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Integro-Differential Equations Generated by Stochastic Problems

I. V. Melnikova[1*], V. A. Bovkun[1], and U. A. Alekseeva[1***]**

[1] *Ural Federal University, Yekaterinburg, 620002 Russia*

*e-mail: * Irina.Melnikova@urfu.ru, ** Vadim.Bovkun@urfu.ru, *** Uliana.Alekseeva@urfu.ru*

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Abstract—The connections between stochastic differential equations in which continuous and discontinuous random processes serve as sources of randomness and deterministic equations for the probabilistic characteristics of solutions of these stochastic equations are studied. In the study, we use various approaches based on the stochastic change of variables formula (Itô's formula), on the analysis of local infinitesimal characteristics of the process, and on the theory of semigroups of operators in combination with the generalized Fourier transform. This allows us to obtain direct and inverse integro-differential equations for various probabilistic characteristics.

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INTRODUCTION

A broad class of processes occurring in various fields of natural science, social phenomena, and economics, can be described mathematically using differential equations with random perturbations—stochastic differential equations (SDEs).

The class of SDEs in which randomness stems from a Wiener process has been studied most thoroughly. Owing to the continuity of trajectories in a Wiener process, solutions of such equations (diffusion processes) also have the property of continuity of trajectories; therefore, modeling based on diffusion-type equations is most suitable for describing processes without jumps.

Classical diffusion processes have a characteristic property: the variance of the deviation of the process from the initial position over the time period Δt is proportional to Δt; such processes are called *normal diffusion* processes. However, for example, for diffusion processes in turbulent media or protein diffusion in a DNA molecule, according to the results of observations, the variance of the deviation is proportional to Δt^μ, $\mu > 1$. Processes with such properties (they are referred to as *superdiffusion* processes), just as processes with jumps, cannot be described in the framework of normal diffusion. They are modeled using Lévy processes and more general Markovian processes—the Lévy-type processes (see, e.g., [1]).

Both in applications and in fundamental research, one is often interested not so much in the process itself as in its various characteristics; therefore, the relationship between stochastic equations and the corresponding deterministic equations for the probabilistic characteristics of these stochastic processes is one of the main directions in stochastic analysis. This connection remains the one most studied for diffusion processes and for the corresponding equations for characteristics, which are partial differential equations of the parabolic type (see, for example, [2, 3]).

In the present paper, we consider ways of passing from SDEs to equations for probabilistic characteristics using the example of a class of equations in which the source of randomness is Lévy processes. A specific feature of the processes under consideration that distinguishes them from diffusion processes is that the resulting deterministic equations are integro-differential (pseudodifferential). We discuss the following main approaches.

1. The approach based on the Itô stochastic integral (see, e.g., [4, 5]) considers characteristics such as the averaging of the Borel function of the process under study. Using the Itô formula, an integro-differential equation is then derived that has the characteristic selected as a solution. There is also an inverse relation from equations for probabilistic characteristics to stochastic equations (see, e.g., [6, 7]).

2. The approach that permits one to obtain equations for probabilistic characteristics under the assumption of existence of the following three limits for the random process under study: the limits as $\Delta t \to 0$ for the "first and second moments" under the condition of proximity of the process values at times t and $t + \Delta t$ (see conditions (9) and (10) below) and the limit (11), responsible for the process continuity properties (see, e.g., [2, 8]).
3. The semigroup approach establishing a connection between homogeneous Markov processes and transition semigroups of these processes (Markov, Feller, and Lévy) (see, e.g., [9–12]).
4. The approach in which the properties of stochastic processes are studied via the behavior of a characteristic function—the Fourier transform (in the general case, the generalized one) of the transition function of the process. In this approach, which is abut with the previous one, the Lévy–Khintchine formula for infinitely divisible distributions allows one to produce a representation of the Fourier transform of the generator of the transition semigroup in the form of a second-order polynomial and some integral term responsible for the behavior of the process in the presence of discontinuous random perturbations.

There are deep and not always obvious connections between all these approaches. Some of these connections have not been studied in the desired completeness despite the huge number of papers dealing with the study of these issues. For example, as we will see in the present paper, one and the same function determines a certain probabilistic characteristic of the process in the first approach, can serve as a test function when studying equations in spaces of generalized functions in the second approach, and can be used as the initial condition under the action of a semigroup in the third one.

The paper consists of four sections. In Sec. 1, we answer the questions of what should be understood by an SDE in which the source of randomness is a Lévy process and what are the conditions for the existence of a solution of this equation. In Sec. 2, based on a generalization of the Itô formula to the case of discontinuous processes, we derive a backward integro-differential equation for a probabilistic characteristic of the form $g(t,x) := \mathsf{E}^{t,x}[h(X(T))]$, determined in the general case by a Borel function h of the random process $X = \{X(t) : t \in [0,T]\}$. In Sec. 3, we discuss the approach based on the assumption about the existence of three limits. The example considered in this section allows us to demonstrate the possibilities of generalized differentiation when treating the equation for the density of the transition probability of the process X. In Sec. 4, we show how Fourier transform methods can be used to obtain the generator of the transition semigroup $\{S(t) : t \geq 0\}$ of the process X, which is a pseudodifferential operator. Based on this, we write an equation for the characteristics of a process of the form $u(t,x) := S(t)h(x)$ and show the link between the properties of smoothness of the characteristic $g(t,x)$ and the function $h(x)$ that determines it.

This paper is aimed at clarifying the connections between the properties of stochastic processes modeled by stochastic equations, integro-differential equations for these processes, semigroups of operators, generalized solutions, and the Fourier transform; therefore, for transparency, all reasoning is carried out for the case of processes ranging in $\mathbb{R}$. The results obtained can be carried over to the case of values in $\mathbb{R}^n$. Research into infinite-dimensional problems, to which recent papers by the present authors have been devoted, becomes complicated already at the stage of stating the problem (see, e.g., [13–15]); therefore, it has required a thorough preliminary analysis in the finite-dimensional case.

1. STATEMENT OF THE PROBLEM

Consider the stochastic differential equation

$$dX(t) = \alpha\big(X(t)\big)\,dt + \beta\big(X(t)\big)\,dL(t), \quad t \geq 0, \tag{1}$$

in which a Lévy process $L = \{L(t) : t \geq 0\}$ is the source of randomness.

Definition 1 [16]. A random process $L = \{L(t) : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and ranging in the state space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called a *Lévy process* if it satisfies the following conditions:

(L1) It is a process with independent increments; i.e., for any $n \in \mathbb{N}$ and $0 \leq t_1 < \ldots < t_n$ the random variables $L(0), L(t_1) - L(0), L(t_2) - L(t_1), \ldots, L(t_n) - L(t_{n-1})$ are independent.

(L2) It starts from zero a.s.; i.e., a.s. $L(0) = 0$.

(L3) It is time-homogeneous; i.e., the distribution law of the increment $\mathcal{L}(L(s+t)-L(s))$, $s,t\geq 0$, is independent of s.

(L4) It is stochastically continuous; i.e., $\mathbf{P}(|L(s+t)-L(s)|>\varepsilon)\to 0$ as $t\to 0$ for any $s\geq 0$ and $\varepsilon>0$.

(L5) Its trajectories are right continuous and have finite left limits a.s.

If conditions (L1)–(L4) are satisfied for a process L, then it is called a *Lévy process over distribution*.

Condition (L5) is often dropped when considering Lévy processes. This is related to the fact that each Lévy process over distribution has a modification whose trajectories are a.s. right continuous and have finite left limits (see, e.g., [10]). In the present paper, condition (L5) is assumed to be satisfied.

Following [4], we introduce the quantity

$$N(t,A)=\#\big\{0\leq s\leq t:\Delta L(s)\in A\big\},\quad t\geq 0,\quad A\in\mathcal{B}\big(\mathbb{R}\setminus\{0\}\big),$$

equal to the number of those jumps of the process L within time interval $[0,t]$ whose height $\Delta L(s):=L(s)-L(s-)$ belongs to the set A. For any $\omega\in\Omega$ and $t\geq 0$, the function $N(t,\cdot)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}\setminus\{0\})$, and $\nu(\cdot)=\mathsf{E}[N(1,\cdot)]$ is an intensity measure associated with the process L. If the set A is bounded below,[1] then the process $\{N(t,A):t\geq 0\}$ is Poisson with intensity $\lambda=\nu(A)$, with the measure $N(t,A)$ being a Poisson random measure on the space $(\mathbb{R}_+\times(\mathbb{R}\setminus\{0\}),\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}\setminus\{0\}))$ and the measure $\widetilde{N}(t,A):=N(t,A)-t\nu(A)$ being a martingale-valued Poisson random measure on this space.

In the notation introduced, the structure of a Lévy process is described by the Lévy–Itô decomposition (see, e.g., [4, p. 126])

$$L(t)=at+bW(t)+\int\limits_{|q|\geq 1}qN(t,dq)+\int\limits_{|q|<1}q\widetilde{N}(t,dq),\tag{2}$$

where $\{W(t):t\geq 0\}$ is a standard Wiener process and a and b are constants. This representation permits one to assign meaning to the differential $dL(t)$ in Eq. (1) and produce an SDE-based mathematical model of the process X of the form

$$\begin{aligned}X(t)-\xi&=\int\limits_0^t a\big(X(s-)\big)\,ds+\int\limits_0^t b\big(X(s-)\big)\,dW(s)\\&\quad+\int\limits_0^t\int\limits_{|q|\geq 1}K\big(X(s-),q\big)N(ds,dq)+\int\limits_0^t\int\limits_{|q|<1}F\big(X(s-),q\big)\widetilde{N}(ds,dq),\quad t\in[0,T],\end{aligned}\tag{3}$$

which generalizes this equation, where the coefficients a, b, K, and F satisfy the conditions of existence of integrals. The second term on the right-hand side in Eq. (3) is the Itô integral over a Wiener process, and the third term is a finite sum (because the process L cannot have an infinite number of jumps within a finite time interval by virtue of condition (L5)),

$$\int\limits_0^t\int\limits_{|q|\geq 1}K\big(X(s-),q\big)N(ds,dq)=\sum_{0\leq s\leq t}K\big(X(s-),\Delta L(s)\big)\cdot\chi_A\big(\Delta L(s)\big),$$

where $A=\{q\in\mathbb{R}:|q|\geq 1\}$ and $\chi_A(\cdot)$ is the characteristic function of the set A. Finally, the last term in (3) can be rewritten as

$$\int\limits_0^t\int\limits_{|q|<1}F\big(X(s-),q\big)\widetilde{N}(ds,dq)=\int\limits_0^t\int\limits_{|q|<1}F\big(X(s-),q\big)N(ds,dq)-\int\limits_0^t\int\limits_{|q|<1}F\big(X(s-),q\big)\nu(dq)\,ds.$$

[1] $A\in\mathcal{B}(\mathbb{R}\setminus\{0\})$ and $0\notin\overline{A}$.

The process $X = \{X(t) : t \geq 0\}$ defined by Eq. (3) has a structure more general than the process (2) and is called a *Lévy type process*. Quite general conditions for its existence are known; they are given [4, pp. 374, 388] by the following assertion.

Theorem 1. *Let a mapping K be predictable, and let mappings a, b, and F satisfy the Lipschitz conditions and the conditions of sublinear growth—there exist positive constants C_1 and C_2 for which one has the inequalities*

$$\big|a(y) - a(z)\big| + \big|b(y) - b(z)\big| + \int\limits_{|q|<1} \big|F(y,q) - F(z,q)\big|\nu(dq) \leq C_1|y - z|, \quad y, z \in \mathbb{R},$$

$$a^2(y) + b^2(y) + \int\limits_{|q|<1} F^2(y,q)\nu(dq) \leq C_2(1 + y^2), \quad y \in \mathbb{R}.$$

Then there exists a unique strong[2] *solution of problem* (3) *that is a homogeneous Markov process.*

In the present paper, we derive integro-differential (pseudodifferential) equations for the probabilistic characteristics of the process X defined by Eq. (3). To this end, we use the approach based on the Itô formula, Kolmogorov's approach based on calculating the limits, and the semigroup approach in conjunction with the Fourier transform.

2. APPROACH BASED ON THE ITÔ FORMULA

Itô's formula for a discontinuous process, which in the general case is a Lévy type process, has a more complex structure than in the case of continuous processes.

Let $\{X(t) : t \geq 0\}$ be a Lévy type process defined by the relation

$$\begin{aligned} X(t) - \xi = \int\limits_0^t \mathfrak{a}(s)\,ds + \int\limits_0^t \mathfrak{b}(s)\,dW(s) + \int\limits_0^t \int\limits_{|q|\geq 1} \mathfrak{K}(s,q)N(ds,dq) \\ + \int\limits_0^t \int\limits_{|q|<1} \mathfrak{F}(s,q)\widetilde{N}(ds,dq), \quad t \in [0,T]. \end{aligned} \tag{4}$$

Then for each function $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$, with probability 1, one has the relation

$$\begin{aligned} f\big(t, X(t)\big) - f\big(0, X(0)\big) &= \int\limits_0^t \Big(f'_s\big(s, X(s-)\big) + \mathfrak{a}(s)f'_x\big(s, X(s-)\big) + \frac{1}{2}\mathfrak{b}^2(s)f''_{xx}\big(s, X(s-)\big)\Big)\,ds \\ &+ \int\limits_0^t \int\limits_{|q|\geq 1} \Big[f\big(s, X(s-) + \mathfrak{K}(s,q)\big) - f\big(s, X(s-)\big)\Big]N(ds,dq) \\ &+ \int\limits_0^t \mathfrak{b}(s)f'_x\big(s, X(s-)\big)\,dW(s) + \int\limits_0^t \int\limits_{|q|<1} \Big[f\big(s, X(s-) + \mathfrak{F}(s,q)\big) - f\big(s, X(s-)\big)\Big]\widetilde{N}(ds,dq) \\ &+ \int\limits_0^t \int\limits_{|q|<1} \Big[f\big(s, X(s-) + \mathfrak{F}(s,q)\big) - f\big(s, X(s-)\big) - \mathfrak{F}(s,q)f'_x\big(s, X(s-)\big)\Big]\nu(dq)\,ds, \end{aligned} \tag{5}$$

called the *Itô formula for a Lévy type process* (4) (see, e.g., [17; 5, p. 278]).

[2] As is customary in stochastic analysis, a *strong* solution is a process that a.s. satisfies relation (3) and is consistent with the filtering $(\mathcal{F}_t)_{t\geq 0}$ generated by random perturbations entering the equation.

Consider the process $X = \{X(t) : t \geq 0\}$—a strong solution of problem (3). Since X is Markovian, associated with it is the transition probability function $P(t, x; T, A)$—the probability of transition of the process X from the position x at time t into one of the states of the Borel set A in time $T - t$. Important characteristics of the current position of the process, $X(t) = x$, are functions of the form

$$g(t,x) := \mathsf{E}^{t,x}\Big[h\big(X(T)\big)\Big] = \int\limits_{\mathbb{R}} h(y)P(t,x;T,dy), \quad t \in [0,T], \quad x \in \mathbb{R}, \tag{6}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function. In the next theorem, based on the Itô formula, we derive an integro-differential equation satisfied by the function $g(t, x)$. This is a backward equation, and for this equation we state the backward Cauchy problem with the condition $g(T, x) = h(x)$.

Theorem 2. *Let $X = \{X(t) : t \geq 0\}$ be a strong solution of problem (3) with the initial condition $\xi \in \mathbb{R}$. If the function $g(t, x)$ defined by relation (6) has continuous partial derivatives g'_t, g'_x, and g''_{xx}, then it is a solution of the Cauchy problem*

$$\begin{gathered} -g'_t(t,x) = a(x)g'_x(t,x) + \frac{1}{2}b^2(x)g''_{xx}(t,x) + \int\limits_{|q|\geq 1}\Big[g\big(t,x+K(x,q)\big) - g(t,x)\Big]\nu(dq) \\ + \int\limits_{|q|<1}\Big[g\big(t,x+F(x,q)\big) - g(t,x) - F(x,q)g'_x(t,x)\Big]\nu(dq), \\ g(T,x) = h(x), \quad x \in \mathbb{R}, \quad t \in [0,T]. \end{gathered} \tag{7}$$

Proof. Applying the Itô formula (5) to the function g and the process X, we obtain

$$\begin{aligned} &g\big(t,X(t)\big) - g\big(0,X(0)\big) \\ &\quad= \int\limits_0^t \Big(g'_s\big(s,X(s-)\big) + a\big(X(s-)\big)g'_x\big(s,X(s-)\big) + \frac{1}{2}b^2\big(X(s-)\big)g''_{xx}\big(s,X(s-)\big)\Big)\,ds \\ &\qquad+ \int\limits_0^t\int\limits_{|q|\geq 1}\Big[g\Big(s,X(s-)+K\big(X(s-),q\big)\Big) - g\big(s,X(s-)\big)\Big]N(ds,dq) \\ &\qquad+ \int\limits_0^t b\big(X(s-)\big)g'_x\big(s,X(s-)\big)\,dW(s) \\ &\qquad+ \int\limits_0^t\int\limits_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big) - g\big(s,X(s-)\big)\Big]\widetilde{N}(ds,dq) \\ &\qquad+ \int\limits_0^t\int\limits_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big) - g\big(s,X(s-)\big) \\ &\qquad\qquad - F\big(X(s-),q\big)g'_x\big(s,X(s-)\big)\Big]\nu(dq)\,ds. \end{aligned} \tag{8}$$

By virtue of the definition of the function g and the Markov property of the process X, we have

$$\begin{aligned} \mathsf{E}\Big[g\big(t,X(t)\big)\Big] &= \mathsf{E}\Big[\mathsf{E}^{t,X(t)}\Big[h\big(X(T)\big)\Big]\Big] \\ &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} h(y)P(t,x;T,dy)P(0,\xi;t,dx) = \int\limits_{\mathbb{R}} h(y)P(0,\xi;T,dy) = \mathsf{E}\Big[g\big(0,X(0)\big)\Big]; \end{aligned}$$

consequently, the expectation of the right-hand side of the representation (8) is zero. Then, according to the stochastic Fubini theorem, we have the relation

$$\int_0^t \mathsf{E}\Big[g'_s\big(s,X(s-)\big)+a\big(X(s-)\big)g'_x\big(s,X(s-)\big)+\frac{1}{2}b^2\big(X(s-)\big)g''_{xx}\big(s,X(s-)\big)\Big]ds$$
$$+\mathsf{E}\left[\int_0^t\int_{|q|\ge 1}\Big[g\Big(s,X(s-)+K\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)\Big]N(ds,dq)\right]$$
$$+\mathsf{E}\left[\int_0^t b\big(X(s-)\big)g'_x\big(s,X(s-)\big)\,dW(s)\right]$$
$$+\mathsf{E}\left[\int_0^t\int_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)\Big]\widetilde{N}(ds,dq)\right]$$
$$+\int_0^t \mathsf{E}\left[\int_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)-F\big(X(s-),q\big)g'_x\big(s,X(s-)\big)\Big]\nu(dq)\right]ds=0.$$

Since the Wiener process and the measure $\widetilde{N}$ are martingale, the third and fourth terms in this relation are zero. Let us use the following property of the Poisson random measure N:

$$\mathsf{E}\left[\int_A f(q)N(ds,dq)\right]=ds\int_A f(q)\nu(dq),\quad f\in L_1(A);$$

then we obtain

$$\int_0^t \mathsf{E}\Bigg[g'_s\big(s,X(s-)\big)+a\big(X(s-)\big)g'_x\big(s,X(s-)\big)+\frac{1}{2}b^2\big(X(s-)\big)g''_{xx}\big(s,X(s-)\big)$$
$$+\int_{|q|\ge 1}\Big[g\Big(s,X(s-)+K\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)\Big]\nu(dq)$$
$$+\int_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)-F\big(X(s-),q\big)g'_x\big(s,X(s-)\big)\Big]\nu(dq)\Bigg]ds=0.$$

Since $t\in[0,T]$ is arbitrary, it follows that the integrand vanishes for each $0\le s\le t\le T$,

$$\mathsf{E}\Bigg[g'_s\big(s,X(s-)\big)+a\big(X(s-)\big)g'_x\big(s,X(s-)\big)+\frac{1}{2}b^2\big(X(s-)\big)g''_{xx}\big(s,X(s-)\big)$$
$$+\int_{|q|\ge 1}\Big[g\Big(s,X(s-)+K\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)\Big]\nu(dq)$$
$$+\int_{|q|<1}\Big[g\Big(s,X(s-)+F\big(X(s-),q\big)\Big)-g\big(s,X(s-)\big)-F\big(X(s-),q\big)g'_x\big(s,X(s-)\big)\Big]\nu(dq)\Bigg]=0.$$

If the evolution of the process starts at time $t \in [0, T]$ from the point $X(t) = x \in \mathbb{R}$, then the last relation turns into the desired backward equation (7). The proof of the theorem is complete.

3. APPROACH VIA LIMIT RELATIONS

This approach dates back to the ideas due to Kolmogorov [18] for diffusion processes and makes use of the three limit variables [8, p. 56].

Let $p(t, x; T, y)$ be the transition probability density[3] of the process $\{X(t) : t \geq 0\}$, and assume that for each $\varepsilon > 0$ there exist finite limits

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|z-x|<\varepsilon} (z - x) p(t, x; t + \Delta t, z)\, dz = A(t, x) + O(\varepsilon), \tag{9}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|z-x|<\varepsilon} (z - x)^2 p(t, x; t + \Delta t, z)\, dz = B(t, x) + O(\varepsilon), \tag{10}$$

$$\lim_{\Delta t \to 0} \frac{p(t, x; t + \Delta t, z)}{\Delta t} = G(t, x; z) \quad \text{under the condition} \quad |z - x| > \varepsilon. \tag{11}$$

Then the density $p(t, x; T, y)$ satisfies the backward equation

$$\begin{aligned} -p'_t(t, x; T, y) = A(t, x) p'_x(t, x; T, y) + \frac{1}{2} B(t, x) p''_{xx}(t, x; T, y) \\ + \int\limits_{\mathbb{R}} \big(p(t, z; T, y) - p(t, x; T, y)\big) G(t, x; z)\, dz, \quad t \in (0, T). \end{aligned} \tag{12}$$

In relations (9) and (10), the convergence is assumed to be uniform with respect to x and t, and in relation (11), with respect to x, z, and t. Note that in the case of diffusion processes, the limit (11) is zero—this limit serves as a characteristic of continuity/discontinuity of the process.

As an example illustrating this approach, we obtain the backward equation for the transition probability density of the process

$$X(t) = at + bW(t) + c\mathbf{N}(t), \quad t \geq 0, \tag{13}$$

where $W = \{W(t) : t \geq 0\}$ is a standard Wiener process, $\{\mathbf{N}(t) : t \geq 0\}$ is a Poisson process with intensity λ, and the quantities a, b, and c are constants. We will assume that the Wiener and Poisson processes are prescribed independently of each other and find the probability distribution density of the process (13) as the convolution of the distribution densities of each term. The density of the degenerate distribution is a delta function,

$$p_{a\tau}(z) = \delta(z - a\tau),$$

the probability distribution density of the random variable $bW(\tau)$ has the form

$$p_{bW(\tau)}(z) = \frac{1}{b\sqrt{2\pi\tau}} e^{-z^2/(2\tau b^2)},$$

and the probability distribution density of the random variable $c\mathbf{N}(\tau)$ has the form

$$p_{c\mathbf{N}(\tau)}(z) = \sum_{k=0}^{[z/c]} \frac{(\lambda\tau)^k}{k!} \delta(z - ck) e^{-\lambda\tau}.$$

[3] Further, to be concise, instead of the "transition probability density function" we write "density" everywhere where this does not lead to confusion.

Therefore, for the probability distribution density of the random variable $X(\tau)$ we obtain

$$p_{X(\tau)}(z) = p_{a\tau} * p_{bW(\tau)} * p_{c\mathbf{N}(\tau)}(z) = \frac{e^{-\lambda\tau}}{b\sqrt{2\pi\tau}} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} e^{-(z-ck-a\tau)^2/(2\tau b^2)}.$$

Using the fact that the processes under consideration and, as a consequence, the process (13) are homogeneous in time and space, $p(t,x;T,y) = p(0,0;T-t,y-x)$, and that the function $p(0,0;T-t,y-x)$ is the probability distribution density $p_{X(T-t)}(y-x)$, we obtain

$$p(t,x;T,y) = \frac{e^{-\lambda(T-t)}}{b\sqrt{2\pi(T-t)}} \sum_{k=0}^{\infty} \frac{\big(\lambda(T-t)\big)^k}{k!} e^{-(y-x-ck-a(T-t))^2/(2b^2(T-t))}.$$

Calculating the local moments (9) and (10) for the process X and the function $G(t,x;z)$, we arrive at the formulas

$$A(t,x) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int\limits_{|z-x|<\varepsilon} (z-x)p(t,x;t+\Delta t,z)\,dz = \begin{cases} a, & \varepsilon \le c \\ a + c\lambda, & \varepsilon > c, \end{cases}$$

$$B(t,x) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int\limits_{|z-x|<\varepsilon} (z-x)^2 p(t,x;t+\Delta t,z)\,dz = \begin{cases} b^2, & \varepsilon \le c \\ b^2 + c^2\lambda, & \varepsilon > c, \end{cases}$$

$$G(t,x;z) = \lim_{\Delta t\to 0} \frac{p(t,x;t+\Delta t,z)}{\Delta t} = \begin{cases} \lambda\delta(z-x-c), & \varepsilon \le c \\ 0, & \varepsilon > c. \end{cases}$$

The established limit values permit one to write the backward equation (12) for the function $p(t,x;T,y)$—the density of process (13),

$$-p'_t(t,x;T,y) = ap'_x(t,x;T,y) + \frac{b^2}{2}p''_{xx}(t,x;T,y) + \lambda(p(t,x+c;T,y) - p(t,x;T,y)). \tag{14}$$

In the example considered, the density of the process X is differentiable only in the sense of generalized functions. Since, in this case, the functions $A(t,x)$ and $B(t,x)$ are constant for small ε ($\varepsilon \le c$) and $G(t,x;\cdot)$ is a delta function, Eq. (14) itself admits formalization in the generalized sense; at the same time, for the test functions it suffices to consider twice continuously differentiable functions of the variable x with compact support. It is more difficult to formalize Eq. (12) in the general case, in particular, owing to the behavior of the functions $A(t,x)$ and $B(t,x)$, which are not multipliers in this space of test functions.

Remark. The approach discussed in this section permits one to obtain, along with the backward equation, the forward equation for the transition probability density. Suppose that for each $\varepsilon > 0$, uniformly in x and t, there exist limits (9) and (10) and, uniformly in x, z, and t, there exists a limit (11). Then the density $p(t,x;T,y)$ satisfies the forward equation [8, p. 50]

$$\begin{aligned} p'_T(t,x;T,y) = &-\frac{\partial}{\partial y}\big(A(T,y)p(t,x;T,y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(B(T,y)p(t,x;T,y)\big) \\ &+ \int\limits_{\mathbb{R}} \big(G(T,z;y)p(t,x;T,z) - G(T,y;z)p(t,x;T,y)\big)\,dz, \quad 0 \le t < T. \end{aligned}$$

This equation admits formalization in the generalized sense on the same test functions as Eq. (14), because $A(T,y)$ and $B(T,y)$ are multipliers in the space of continuous functions with compact support.

4. SEMIGROUP APPROACH

Let us discuss one more approach to producing a deterministic equation generated by the stochastic equation (3). This approach is related to the theory of semigroups of operators and the Fourier transform.

We denote the space of bounded Borel measurable functions on $\mathbb{R}^n$by $B_b(\mathbb{R}^n)$ and the space of continuous functions f on $\mathbb{R}^n$ for which $\lim\limits_{x\to\infty} f(x) = 0$ by $C_0(\mathbb{R}^n)$. Let $\mathcal{D}$ be the Schwartz space on $\mathbb{R}^n$, and let $\mathcal{F}[f](\sigma) = \widehat{f}(\sigma)$ and $\mathcal{F}^{-1}[f](x) = \check{f}(x)$ be the direct and inverse Fourier transforms, respectively. Let us introduce several concepts.

Definition 2 [10, p. 2]. A one-parameter family $S = \{S(t) : t \ge 0\}$ of linear operators in the space $B_b(\mathbb{R}^n)$ that satisfy, for any $t, s \ge 0$ and $f \in B_b(\mathbb{R}^n)$, the conditions

(S1) $S(t+s)f = S(t)S(s)f$ (the semigroup property) and $S(0) = \mathrm{id}$

(S2) If $f \ge 0$, i.e., $f(x) \ge 0$ for all $x \in \mathbb{R}^n$, then $S(t)f \ge 0$ (the preservation of positivity)

(S3) If $f \le 1$, i.e., $f(x) \le 1$ for all $x \in \mathbb{R}^n$, then $S(t)f \le 1$ (the sub-Markovian property)

(S4) $S(t)1 = 1$ (the persistence property)

is called a *Markov semigroup*. A family S satisfying conditions (S1)–(S3) is called a *sub-Markov semigroup*.

If $X = \{X(t) : t \ge 0\}$ is a homogeneous Markov process in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with transition function $P(s, x; t, A)$, then the family of operators

$$S(t)f(x) := \mathsf{E}^{0,x}\Big[f\big(X(t)\big)\Big] = \int\limits_{\mathbb{R}^n} f(y)P(0, x; t, dy), \quad f \in B_b(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t \ge 0, \tag{15}$$

forms a semigroup in the space $B_b(\mathbb{R}^n)$, called the *transition semigroup of process* X, and it is Markovian. In particular, this is true for a process X that is a solution of problem (3).

The transition semigroup gives various characteristics of the current position of the process $\{X(t) : t \ge 0\}$ for a known initial position $X(0) = x$. Recall that the function $g(t, x)$ defined by relation (6) is a characteristic of the position of the process $X(t) = x$ for a known terminal position $X(T)$. By comparing the function (6) with the family (15), it becomes clear why the integro-differential equation for the function $g(t, x)$ describing the history of development of the process until the time T is backward and why the Cauchy problem for this equation is naturally produced backward, whereas the evolution equation, which will be obtained below for the function $u(t, x) := S(t)f(x)$, is direct with the initial condition $f(x)$.

A subclass of the semigroups introduced is formed by *Feller semigroups*, that is, sub-Markov semigroups mapping $C_0(\mathbb{R}^n)$ to $C_0(\mathbb{R}^n)$ and strongly continuous at the origin: $\|S(t)f - f\| \to 0$ as $t \to 0$ for each $f \in C_0(\mathbb{R}^n)$. The corresponding time-homogeneous Markov random processes are called *Feller* processes.

A special place among transition semigroups is occupied by semigroups that can be represented in the form of convolution operators as

$$S(t)f(x) = \int\limits_{\mathbb{R}^n} f(y)\eta_t(x - dy), \quad f \in B_b(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t \ge 0. \tag{16}$$

This kind of operators opens up opportunities for the effective use of the Fourier transform. The possibility of the representation (16) is determined by the structure of the transition probability function, namely, its invariance with respect to shifts of the state space; this corresponds to the property of the semigroup to be invariant with respect to spatial shifts. We have the following chain of statements:

1. A continuous linear operator S acting from $\mathcal{D}(\mathbb{R}^n)$ to $C(\mathbb{R}^n)$ is a convolution operator (with kernel $\eta \in \mathcal{D}'(\mathbb{R}^n)$) if and only if it is invariant with respect to shifts, $\tau_y(Sf) = S(\tau_y f)$, where $\tau_y f(x) := f(x - y)$, $x, y \in \mathbb{R}^n$ [19, p. 184].
2. A (Feller) transition semigroup of the process X is invariant with respect to shifts if and only if X is a Lévy process [10, p. 35].

3. The class of homogeneous Markov processes whose transition function is invariant with respect to shifts of the state space (the property of spatial additivity, $P(s,x;t,dy) = P(s,0;t,dy-x)$) coincides with the class of homogeneous processes with independent increments [20, p. 263] and necessitates considering homogeneous (in time and space) Markov processes with independent increments—Lévy processes, for which the kernel of the operator (16) is determined by the relation
$$\eta_t(x-dy) := P(0,x;t,dy) = P(0,0;t,dy-x).$$
Denoting the distribution law for a process that has a.s. started from origin by $\mu_t := \mathcal{L}(X(t))$, $t \geq 0$, we obtain the representation
$$\eta_t(-dy) = \mu_t(dy) = P(0,0;t,dy), \tag{17}$$
which is more convenient for the reasoning to follow.

It is well known that the distribution law μ_t of a stochastically continuous process with independent increments that a.s. starts from the origin (i.e., a process with properties (L1), (L2), and (L4)) is infinitely divisible [16, p. 4], $\mu_t = (\mu_{t/n})^{*n}$, or, in terms of Fourier transform, $\widehat{\mu_t} = (\widehat{\mu_{t/n}})^n$.

According to the Lévy–Khintchine formula, the characteristic function of an infinitely divisible distribution μ has the form
$$2\pi\check{\mu}(\sigma) = 2\pi\mathcal{F}^{-1}[\mu] = \int\limits_{\mathbb{R}^n} e^{i\langle\sigma,y\rangle}\mu(dy) = e^{\psi(\sigma)}, \quad \sigma \in \mathbb{R}^n, \tag{18}$$
where
$$\psi(\sigma) = i\langle a,\sigma\rangle - \frac{1}{2}\langle\sigma,B\sigma\rangle + \int\limits_{\mathbb{R}^n}\big(e^{i\langle\sigma,y\rangle} - 1 - i\langle\sigma,y\rangle\cdot\chi_{|y|\leq 1}(y)\big)\nu(dy), \tag{19}$$
$a \in \mathbb{R}^n$, B is a symmetric nonnegative defined $n\times n$ matrix, and ν is a measure on $\mathbb{R}^n$ satisfying the conditions $\nu(\{0\}) = 0$ and $\int\limits_{\mathbb{R}^n}(|y|^2 \wedge 1)\nu(dy) < \infty$. The triple (a,B,ν) is determined uniquely by the measure μ. The converse is also true; namely, for each triple (a,B,ν) satisfying the above conditions there exists an infinitely divisible measure μ with the characteristic function (18), (19).

Thus, if X is a stochastically continuous process with independent increments a.s. starting from the origin, then its distribution law μ_t is characterized at each time by the triple $(a(t),B(t),\nu_t)$, and the process is thereby determined by the tuple of triples $\{(a(t),B(t),\nu_t) : t \geq 0\}$. In this case, if the process X is time-homogeneous (property (L3)), then the triple is also homogeneous, $(a(t),B(t),\nu_t) = (ta,tB,t\nu)$, and the characteristic function acquires the form
$$2\pi\check{\mu}_t(\sigma) = \mathsf{E}\big[e^{i\langle\sigma,X(t)\rangle}\big] = e^{t\psi(\sigma)},$$
where the function $\psi(\sigma)$ is determined by relation (19); i.e., the Lévy process is fully characterized by the triple (a,B,ν).

Let us return to semigroups corresponding to Lévy processes and derive an evolution equation for the characteristic of the form $u(t,x) = S(t)f(x)$. To this end, we find the infinitesimal generator of the transition semigroup
$$Af(x) := \lim_{t\to 0}\frac{S(t)f(x) - f(x)}{t}$$
using the Fourier transform, following [11, p. 59]. To ensure the existence of direct and inverse Fourier transforms, first, for simplicity, we assume that $f \in \mathcal{D}$. It follows from the representation (16) that
$$\mathcal{F}\big[S(t)f(x)\big] = \mathcal{F}[f * \eta_t] = \widehat{f}(\sigma)\widehat{\eta_t}(\sigma),$$
where, by virtue of relation (17),
$$\widehat{\eta_t}(\sigma) = \mathsf{E}\big[e^{-i\langle\sigma,X(t)-x\rangle}\big] = \int\limits_{\mathbb{R}^n} e^{-i\langle\sigma,y\rangle}\eta_t(dy) = \int\limits_{\mathbb{R}^n} e^{i\langle\sigma,z\rangle}\mu_t(dz) = 2\pi\check{\mu}_t(\sigma).$$

Then

$$Af(x) = \mathcal{F}^{-1}\left[\lim_{t\to 0}\frac{\check{\mu}_t(\sigma)-1}{t}\widehat{f}(\sigma)\right] = \mathcal{F}^{-1}\left[\lim_{t\to 0}\frac{e^{t\psi(\sigma)}-1}{t}\widehat{f}(\sigma)\right] = \mathcal{F}^{-1}\big[\psi(\sigma)\widehat{f}(\sigma)\big].$$

This representation shows that the generator of the transition semigroup of a Lévy process is a pseudodifferential operator with the symbol $\psi(\sigma)$ defined in (19). Hence, using the properties of the Fourier transform, we find the generator of the semigroup,

$$Af(x) = \big\langle a, \nabla f(x)\big\rangle + \frac{1}{2}\operatorname{div} B\nabla f(x) + \int\limits_{\mathbb{R}^n}\Big(f(x+y) - f(x) - \Big\langle \nabla f(x), y\chi_{|y|\le 1}\big(|y|\big)\Big\rangle\Big)\nu(dy).$$

It was proved in [9, p. 208] that $C_0^2(\mathbb{R}^n) \subset \operatorname{Dom}(A)$ and the resulting representation takes place for each $f \in C_0^2(\mathbb{R}^n)$.

If the process X is not a Lévy process but is homogeneous Markovian (in particular, a Lévy type process), then, under some auxiliary conditions, the generator of the corresponding operator semigroup $\{S(t) : t \ge 0\}$ can also be found using the Lévy–Khintchine formula with the help of the inverse Fourier transform. Namely [10, p. 47], if A is the generator of a Feller semigroup and $\mathcal{D} \subset \operatorname{Dom}(A)$, then

$$\begin{aligned} Af(x) = {} & \big\langle c(x), f(x)\big\rangle + \big\langle a(x), \nabla f(x)\big\rangle + \frac{1}{2}\operatorname{div} B(x)\nabla f(x) \\ & + \int\limits_{\mathbb{R}^n}\Big(f(x+y) - f(x) - \Big\langle \nabla f(x), y\chi_{|y|\le 1}\big(|y|\big)\Big\rangle\Big)\nu(x, dy). \end{aligned} \tag{20}$$

Thus, for the probabilistic characteristic

$$u(t,x) = S(t)f(x) = \int\limits_{\mathbb{R}^n} f(y)P(0,0;t,dy-x), \quad t \ge 0, \quad x \in \mathbb{R}^n,$$

of the process X we have the following assertion.

Proposition 1. *Let X be a Feller process, and let $\mathcal{D} \subset \operatorname{Dom}(A)$; then for each $f \in \operatorname{Dom}(A)$ the function $u(t,x)$ is a solution of the Cauchy problem*

$$u_t'(t,x) = Au(t,x), \quad u(0,x) = f(x), \quad t \ge 0, \quad x \in \mathbb{R}^n,$$

with the operator A determined by relation (20).

In conclusion, we use the semigroup technique to prove the following assertion.

Proposition 2. *Let the function $g = g(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^n$, be defined by relation* (6). *If X is a Feller process, then for each $h \in \operatorname{Dom}(A) \cap C_0(\mathbb{R}^n)$ the function g is twice continuously differentiable with respect to x.*

Proof. By virtue of the homogeneity of the process in time, the function g can be represented as the action of the semigroup

$$g(t,x) = \int\limits_{\mathbb{R}^n} h(y)P(t,x;T,dy) = \int\limits_{\mathbb{R}^n} h(y)P(0,x;T-t,dy) = S(T-t)h(x), \quad h \in C_0(\mathbb{R}^n).$$

It follows from the commutativity of the semigroup with its generator on its domain that $g \in \operatorname{Dom}(A)$ for $h \in \operatorname{Dom}(A)$, $Ag(t,x) = AS(T-t)h(x) = S(T-t)Ah(x)$. It remains to note that the functions lying in the domain of A are twice continuously differentiable. The proof of the proposition is complete.

Thus, in the present paper,

– Based on the Itô formula, we have derived an integro-differential equation for the function g determined by formula (6).
– Based on the approach using the limit relations (9)–(11), we have obtained forward and backward equations for the transition probability density, which, in essence, are generalized.
– Based on the semigroup approach, we have obtained the direct Cauchy problem for probabilistic characteristics of the type $S(t)f(x)$.
– Using the semigroup technique, we have shown that the condition of differentiability of the function $g(t,x)$ with respect to x can be carried over to the function $h(x)$ and consequently, we may not require the differentiability of the transition probability function $P(t,x;T,dy)$.

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