, we shall consider the case of identical impurities, with random coordinate distribution. The average ($\langle\ldots\rangle$) in rhs. of eq. (1) then reduces to the multiplication by the impurity concentration n_i. Substituting eqs. (2-4) with (6) into (1) yields the nonlinear equation on the memory function, where all quantities are reduced to the dimensionless form and tilda's are omitted:

$$M = \frac{8\Lambda}{\pi}\int_0^\infty \frac{(z+M)k^5\varphi(k)^2}{(1+k^4)(1+k^4-z(z+M))}dk. \qquad (7)$$

Thus, the optical response of the model is governed by a single parameter

$$\Lambda = \frac{\pi}{2}\frac{n_i}{n}\left(\frac{U_0}{\varepsilon_0}\right)^2 = \frac{n_i m}{4n^2}\left(\frac{U_0}{\hbar Q}\right)^2, \qquad (8)$$

where U_0 measures the strength of the impurity potential, $U(k) = U_0\varphi(k)$. Note, that U_0 has the dimension of energy, and that of $\varphi(k)$ is $1/k^2$. The similar non-linear equations on $M(z)$ have been derived early on for the 2D and 3D Bose gas with the Coulomb repulsion[6,7]. However, their analytic solution were never possible because of the complexity of the integrand.

In what follows, we focus on the special case of the short-range impurity centers, that is usually considered in the theory of pinned density waves and Wigner crystals: $\varphi(r) = \delta(r)$, whence $\varphi(k) = 1$. Then, the integral in rhs. of (7) can be solved to yield the quadratic equation for M, whence we obtain the explicit expression for the memory function:

$$M = 2\Lambda\left(\frac{1-\Lambda}{z} - \sqrt{\left(\frac{1-\Lambda}{z}\right)^2 - 1}\right), \qquad (9)$$

where the root with the non-negative imaginary part is to be chosen. This is the main result of the present article. Note, that within the mode-coupling theory eq. (9) is an *exact* relation.

III. OPTICAL CONDUCTIVITY AND THE SUPERFLUID - GLASS TRANSITION

The optical response of the system is most conveniently studied in terms of the optical conductivity. Within the memory function approach this is expressed in the generalized Drude form:

$$\sigma' + i\sigma'' = i\frac{\omega_p^2}{z + M(z)}, \qquad (10)$$

where $z = \omega + 0i$ is the complex frequency, ω_p is the plasma frequency ($\omega_p^2 = Q^2 n/m$, where n, Q and m are the concentration, charge and mass of bosons, respectively).
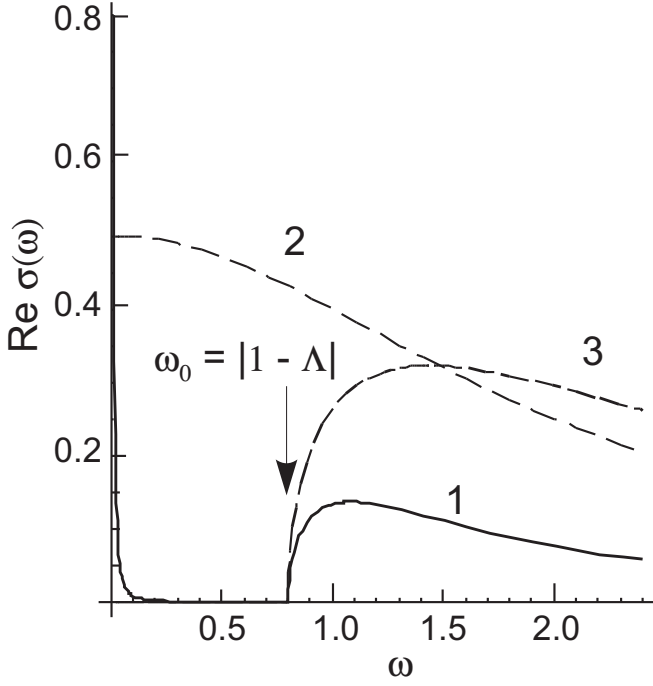
FIG. 1: The optical conductivity spectra of 2D Bose gas with point-like impurities at different levels of the disorder: 1 - $\Lambda = 0.2$ (superfluid), 2 - $\Lambda = 1.0$ (metal), 3 - $\Lambda = 1.8$ (insulator). The arrow indicates the edge of the incoherent wing, that falls to the same frequency for both curves 1 and 3. For the sake of clearness, the infinitely narrow δ-like peak in the superfluid phase is shown broadened.

FIG. 2: The dependence of the spectral weight S of the incoherent wing, eq.(15), on the disorder parameter Λ. The knee at $\Lambda_c = 1$ indicates the superfluid-insulator transition.

The explicit analytic expression for the memory function (9) makes it possible to explore various regimes of the conductivity of the system. First of all, it is straightforward to show, that this results preserves the sum-rule:

$$\int\limits_0^\infty \sigma'(\omega)d\omega = \frac{1}{2}\pi\omega_p^2, \qquad (11)$$

that we notice as an important sign of the sound behaviour of the present model.

Turning to the detailed analysis of the conductivity spectrum, let us consider first the weak disorder limit, $\Lambda \ll 1$. Expanding the generalized Drude formula (10) to first order in Λ, we arrive at the following expression for the real part of the optical conductivity at finite frequencies:

$$\sigma'(\omega)_{incoh} = \begin{cases} 2\Lambda\dfrac{\sqrt{\omega^2-1}}{\omega^3}, & \omega \geqslant 1 \\ 0, & \omega < 1. \end{cases} \qquad (12)$$

Thus, the spectrum shows a gap, that a more accurate treatment finds to extend at $\omega_0 = |1-\Lambda|$. The free system parameters being fixed, this gap frequency is governed by the amplitude of the disorder (pinning) potential. Above the gap, the optical conductivity σ'_{incoh} displays a broad asymmetric feature, shown in fig. II. It corresponds to
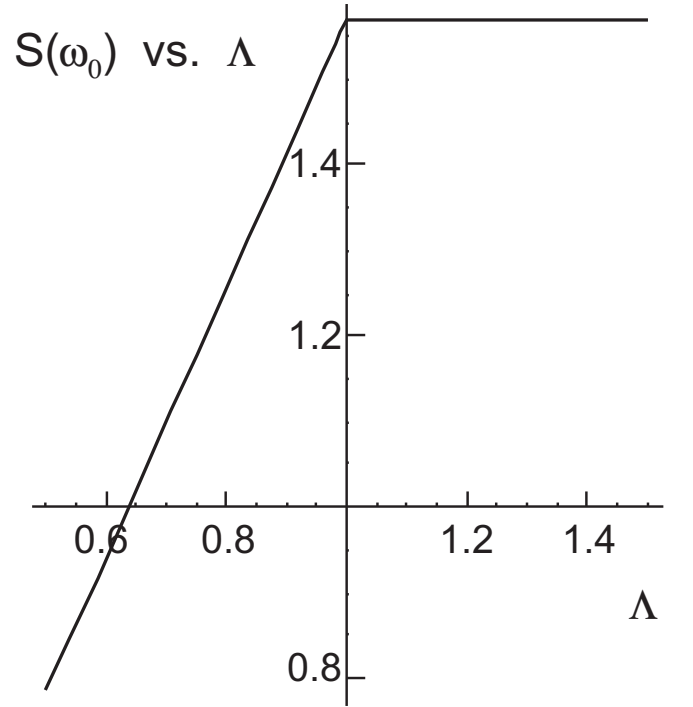
the localized plasma oscillations of the charge density and is hereafter called the incoherent wing. Such a behaviour is typical of the charge density waves. Interestingly, we recover in (12) the expression for the conductivity of pinned CDW, first derived by Fukuyama and Lee[10]. We believe this result to be the consequence of a nontrivial analogy between the pinned CDW and the disordered Bose condensate, albeit such a literal coincidence of the formulas is rather accidental. The point is that pinned CDW is a collective state, that can be described by a model of a particle in a potential well[16]. On the other hand, the condensate at $T = 0$ is also a collective (coherent) state, that behaves as a single particle.

However, it follows from the sum-rule (11), that the contribution of σ_{incoh} (12) to the optical conductivity spectrum cannot be the only one, since it vanishes as Λ tends to zero. As eq.(12) is reasonable at ω large, the careful examination of the conductivity at low frequencies is required. The corresponding expansion of the memory function (9) is:

$$M(z) \simeq (1 - sign(1-\Lambda))\frac{\Lambda(1-\Lambda)}{z} + \frac{\Lambda}{1-\Lambda}sign(1-\Lambda)z. \qquad (13)$$

The behaviour of the system depends critically on the magnitude of Λ. According to eq. (13), the memory function is linear near the zero frequency for small disorder parameter, $(\Lambda < 1)$. The corresponding optical

conductivity is:

$$\sigma(\omega)_{coh} = (1 - \Lambda)\frac{i}{z}\Big|_{z=\omega+0i}. \qquad (14)$$

This expression describes the coherent response of the condensate. Its spectral weight drops linearly as the disorder parameter Λ increases. According to the sum-rule (11), this implies the emergence of the new spectral feature at higher frequencies, described by σ_{incoh} (12). The full conductivity spectrum is the sum of the coherent (14) and incoherent (12) contributions, as depicted in fig. II.

In the strong disorder regime ($\Lambda > 1$), one obtains from (13) in the leading order $M \sim -1/z$, and $\sigma'(0) = 0$. Within the theory of the liquid-glass transitions[17] this pole of the memory function is known as the non-ergodicity pole and its appearance is identified with the onset of the glass phase as it calls forth the long-lived density fluctuations $\langle \delta\rho(k,t)\delta\rho(k,0)\rangle$, that do not decay to zero at large times. Thus, we call the insulating phase of our model the Bose glass.

The superfluid-glass transition may be clearly visualized by considering the partial spectral weight of the incoherent contribution,

$$S_{incoh} = \int_{|1-\Lambda|}^{\infty} \sigma'(\omega)d\omega. \qquad (15)$$

This quantity is shown in fig.2 against the disorder parameter Λ. It can be seen, that S grows progressively as Λ approaches unity from below, i.e. the spectral weight moves away from the condensate to the localized modes. However, it does no longer change as soon as $\Lambda > 1$, although ω_0 continues to grow. This is because no spectral weight is now contained below ω_0 and the superfluidity is completely suppressed by disorder.

The system undergoes the superfluid-insulator transition via an intermediate state, that may be called metallic. The high-frequency asymptotics of the optical conductivity in both phases follows the Drude law with the damping parameter equal to 2Λ, as can be seen from the limiting form of (9). At the critical transition point, $\Lambda = 1$, the spectral weight of the condensate vanishes, and so does the gap frequency ω_0, while the memory function is $M = 2i$. This obviously corresponds to the conventional Drude optical conductivity with the dispersionless relaxation rate in the full spectral range.

IV. SUMMARY AND CONCLUSIONS

In this paper we have studied analytically the superfluid-glass transition within the purely two-dimensional Bose gas with logarithmic interparticle interaction at $T = 0$ K with static disorder, represented by the point-like random impurities. The dielectric properties of the system are considered proceeding from the explicit expression for the memory function $M(z)$, obtained within the mode-coupling theory. The onset of the glass phase manifests itself as the appearance of the nonergodicity pole[17] of the memory function, $M(z) \sim 1/z$.

The above analysis shows, that the phase diagram of the model is delimited by the following combination of the physical parameters:

$$\Lambda_c = \frac{n_i m}{4n^2}\left(\frac{U_0}{\hbar Q}\right)^2 = 1, \qquad (16)$$

Untill $\Lambda < 1$, the optical conductivity $\sigma(\omega)$ spectrum consists of the response of the condensate $\propto \delta(\omega)$, where $\delta(w)$ is the Dirac delta function, and an inhomogeneously broadened feature at $\omega \geq |1-\Lambda|$. The latter one is typical to the pinned density waves and falls off as $1/\omega^2$ at large frequencies.

At $\Lambda > 1$ the Dirac delta function contribution disappears and the system undergoes the transition to a localized phase. Hence, large value of either the boson effective mass m, the amplitude of the impurity potential U_0 or the concentration of impurities n_i, other parameters being fixed, favours the localization and suppresses the superfluid phase, while strong boson interparticle repulsion Q and large boson concentration n have an opposite effect.

A natural question arises on the internal structure of the novel localized phase. Within the present hydrodynamic approach this problem cannot be resolved on the microscopic level. Qualitatively, the phase transition may be realized as a condensate fragmentation in the rugged potential landscape, that entails the breakdown of the long range non-diagonal order. The system breaks up into the patches, trapped by the randomly distributed pinning centers. By analogy with the mode-coupling theory of simple liquids, where the similar transition is referred to as the liquid-glass transition, we call the localized phase of the present model the Bose-glass phase (see also[6,12], where this name was first introduced).

Now, let us dwell in more details on the comparison of our results with some other approaches to the localization and superfluid-insulator transition in a system of 2D bosons. In Ref.[9] the phase diagram of the disordered 2D Bose liquid is discussed in terms of two dimensionless parameters, J/V and Δ/V, where J is the zero point energy of the bosons, V measures their short-range repulsion and Δ is a disorder parameter. It was found, that large value of J/V favours the superfluid (SF) phase, while the decreas of J/V or the increase of Δ/V leads to the localization and the supression of SF phase. In our case the only parameter Λ, eq.(8), governs the phase diagram. The ratio n^2/n_i, that enters therein may be understood as the concentration of bosons per a patch around an impurity. Thus, the first factor in Λ is proportional to the inverse energy per boson in such a patch[18], $n_i m/(\hbar^2 n^2) \sim J^{-1}$. Hence, Λ may be written in terms of Ref.[9] as: $\Lambda \sim \Delta^2/(J V^2)$. It can be seen, that the increase of Λ, that describes the gradual supression of SF phase and the onset of the localization, qualitatively agrees with the mentioned general results of Ref.[9].

The problem of the relevance of the model to the realistic physical systems is subtle, since the real particles, either boson or fermions, are likely to interact via not logarithmic, but rather the screened Coulomb potential. However, the effective model of 2D Bose liquid with the logarithmic interaction at $T = 0$ is related to the physics of the pinned vortices in the bulk type II superconductors[19]. We do not scrutinize this question any further in the present article.

We believe the model considered to be of importance on its own right, as it exhibits a rather non-trivial behaviour and allows a fully analytic treatment. The comparison of our results with those, obtained for the other model of the disordered Bose systems numerically and semi-analytically within the same theoretical framework[6,7], leads to the conclusion, that in both cases the phase diagram and the optical spectra are essentially the same and are fairly weakly sensitive to the character of the interparticle interaction, the impurity potential and the dimensionality of the systems. Thus, we hope the present study may shed light on the behaviour of more realistic models, that do not permit such an elementary analysis.

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[20] The account of nonzero bare damping does not change the obtained results in a qualitative way, but merely broadens the incoherent wing (see below).

[21] Note, that in 2D the dimensionality of charge (Q) differs from that of the conventional 3D charge by a factor $length^{-1/2}$