Hidden Galilean symmetry, conservation laws and emergence of spin current in the soliton sector of chiral helimagnet

I.G. Bostrem[1], J.Kishine[2], R.V. Lavrov[1], A.S. Ovchinnikov[1]
[1]Department of Physics, Ural State University,
Ekaterinburg, 620083, Russia
[2]Faculty of Engineering, Kyushu Institute of Technology,
Kitakyushu 804-8550,Japan

October 25, 2018

1 Introduction

The multidisciplinary field of spintronics has attracted keen scientific interest from the viewpoints of both fundamental and applied physics.[1] The key notion there is how to carry the intrinsic spin magnetization in a well defined fashion.[2, 3, 4] The current induced motion of a magnetic domain involves a torque transfer process from the conduction electrons which generates a sliding mode of the domain wall, i.e. a motion without distortion and without tilting and twisting[5, 6]. Naturally speaking, the spin current is understood as the deviation of the spin projection from its equilibrium value.

Quite recently, we proposed a new way to generate a spin current in the so-called chiral helimagnets where the axial coupling between the neighboring spins called Dzyaloshinskii-Morya (DM) coupling fixes the left- or right-handedness of the helical spin arrangement over macroscopic scales.[7] By applying the static magnetic field perpendicular to the helical axis, the magnetic kink crystal (chiral soliton lattice) is formed[8, 9] and the inertial motion of the kinks (Galilean boost) triggers of the transport of magnon density (spin current) inside the magnetic background[7]. Starting with the effective lattice Hamiltonian and its continuum version, we wrote down the Lagrangian including the appropriate Berry phase term and introduced the sliding solution of the soliton lattice by inspection. Then, we constructed the canonical Hamiltonian for the collective coordinate variables which describe the center-of-mass motion of the whole soliton lattice, based on the Dirac's prescription for the singular Lagrangian. After these procedures, we succeeded in describing the internal motion of the kink crystal state. The most important notion is that the "spin phase" directly comes up in the observable effects through the soliton lattice formation. In our mechanism, the current is carried by the moving magnetic kink crystal, where the linear momentum has a form, $P = 2\pi S\mathcal{Q} + M\dot{X}$. The topological magnetic charge, $S\mathcal{Q}$, merely enters the equilibrium background momentum $2\pi S\mathcal{Q}$, while the collective translation of the kinks with the velocity $\dot{X}$ gives the mass M.

From a mathematical viewpoint, however, the starting Lagrangian with the Berry phase and the axial coupling [see Eq. (3) below] is not manifestly Galilean invariant. Then, the following problems remain highly non-trivial: (1) how to describe the stability of the ground state under the Galilean boost, and (2) how to justify the existence of the linear momentum as a conserved Noether current. To guarantee the stability, Lie point group symmetries of nonlinear differential equations (DE) describing dynamics of the system should embrace the whole transformation. To justify the existence of the conserved current, we need to establish appropriate conservation laws from the analyses of variational symmetries. The latter plays an important role in the study of magnon transport. In this paper, to clarify the above mentioned problems (1) and (2), we start with the continuum version of the chiral XY model under the magnetic field that describes the chiral helimagnet with soliton lattice. Then, we apply the Lie group methods and symmetry analysis to the model. Although the model and the method of the present analysis have been well established, we are definitely motivated by how to justify the emergence of the spin current in real chiral helimagnets that serve as potentially promising devices in spintronics.

The paper is organized as follows. We discuss the model of chiral helimagnet in Sec. I. In Sec. II we consider the problem of finding the Lie point symmetries of the DE system of the continuum theory. The main question studied here is how to find group-invariant solutions of the differential equations. The variational symmetries are deduced in Sec. II. and we establish here explicit formulae for the conserved densities and density currents involved in the conservation laws. Conclusions are given in Sec. IV.

2 Model

To describe chiral spin ordering that occurs along a crystallographic axis, say z-axis, the relevant t model Hamiltonian is given by

$$H = -\mathcal{J}\sum_j \vec{S}_j\vec{S}_{j+1} - \vec{\mathcal{D}}\cdot\sum_j\left[\vec{S}_j\times\vec{S}_{j+1}\right] - 2\mu_0 h\sum_j S_j^x. \tag{1}$$

The ferromagnetic interaction with the exchange coupling $\mathcal{J}$ (first term), and the parity-violating DM interaction, described by the uniform axial vector $\vec{\mathcal{D}}$, between the nearest-neighbor spins lead to the long-period incommensurate helimagnetic structure with either left-handed or right-handed chirality. The direction of the mono-axial $\vec{\mathcal{D}}$ vector is fixed by the crystallographic symmetry that we do not get into the detail[7]. The third term is the Zeeman term with a transverse magnetic field with its strength being h.

In a continuum approximation, we obtain the stationary conditions for the θ and φ variables that lead to the coupled nonlinear partial differential equations given by

$$S\,\theta_t + \mathcal{J}S^2\left(\sin\theta\,\varphi_{zz} + 2\cos\theta\,\theta_z\varphi_z\right) - 2\mathcal{D}S^2\cos\theta\,\theta_z - 2\mu_0 hS\sin\varphi = 0,$$

$$S\sin\theta\,\varphi_t + \mathcal{J}S^2\left(\sin\theta\cos\theta\varphi_z^2 - \theta_{zz}\right) - 2\mathcal{D}S^2\sin\theta\cos\theta\varphi_z - 2\mu_0 hS\cos\theta\cos\varphi = 0. \tag{2}$$

The energy functional is then given by

$$\tilde{H} = \int dz\left\{\frac{\mathcal{J}S^2}{2}\left[\theta_z^2 + \sin^2\theta\varphi_z^2\right] - \mathcal{D}S^2\sin^2\theta\varphi_z - 2\mu_0 hS\sin\theta\cos\varphi\right\}.$$

To describe the dynamics in an appropriate way, we need the Lagrangian density,

$$\begin{aligned} L &= S\left(\cos\theta - 1\right)\varphi_t - \frac{\mathcal{J}S^2}{2}\left[\theta_z^2 + \sin^2\theta\varphi_z^2\right] \\ &\quad + \mathcal{D}S^2\sin^2\theta\varphi_z + 2\mu_0 hS\sin\theta\cos\varphi \end{aligned} \tag{3}$$

where the first term is the Berry phase term. The ground state configuration is then the kink crystal (chiral soliton lattice) state specified by the stationary solution,

$$\varphi(z) = 2\cos^{-1}\left[\mathrm{sn}\left(\frac{m}{\kappa}z, \kappa\right)\right], \quad \theta = 0, \tag{4}$$

where the helical pitch in the zero field is given by $q_0 = \mathcal{D}/\mathcal{J}$, $m = \sqrt{2\mu_0 h/\mathcal{J}S}$, and sn is the Jacobi elliptic function with the elliptic modulus κ ($0 < \kappa < 1$) determined through minimization of energy per unit length, $\kappa/m = 4E(\kappa)/\pi q_0$, with $E(\kappa)$ being the elliptic integrals of the second kind.[8] In. Ref. [7], we constructed the sliding solution of the kink crystal by introducing the position of the kink center Z as a dynamical variable. Based on the collective coordinate method and the Dirac's canonical formulation for the singular Lagrangian system, we derived the closed formulae for the mass, spin current and induced magnetic dipole moment accompanied with the kink crystal motion.

3 Classical Lie symmetries and Galilean boosts

Now, we perform the classical Lie symmetry analysis to manifest the existence of the Galilei-boosted solution of the stationary solution (4). The system of the second order DEs, (2), are eppressed as $F_1 = 0$, $F_2 = 0$, where

$$F_1 = S\sin\theta\,\theta_t + \mathcal{J}S^2\sin\theta\,(\sin\theta\,\varphi_{zz} + 2\cos\theta\,\theta_z\varphi_z) - \mathcal{D}S^2\sin 2\theta\,\theta_z - 2\mu_0 hS\sin\theta\sin\varphi,$$

$$F_2 = S\sin\theta\,\varphi_t + \mathcal{J}S^2\left(\sin\theta\cos\theta\varphi_z^2 - \theta_{zz}\right) - 2\mathcal{D}S^2\sin\theta\cos\theta\varphi_z - 2\mu_0 hS\cos\theta\cos\varphi, \tag{5}$$

with two dependent variables $u = (\theta, \varphi)$, and two independent variables $w = (z, t)$. The first equation in (5) is multiplied by the factor $\sin\theta$ that is necessary for the further symmetry analysis. The symmetry condition requires that the system must hold in the transformed variables whenever it holds in the original variables.

In order to take into account the derivative terms involved in Eq.(5), the infinitesimal generator is the prolongation introduced as[13] ($i = 1$ means "z", $i = 2$ does "t")

$$\underset{2}{\hat{X}} = \xi\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial\theta} + \chi\frac{\partial}{\partial\varphi} + \sum_{i=1}^{2}\zeta_i\frac{\partial}{\partial\theta_i} + \sum_{i=1}^{2}\chi_i\frac{\partial}{\partial\varphi_i} + \sum_{\substack{i,j=1\\(i\le j)}}^{2}\zeta_{ij}\frac{\partial}{\partial\theta_{ij}} + \sum_{\substack{i,j=1\\(i\le j)}}^{2}\chi_{ij}\frac{\partial}{\partial\varphi_{ij}},$$

where the first prolongation is given by

$$\zeta_i = D_i(\zeta) - \theta_z D_i(\xi) - \theta_t D_i(\eta)\,,\ \chi_i = D_i(\chi) - \varphi_z D_i(\xi) - \varphi_t D_i(\eta)\,,\ (i = z, t)$$

and the second one is determined from the first ones

$$\zeta_{1j} = D_j(\zeta_1) - \theta_{zz} D_j(\xi) - \theta_{zt} D_j(\eta), \; \chi_{1j} = D_j(\chi_1) - \varphi_{zz} D_j(\xi) - \varphi_{zt} D_j(\eta),$$

$$\zeta_{2j} = D_j(\zeta_2) - \theta_{zt} D_j(\xi) - \theta_{tt} D_j(\eta), \; \chi_{2j} = D_j(\chi_2) - \varphi_{zt} D_j(\xi) - \varphi_{tt} D_j(\eta).$$

Here, the D_i are the total derivatives with respect to z and t relevant for the system (5).

The system of DEs is invariant under the group G if

$$\underset{2}{\hat{X}} F_i|_F = 0, \; i = 1, 2. \tag{6}$$

The invariance condition provides a partial differential equations involving the unknown infinitesimal functions with their derivatives and products of the partial derivatives of the dependent variables. After replacing θ_{zz} and φ_{zz} from Eq.(5) and splitting the resulting equation into powers of derivatives of the dependent variables (shown below at left), handled as independent ones, the overdetermined system $\underset{2}{\hat{X}} F_1|_F = 0$ is obtained

$$\varphi_{zt} : \sin^2\theta \, \eta_z = 0 \tag{7}$$

$$\varphi_{zt}\theta_z : \sin^2\theta \, \eta_\theta = 0 \tag{8}$$

$$\varphi_{zt}\varphi_z : \sin^2\theta \, \eta_\varphi = 0 \tag{9}$$

$$\theta_z\theta_t : \xi_\theta \sin\theta = 0 \tag{10}$$

$$\theta_z\varphi_t : \xi_\varphi \sin\theta = 0 \tag{11}$$

$$\theta_t : -\zeta\cos\theta - \eta_t\sin\theta + 2\xi_z\sin\theta - \chi_\varphi\sin\theta + \zeta_\theta\sin\theta = 0 \tag{12}$$

$$\varphi_t : \zeta_\varphi\sin\theta + \chi_\theta\sin^3\theta = 0 \tag{13}$$

$$\theta_z\varphi_z : -\chi_{\theta\varphi} + 2\zeta + \chi_{\theta\varphi}\cos 2\theta - \zeta_\theta\sin 2\theta = 0 \tag{14}$$

$$\theta_z^2 : 2\chi_\theta\cos\theta + \chi_{\theta\theta}\sin\theta = 0 \tag{15}$$

$$\varphi_z^2 : \chi_{\varphi\varphi}\sin^2\theta + \chi_\theta\cos\theta\sin^3\theta + \zeta_\varphi\sin 2\theta = 0 \tag{16}$$

$$\begin{aligned}\theta_z : &-\mathcal{J}S\chi_{\theta z}\cos 2\theta - \xi_t\sin\theta + \mathcal{J}S\chi_{\theta z} + 2\mathcal{D}S\zeta + \mathcal{J}S\chi_z\sin 2\theta \\ &- \mathcal{D}S\sin 2\theta\left(\xi_z - \chi_\varphi + \zeta_\theta\right) = 0\end{aligned} \tag{17}$$

$$\varphi_z : -\mathcal{J}S\sin^2\theta\xi_{zz} + 2\mathcal{J}S\sin^2\theta\chi_{\varphi z} - 2\mathcal{D}S\cos\theta\sin^3\theta\chi_\theta + \mathcal{J}S\sin 2\theta\zeta_z - \mathcal{D}S\sin 2\theta\zeta_\varphi = 0 \tag{18}$$

$$1:\zeta_t\sin\theta-2\mu_0 h\cos\varphi\sin\theta\chi+\mathcal{J}S\sin^2\theta\chi_{zz}-2\mu_0 h\sin^2\theta\cos\theta\cos\varphi\chi_\theta$$
$$-\mathcal{D}S\sin 2\theta\zeta_z+2\mu_0 h\cos\theta\sin\varphi\zeta-4\mu_0 h\sin\theta\sin\varphi\xi_z+2\mu_0 h\sin\theta\sin\varphi\chi_\varphi=0. \tag{19}$$

The overdetermined system $\underset{2}{\hat{X}}F_2|_F=0$ is written as

$$\theta_{zt}:\eta_z=0 \tag{20}$$

$$\theta_{zt}\theta_z:\eta_\theta=0 \tag{21}$$

$$\theta_{zt}\varphi_z:\eta_\varphi=0 \tag{22}$$

$$\varphi_z\varphi_t:\xi_\varphi\sin\theta=0 \tag{23}$$

$$\varphi_z\theta_t:\xi_\theta\sin\theta=0 \tag{24}$$

$$\varphi_t:\zeta\cos\theta-\eta_t\sin\theta+2\xi_z\sin\theta+\chi_\varphi\sin\theta-\zeta_\theta\sin\theta=0 \tag{25}$$

$$\theta_t:\zeta_\varphi+\chi_\theta\sin^2\theta=0 \tag{26}$$

$$\theta_z\varphi_z:\zeta_{\theta\varphi}-\zeta_\varphi\cot\theta-\chi_\theta\cos\theta\sin\theta=0 \tag{27}$$

$$\theta_z^2:\zeta_{\theta\theta}=0 \tag{28}$$

$$\varphi_z^2:2\zeta_{\varphi\varphi}-2\zeta\cos 2\theta+(\zeta_\theta-2\chi_\varphi)\sin 2\theta=0 \tag{29}$$

$$\varphi_z:-2\mathcal{J}S\zeta_{\varphi z}-2\mathcal{D}S\cos 2\theta\zeta-\xi_t\sin\theta-\mathcal{D}S\sin 2\theta\xi_z+\mathcal{J}S\sin 2\theta\chi_z$$
$$-\mathcal{D}S\sin 2\theta\chi_\varphi+\mathcal{D}S\sin 2\theta\zeta_\theta=0 \tag{30}$$

$$\theta_z:\mathcal{J}\left(\xi_{zz}-2\zeta_{\theta z}\right)-2\mathcal{D}\cot\theta\zeta_\varphi-\mathcal{D}\sin 2\theta\chi_\theta=0 \tag{31}$$

$$1:-\mathcal{J}S\zeta_{zz}-4\mu_0 h\cos\theta\cos\varphi\xi_z+2\mu_0 h\cos\theta\cos\varphi\zeta_\theta+\chi_t\sin\theta+2\mu_0 h\sin\theta\cos\varphi\zeta$$
$$-\mathcal{D}S\sin 2\theta\chi_z+2\mu_0 h\cos\theta\sin\varphi\chi-2\mu_0 h\frac{\sin\varphi}{\sin\theta}\zeta_\varphi=0 \tag{32}$$

The general solution of the determining equations is presented in Appendix A. The final result is

$$\xi=C_1,\ \eta=C_2,\ \chi=0,\ \zeta=0, \tag{33}$$

where $C_{1,2}$ are the constants. Therefore, every infinitesimal generator of Lie point symmetries is a linear combination of $\hat{X}_1=\partial_z$ and $\hat{X}_2=\partial_t$

$$\hat{X}=C_1\partial_z+C_2\partial_t. \tag{34}$$

We seek invariant solutions of the Eqs.(5) under symmetries generated by (34). The characteristic equations are

$$\frac{dz}{C_1} = \frac{dt}{C_2} = \frac{d\theta}{0} = \frac{d\varphi}{0},$$

which have three functionally independent first integrals

$$\mathcal{I}_1 = z - vt,\ \mathcal{I}_2 = \theta,\ \mathcal{I}_3 = \varphi,$$

where $v = C_1/C_2$ is a velocity, and $\hat{X}\mathcal{I}_i = 0$ $(i = 1, 2, 3)$.

We can insert first integrals having the form

$$\theta = \theta\,[z - vt]\,,\ \varphi = \varphi\,[z - vt] \tag{35}$$

into the original system (5). This yields the simplified partial DE system after introducing the new coordinate $\tilde{z} = z - vt$. Thus, the invariant solution includes the sliding mode (35) which can be found by inspection, but we classify *all* invariant solutions, using the structure of the Lie algebra. Therefore, we proved that the Galilei-boosted solution certainly exists.

4 Variational symmetries and conservation laws

Next, we construct the conserved Noether current to justify the existence of the linear momentum that eventually carries the spin current in chiral helimagnets. A derivation of conservation laws is based on so-called Noether identity[14]

$$\hat{Y}\,(L) + \hat{D}_i \cdot \left(\xi^i L\right) = W^\alpha E^\alpha(L) + D_i \hat{N}^i(L),\ (\alpha = 1, 2;\ i = z, t) \tag{36}$$

where the canonical vector field eligible for the Lagrangian (3)

$$\hat{Y} = W^1\partial_\theta + W^2\partial_\varphi + D_z\left(W^1\right)\partial_{\theta_z} + D_z\left(W^2\right)\partial_{\varphi_z} + D_t\left(W^2\right)\partial_{\varphi_t},$$

with the characteristics

$$W^1 = \eta^1 - \xi^t\theta_t - \xi^z\theta_z,\ W^2 = \eta^2 - \xi^t\varphi_t - \xi^z\varphi_z.$$

The dot in Eq. (36) means the differentiation rule $\hat{D}_i \cdot \left(\xi^i L\right) \equiv \hat{D}_i\left(\xi^i\right)L + \xi^i\hat{D}_i\,(L)$.

In Eq. (36) N^i are the Noether operators given by the expressions

$$\hat{N}^i = \xi^i + \left(\eta^\alpha - \xi^l u_l^\alpha\right) \left(\frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{ij_1 \ldots j_s}^\alpha} \right)$$

$$+ \sum_{r \geq 1} (-1)^r D_{k_1} \ldots D_{k_r} \left(\eta^\alpha - \xi^l u_l^\alpha\right) \left(\frac{\partial}{\partial u_{ik_1 \ldots k_r}^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{ik_1 \ldots k_r j_1 \ldots j_s}^\alpha} \right), \tag{37}$$

and Euler-Lagrange operators are

$$E^\alpha = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha}.$$

The variational symmetries on the Euler-Lagrange equations may be found via the Noether identity

$$\hat{Y}(L) + D_z \cdot (\xi^z L) + D_t \cdot \left(\xi^t L\right) = 0.$$

In explicit form it is written as

$$0 = \eta^1 L_\theta + \eta^2 L_\varphi + \left(D_z\left(W^1\right) + \xi^t \theta_{zt} + \xi^z \theta_{zz}\right) L_{\theta_z} + \left(D_z\left(W^2\right) + \xi^t \varphi_{zt} + \xi^z \varphi_{zz}\right) L_{\varphi_z}$$
$$+ \left(D_t\left(W^2\right) + \xi^t \varphi_{tt} + \xi^z \varphi_{zt}\right) L_{\varphi_t} + \left(D_z\left(\xi^z\right) + D_t\left(\xi^t\right)\right) L$$

with the derivatives

$$L_\theta = -S \sin\theta \varphi_t - \frac{\mathcal{J} S^2}{2} \sin 2\theta \varphi_z^2 + \mathcal{D} S^2 \sin 2\theta\, \varphi_z + 2\mu_0 h S \cos\theta \cos\varphi,$$

$$L_\varphi = -2\mu_0 h S \sin\theta \sin\varphi,\ L_{\theta_z} = -\mathcal{J} S^2 \theta_z,$$
$$L_{\varphi_z} = -\mathcal{J} S^2 \sin^2\theta \varphi_z^2 + \mathcal{D} S^2 \sin^2\theta,\ L_{\varphi_t} = S\left(\cos\theta - 1\right).$$

After a little manipulation one gets the following equation for the unknowns

$$0 = \eta^1 \left(-S \sin\theta\, \varphi_t - \frac{\mathcal{J} S^2}{2} \sin 2\theta \varphi_z^2 + \mathcal{D} S^2 \sin 2\theta\, \varphi_z + 2\mu_0 h S \cos\theta \cos\varphi \right)$$

$$-\eta^2\, 2\mu_0 h S \sin\theta \sin\varphi$$
$$-\mathcal{J} S^2 \theta_z \left(\eta_z^1 + \theta_z \eta_\theta^1 + \varphi_z \eta_\varphi^1 - \theta_t \left[\xi_z^t + \theta_z \xi_\theta^t + \varphi_z \xi_\varphi^t\right] - \theta_z \left[\xi_z^z + \theta_z \xi_\theta^z + \varphi_z \xi_\varphi^z\right]\right)$$

$$
\begin{aligned}
&- \left(\mathcal{J} S^2 \sin^2\theta \varphi_z - \mathcal{D} S^2 \sin^2\theta\right) \\
&\times \left(\eta^2_z + \theta_z \eta^2_\theta + \varphi_z \eta^2_\varphi - \varphi_t \left[\xi^t_z + \theta_z \xi^t_\theta + \varphi_z \xi^t_\varphi\right] - \varphi_z \left[\xi^z_z + \theta_z \xi^z_\theta + \varphi_z \xi^z_\varphi\right]\right) \\
&+ S(\cos\theta - 1)\left(\eta^2_t + \theta_t \eta^2_\theta + \varphi_t \eta^2_\varphi - \varphi_t \left[\xi^t_t + \theta_t \xi^t_\theta + \varphi_t \xi^t_\varphi\right] - \varphi_z \left[\xi^z_t + \theta_t \xi^t_\theta + \varphi_t \xi^t_\varphi\right]\right) \\
&+ \left(S(\cos\theta - 1)\varphi_t - \frac{\mathcal{J} S^2}{2}\left(\theta^2_z + \sin^2\theta \varphi^2_z\right) + \mathcal{D} S^2 \sin^2\theta \, \varphi_z + 2\mu_0 h S \sin\theta \cos\varphi\right) \\
&\times \left(\xi^t_t + \theta_t \xi^t_\theta + \varphi_t \xi^t_\varphi + \xi^z_z + \theta_z \xi^z_\theta + \varphi_z \xi^z_\varphi\right).
\end{aligned}
$$

The equation is solved as an algebraic equation with respect to the partial derivatives of the dependent variables, handled as independent variables. The equation leads to a determining equations with respect to the unknown infinitesimal functions ξ^z, ξ^t, η^1 and η^2. The details of the calculations are relegated into Appendix B. As a result, we obtain, $C_{1,2}$ are constants,

$$
\xi^t = C_1, \ \xi^z = C_2, \ \eta^1 = \eta^2 = 0. \tag{38}
$$

Using the found variational symmetries the operators $\hat{N}^i$ are firstly constructed from formulae (37) and the corresponding conservation laws are then calculated from

$$
D_i\left(\hat{N}^i L\right) = 0.
$$

The Noether operators modify the Lagrangian density into a conserved quantity $\mathcal{C}^i = N^i(L)$ with a zero total divergence.

(i) The choice $C_1 = 1$ and $C_2 = 0$ corresponds to the symmetry group under translations in time $t \to t + a$. Then

$$
\hat{N}^t = 1 - \theta_t \frac{\partial}{\partial \theta_t} - \varphi_t \frac{\partial}{\partial \varphi_t}, \ \hat{N}^z = -\theta_t \frac{\partial}{\partial \theta_z} - \varphi_t \frac{\partial}{\partial \varphi_z}.
$$

By introducing the functions

$$
\mathcal{C}^t = N^t(L) = -\frac{\mathcal{J} S^2}{2}\sin^2\theta \, \varphi^2_z - \frac{\mathcal{J} S^2}{2}\theta^2_z + \mathcal{D} S^2 \sin^2\theta \, \varphi_z + 2\mu_0 h S \sin\theta \cos\varphi,
$$

and

$$
\mathcal{C}^z = N^z(L) = \mathcal{J} S^2 \, \theta_z \theta_t + \mathcal{J} S^2 \sin^2\theta \, \varphi_z \varphi_t - \mathcal{D} S^2 \sin^2\theta \ \varphi_t.
$$

one gets the energy conservation law $D_t\left(\mathcal{C}^t\right) + D_z\left(\mathcal{C}^z\right) = 0$, or

$$
D_t\left(\mathcal{E}\right) = D_z\left(J^{(\mathcal{E})}_z\right),
$$

where $\mathcal{E} = -\mathcal{C}^t$ and $J_z^{(\mathcal{E})} = \mathcal{C}^z$ are the energy density and the density of energy current, respectively.

(ii) The choice $C_1 = 0$ and $C_2 = 1$ yields the variational symmetry under translations in space $z \to z + a$. Then

$$\hat{N}^t = -\theta_z \frac{\partial}{\partial \theta_t} - \varphi_z \frac{\partial}{\partial \varphi_t}, \; \hat{N}^z = 1 - \theta_z \frac{\partial}{\partial \theta_z} - \varphi_z \frac{\partial}{\partial \varphi_z}.$$

By the same manner one can obtain the momentum conservation law

$$D_t \left(P_z \right) = D_z \left(T_{zz} \right),$$

with the momentum $P_z = -\mathcal{C}^t = -N^t(L)$

$$P_z = S \left(1 - \cos\theta \right) \varphi_z,$$

and the canonical energy-momentum tensor $T_{zz} = \mathcal{C}^z = N^z(L)$

$$T_{zz} = S \left(\cos\theta - 1 \right) \varphi_t + \frac{\mathcal{J} S^2}{2} \left(\theta_z^2 + \sin^2\theta \, \varphi_z^2 \right) + 2\mu_0 h S \sin\theta \cos\varphi.$$

Therefore we proved the existence of the linear momentum as a conserved Noether current. Then, we are ready to define the longitudinal spin current carried by the kink crystal by writing the linear momentum per unit area as

$$\mathcal{P}_z = S \int_0^{L_0} \left(1 - \cos\theta \right) \varphi_z \, dz = 2\pi S \mathcal{Q} + M \dot{z}, \tag{39}$$

where L_0 is the system size. The topological magnetic charge, $S\mathcal{Q}$, merely produces the equilibrium background momentum $2\pi S\mathcal{Q}$, while the collective translation of the kinks with the velocity $\dot{z}$ gives the inertial mass M of the kink crystal[7].

5 Conclusions

In this paper, motivated by the spin current problem in chiral helimagnet, we rigorously proved the hidden Galilean invariance embedded in the chiral XY model under the magnetic field. The Lie group analysis is applied to the differential equations of the continuum theory of the chiral helimagnet with the parity-violating Dzyaloshinskii-Morya coupling under a transversal magnetic field. Lie point symmetries and the invariant solutions under these symmetries are found. They present sliding solutions that come up as a consequence of both a breaking of spin rotational symmetry by the external magnetic field and a parity violation due to DM interaction. We found that variational symmetries are related with translations in space and time, the corresponding energy and momentum conservation laws are derived. We therefore succeeded in justifying the existence of the transport spin current in chiral helimagnet.

Appendix A

From Eqs.(7-11) and (20-24) one obtain $\eta = \eta(t)$ and $\xi = \xi(z,t)$. From Eq.(15) we get $\chi = \chi_0(z,t,\varphi) - \chi_1(z,t,\varphi)\cot\theta$ and $\zeta = \zeta_1(z,t,\varphi)\theta + \zeta_0(z,t,\varphi)$ from Eq.(28).

(i) Let us consider $\chi = \chi_0(z,t,\varphi)$. A substitution of the results for χ and ζ into Eq.(12) yields

$$-(\zeta_1\theta + \zeta_0)\cos\theta - \eta_t\sin\theta + 2\xi_z\sin\theta - \chi_{0\varphi}\sin\theta + \zeta_1\sin\theta = 0$$

and after splitting over the θ variable we obtain $\zeta = 0$ ($\zeta_0 = \zeta_1 = 0$) and

$$-\eta_t + 2\xi_z - \chi_{0\varphi} = 0. \tag{40}$$

From Eq.(13) we get $\chi_\theta = 0$ that agrees with the choice $\chi = \chi_0(z,t,\varphi)$ and transforms (27) into identity. A sum of Eqs.(12) and (25) gives $\eta_t = 2\xi_z$, and, therefore, $\xi_{zz} = 0$ and $\chi_{0\varphi} = 0$ (from Eq.[40]), i.e. $\chi = \chi_0(z,t)$. Then Eqs.(16,29) become identities. Eqs.(17,30) coincide with each other and may be written as

$$-\xi_t\sin\theta - \mathcal{D}S\sin 2\theta\xi_z + \mathcal{J}S\sin 2\theta\chi_z = 0.$$

The relation splits into

$$\xi_t = 0,\ -\mathcal{D}\xi_z + \mathcal{J}\chi_z = 0,$$

that gives $\chi_{zz} = 0$. Eqs.(18,31) are identically fulfilled. The last Eqs.(19,32) take the form

$$\cos\varphi\sin\theta\chi + 2\sin\theta\sin\varphi\xi_z = 0,$$

$$-4\mu_0 h\cos\theta\cos\varphi\xi_z + \chi_t\sin\theta - \mathcal{D}S\sin 2\theta\chi_z + 2\mu_0 h\cos\theta\sin\varphi\chi = 0,$$

respectively. Since χ does not depend on θ and φ, after splitting over these variables we get

$$\chi_t = 0, \ \chi_z = 0, \ \chi = 0, \ \xi_z = 0.$$

As a result we have eventually $\xi =$const, $\eta =$const, $\chi = 0$, $\zeta = 0$.

(ii) Now we take $\chi = -\chi_1(z, t, \varphi) \cot \theta$. We prove that only $\chi_1 = 0$ satisfies the determining equations. Indeed, a substitution of χ into Eq.(25) and a splitting over θ variable gives

$$\zeta_1 = 0, \ \zeta_0 = \chi_{1\varphi}, \ \eta_t = 2\xi_z.$$

This means that $\zeta = \zeta_0(z, t, \varphi)$ and $\xi_{zz} = 0$, and Eq.(12) becomes identity. From Eq.(26) we obtain $\zeta_{0\varphi} = -\chi_1$ that results together with $\zeta_0 = \chi_{1\varphi}$ in

$$\chi = -\left[a(z, t) \cos \varphi + b(z, t) \sin \varphi\right] \cot \theta,$$

$$\zeta = -a(z, t) \sin \varphi + b(z, t) \cos \varphi.$$

Eqs.(13,14,27,16,29,18,31) are identically fulfilled. After splitting over θ and φ variables Eq.(30) turns into

$$\sin \theta : \ \xi_t = 0, \quad \sin 2\theta : \ \xi_z = 0,$$

$$\cos \varphi : \ \mathcal{J} a_z + \mathcal{D} b = 0, \quad \sin \varphi : \ \mathcal{J} b_z - \mathcal{D} a = 0,$$

$$\cos^2 \theta \cos \varphi : \ \mathcal{D} b + \mathcal{J} a_z = 0, \quad \cos^2 \theta \sin \varphi : \ -\mathcal{D} a + \mathcal{J} b_z = 0,$$

that yields $\xi =$const, $\eta =$const, and $a_z = -\left(\mathcal{D}/\mathcal{J}\right) b$, $b_z = \left(\mathcal{D}/\mathcal{J}\right) a$. Eq.(17) gives the same result. From the last Eq.(19) we obtain

$$\sin \theta \sin \varphi : \ a_t = 0, \quad \sin \theta \cos \varphi : \ b_t = 0,$$

i.e. $a = a(z)$ and $b = b(z)$ and

$$\sin \theta \cos \theta \cos \varphi : \ \mathcal{J} a_{zz} + 2\mathcal{D} b_z = 0,$$

$$\sin \theta \cos \theta \sin \varphi : \ \mathcal{J} b_{zz} - 2\mathcal{D} a_z = 0.$$

Together with the previous results these equations give $a_z = 0$ and $b_z = 0$, therefore $a = b = 0$, and we get eventually $\chi = \zeta = 0$. Then Eq.(32) will be an identity. Eventually, the result (33) is again obtained.

Appendix B

A set of equations

$$\begin{aligned} \theta_z^3: &\quad \xi_\theta^z = 0 \\ \theta_z^2\varphi_z: &\quad \xi_\varphi^z = 0 \\ \theta_z^2\theta_t: &\quad \xi_\theta^t = 0 \\ \theta_z\theta_t\varphi_z: &\quad \xi_\varphi^t = 0 \end{aligned}$$

yields immediately $\xi^z = \xi^z(z,t)$ and $\xi^t = \xi^t(z,t)$. A remaining part acquires the form

$$\theta_z\theta_t: \quad \xi_z^t = 0 \tag{41}$$

$$\theta_z^2: \quad \eta_\theta^1 - \tfrac{1}{2}\xi_z^z + \tfrac{1}{2}\xi_t^t = 0 \tag{42}$$

$$\theta_z\varphi_z: \quad \eta_\varphi^1 + \sin^2\theta\eta_\theta^2 = 0 \tag{43}$$

$$\varphi_z^2: \quad -\tfrac{1}{2}\sin 2\theta\,\eta^1 + \sin^2\theta\,\xi_z^z - \sin^2\theta\,\eta_\varphi^2 - \tfrac{1}{2}\sin^2\theta\,\left(\xi_t^t + \xi_z^z\right) = 0 \tag{44}$$

$$\theta_t: \quad (\cos\theta - 1)\eta_\theta^2 = 0 \implies \eta_\theta^2 = 0 \tag{45}$$

$$\theta_z: \quad -\mathcal{J}\eta_z^1 + \mathcal{D}\sin^2\theta\,\eta_\theta^2 = 0 \tag{46}$$

$$\varphi_t: \quad -\sin\theta\,\eta^1 + (\cos\theta - 1)\left(\eta_\varphi^2 + \xi_z^z\right) = 0 \tag{47}$$

$$\varphi_z: \quad \mathcal{D}S^2\sin 2\theta\;\eta^1 - \mathcal{J}S^2\sin^2\theta\eta_z^2 + \mathcal{D}S^2\sin^2\theta\eta_\varphi^2$$

$$-\mathcal{D}S^2\sin^2\theta\xi_z^z - S\left(\cos\theta - 1\right)\,\xi_t^z + \mathcal{D}S^2\sin^2\theta\left(\xi_t^t + \xi_z^z\right) = 0 \tag{48}$$

$$1: \quad 2\mu_0 hS\cos\theta\cos\varphi\,\eta^1 - 2\mu_0 hS\sin\theta\sin\varphi\,\eta^2 + \mathcal{D}S^2\sin^2\theta\,\eta_z^2$$

$$+S\left(\cos\theta - 1\right)\eta_t^2 + 2\mu_0 hS\sin\theta\cos\varphi\,\left(\xi_t^t + \xi_z^z\right) = 0 \tag{49}$$

From Eqs.(41) and (45) we obtain $\xi^t = \xi^t(t)$ and $\eta^2 = \eta^2(z,t,\varphi)$. Together with Eqs.(43) and (46) we get $\eta^1 = \eta^1(t,\theta)$. From Eq.(42) $\eta_{\theta\theta}^1 = 0$, hence $\eta^1 = a(t)\theta + b(t)$. A substitution of the result into Eq.(47) yields $a = 0$, $b = 0$ and $\eta_\varphi^2 + \xi_z^z = 0$. This means $\eta^1 = 0$ and $\xi_t^t = \xi_z^z$ [Eq.(42)]. Together with Eq.(44) this produces $\eta_\varphi^2 = 0$ and $\xi_z^z = \xi_t^t = 0$, i.e. ξ^t =const and $\xi^z = \xi^z(t)$. Using Eq.(48) we get after splitting over θ variable ξ^z =const and $\eta^2 = \eta^2(t)$. It means that $\eta_z^2 = 0$ and we get $\eta^2 = 0$ from Eq.(49). By collecting all results together we obtain (38).

References

[1] I. Žutić, J. Fabian, and S. Das. Sarma, Rev. Mod. Phys. **76**, 323 (2004) and references therein.

[2] J.C. Sloncewski, J. Magn.Magn. Mater. **159**, L1 (1996).

[3] L. Berger, Phys. Rev. B **54**, 9553 (1996).

[4] E. I. Rashba, J. Superconductivity, **18**, 137 (2005).

[5] Y.B. Bazaliy, B.A. Jones, and S.-C. Zhang, Phys. Rev. B **57**, R3213 (1998).

[6] S.E. Barnes and S.Maekawa, Phys. Rev. Lett. **95**, 107204 (2005).

[7] I.G. Bostrem,J. Kishine, A.S. Ovchinnikov, Phys. Rev. B 77, 132405 (2008); Phys. Rev. B 78, 064425 (2008).

[8] I.E. Dzyaloshinskii, J. Phys. Chem.Solids **4**, 241 (1958).

[9] I.E. Dzyaloshinskii, Sov. Phys. JETP **19**, 960 (1964); Sov. Phys. JETP **20**, 665 (1965).

[10] V.L. Pokrovsky and A.L. Talapov, Sov. Phys. JETP **47**, 579 (1978).

[11] W.L. McMillan, Phys. Rev. B 14, 1496 (1976); Phys. Rev. B 16, 4655 (1977).

[12] J. Kishine, K. Inoue, and Y. Yoshida, Prog. Theor. Phys., Suppl. **159**, 82 (2005).

[13] P.J. Olver, Applications of Lie Groups to Differential Equations (2nd edn); Springer-Verlag: New York, 1993.

[14] N.H. Ibragimov (ed.) CRC Handbook of Lie Group Analysis of Differential Equations, Vol. II: Symmetries, Exact Solutions, and Conservation Laws; CRC Press: Boca Raton, 1994.