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Hierarchical Game with Random Second Player and Its Application

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Abstract. A hierarchical game with a random second player is considered, optimal strategies are defined on the base of Stackelberg equilibrium. The random second player is understood as a randomly selected person from a homogeneous set of decision-makers. The proposed model can be used in various problems. First of all, it is may be used for optimal price choosing for a new product. In the article the model is applied to the problem of setting the optimal fare to a new route. A carrier acts as a first player, a randomly selected passenger acts as a second player. It is assumed that the function of passenger's preferences depends on a random parameter. The price for the hierarchical game is an optimal payoff of the first player, this price is compared with the price of a game in which the strategies of both players depend on a random parameter. A model example is considered.

INTRODUCTION

The choice of an optimal strategy for a significant number of applied problems can be formalized as a game theory problem, including under conditions of incomplete information and risk, which are usually modeled using random variables. Games with a random payoff matrix were investigated in [1], matrix games with probability-distribution-valued payoff were studied in [2], where the authors construct a total ordering on a subset of the space of payoff probability distributions and transfer Nash equilibrium concept of solution to these games. Hierarchical models of game theory with asymmetric information conditions for players are often used in mathematical modeling of the behavior of market participants [3, 4]. In this case, the strategies of the players that ensure the Stackelberg equilibrium are considered optimal.

We deal with a hierarchical game with a random second player, in which the first player chooses a deterministic solution, and the second player is represented by a set of decision makers. The strategy of the second player is formalized as a probabilistic solution to an optimization problem with an objective function depending on a continuously distributed random parameter.

PROBABILISTIC SOLUTION TO STOCHASTIC OPTIMIZATION PROBLEM

In many situations, optimal strategies are chosen for optimization problems with many decision makers, each of them chooses a decision based on his (her) criterion. The mathematical formalization of such problems leads to the study of probabilistic solutions to stochastic optimization problems.

Let us consider the optimization problem with an objective function depending on a random parameter

$$\min_{y\in Y} f(y,\xi), \tag{1}$$

here $Y \subset R^n$ is a compact set, $\xi = \xi(\omega)$ is a random vector defined on a probability space $(\Omega,\mathscr{F},\mathscr{P})$ with values in $B \subseteq R^m$. Denote by

$$Y^*(z) = \operatorname{Arg\,min}\{f(y,z)|y \in Y\}.$$

the set of solutions to the optimization problem

$$\min_{y\in Y} f(y,z), \tag{2}$$

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for a fixed value of parameter z.

Definition 1. [5] The random compact set $Y^*(\xi) = Y^*(\xi(\omega))$ is called *a probabilistic solution* to stochastic optimization problem (1).

Theorem 1. [6] Let the following conditions hold

1. the function $f(y,z)$ is strictly convex in y and continuous in the aggregate of variables on $Y \times Z$,
2. $Y \subset R^n$ is a convex compact set, $Z \subseteq R^m$ is an open connected set;
3. $\xi = \xi(\omega)$ is a random vector taking values from the set Z with probability 1.

Then the solution $y^*(z)$ of the problem (2) exists and is unique for all $z \in Z$, the function $Y^*(z) = y^*(z)$ is a continuous vector function of the parameter z and $y^*(\xi) = y^*(\xi(\omega))$ is a random vector.

The theorem follows from Berge's maximum theorem and measurability of continuous mappings [7].

HIERARCHICAL GAME WITH A RANDOM SECOND PLAYER

Definition 2. [8] *Hierarchical game with a random second player*

$$\mathbb{G}(\mathscr{P}_\xi) = \langle X, Y, f_1(x,y), f_2(x,y,\xi), \mathscr{P}_\xi \rangle,$$

is defined, if there are given

1. the sets X and Y of possible strategies of the first and second players;
2. objective functions $f_1(x,y)$, $f_2(x,y,\xi)$ of the first and second players, where function $f_1(x,y,\xi)$ depends on the variables x,y, function $f_2(x,y,\xi)$ depends on the random parameter ξ too;
3. a distribution $\mathscr{P}_\xi$ of the random parameter ξ.

We assume further that the function $f_2(x,y,\xi)$ for every $x \in X$, the set Y and the distribution $\mathscr{P}_\xi$ satisfy the conditions of Theorem 1. In this case, the probabilistic solution to the problem

$$\min_{y \in Y} f_2(x,y,\xi), \tag{3}$$

is a random vector denoted by $y^*(x,\xi)$.

A feature of the game is the number of possible states of nature and the number of possible strategies of players are not finite as a rule.

Definition 3. [8] A couple of first and second player strategies $\{x^*, y^*(x^*,\xi)\}$ is called *Stackelberg equilibrium* in the hierarchical game with a random second player $\mathbb{G}(\mathscr{P}_\xi) = \langle X, Y, f_1(x,y), f_2(x,y,\xi), \mathscr{P}_\xi \rangle$, if

1. $y^*(x^*,\xi)$ is the probabilistic solution to the problem (3) for $x = x^*$,
2. the vector $x^* \in X$ is the solution of the mean value optimization problem

$$\max_{x \in X} E\left(f_1(x, y^*(x,\xi))\right). \tag{4}$$

The strategies $x^* \in X$ and $y^*(x^*,\xi)$ are called *optimal strategies of the first and second players*, the value

$$c(\mathbb{G}) = \max_{x \in X} E f_1\left(x, y^*(x,\xi)\right), \tag{5}$$

is called *the optimal second player's payoff* or *the price of the hierarchical game* $\mathbb{G}(\mathscr{P}_\xi)$.

Theorem 2. Let the following conditions hold

1. the function $f_2(x,y,z)$ is strictly convex in y and continuous in the aggregate of variables on $X \times Y \times Z$, where $Z \subseteq R^m$ is an open connected set;
2. $\xi = \xi(\omega)$ is a random vector taking values from the set Z with probability 1.

3. the function $f_1(x,y)$ is continuous in the aggregate of variables on $X \times Y$,

4. $X \subset R^k$ is compact set, $Y \subset R^n$ is a convex compact set.

Then optimal strategies of the first and second players $x^*, y^*(x,\xi)$ exist, and the price $c(\mathbb{G})$ of the hierarchical game $\mathbb{G} = \mathbb{G}(\mathscr{P}_\xi)$ with the random second player is defined.

Proof. For every fixed $x \in X$ the function $f_2(x,y,z)$ and the sets Y and Z satisfy the conditions of Theorem 1, so the optimal solution $y^*(x,\xi)$ of the stochastic optimization problem (3) is a random vector.

Let us consider the mean value of the function $f_1(x,y^*(x,\xi))$. The continuity of this function and boundedness of the set Y imply the existence of its mean value $F_1(x) = Ef_1(x,y^*(x,\xi))$ for every $x \in X$ [9].

The continuity $f_1(x,y)$ in x results in the continuity of its mean value function $F_1(x)$. A continuous function on a compact set reaches its maximum, so a vector

$$x^* = \arg\max_{x\in X} F_1(x)$$

exists and it is the optimal strategy of the first player.

Remark. In the condition of Theorem 2 the optimal strategy of the first player may be not unique.

COMPARISON WITH OTHER FORMALIZATIONS OF GAMES WITH RANDOM PARAMETERS

Let us consider other games related to the considered game $\mathbb{G}(\mathscr{P}_\xi)$ with a random second player, fist of all, study a parametrical hierarchical game $\mathbb{G}_1(z) = \langle X, Y, f_1(x,y), f_2(x,y,z)\rangle$ depending on a parameter z.

Definition 4. A couple of first and second player strategies $\{x_1^*(z), y^*(x^*(z),z)\}$ is called *Stackelberg equilibrium* in the parametrical hierarchical game $\mathbb{G}_1(z)$, if for every $z \in Z$

1. $y^* = y^*(x_1^*(z),z)$ is a solution to the problem

$$f_2(x,y,z) \to \min_{y\in Y} \tag{6}$$

   for $x = x_1^*(z)$,

2. the vector $x_1^*(z) \in X$ is a solution to the parametrical optimization problem

$$\max_{x\in X} f_1(x,y^*(x,z)). \tag{7}$$

The strategies $x_1^*(z) : Z \mapsto X$ and $y^*(x_1^*(z),z) : Z \mapsto X$ are called *optimal strategies of the first and second players* for the game $\mathbb{G}_1(z)$, the function

$$c_1(z) = \max_{x\in X} f_1\,(x,y^*(x,z)) = f_1(x_1^*(z), y^*(x^*(z),z) \tag{8}$$

is called a price of the parametrical game.

In the conditions of Theorem 2 for every $z \in Z$ there are optimal strategies $x_1^*(z)$ and $y^*(z)$ and the price $c_1(z)$.

If $z = \xi$ is random, for example, it could be an external macroeconomic factor, that affects to the choice of a consumer (the second player), and the first player (the seller) can take this factor into account and choose a strategy depending on ξ.

In this case the optimal price $c_1(\xi)$ is a random value and depends on the random parameter ξ. Let us compare the mean value of the random optimal price with the price in the hierarchical game.

Theorem 4. The optimal price for the hierarchical game $\mathbb{G}(\mathscr{P}_\xi)$ with a random second player is not grater than the mean value of the probabilistic price $c_1(\xi)$ for the random game $\mathbb{G}_1(\xi)$, *i.e.*,

$$c \le \bar{c}_1 = Ec_1(\xi). \tag{9}$$

Theorem 4 results from the property of the mathematical expectation of a random function $\eta(\omega,x)$:

$$\max_x E\eta(\omega,x) \le E\max_x \eta(\omega,x).$$

Theorem 5. If the conditions of Theorem 2 hold and the function $f_1(x,y)$ is convex in x, then the optimal price in the hierarchical game $\mathbb{G}$ satisfy the inequality

$$c \geq c_2, \;\; c_2 = \max_{x \in X} f_1\left(x, Ey^*(x,\xi)\right). \tag{10}$$

This statement results from the property of the mathematical expectation of a convex function. For any convex function $f(x): R^n \mapsto R^1$ and random vector η with mean value $E\eta$ it holds $f(E\eta) \geq Ef(\eta)$ [9]. Thus, $Ef_1(x,y^*(x,\xi)) \geq f_1(x,Ey^*(x,\xi))$ and $c_2 \leq c$.

OPTIMAL FARE CHOICE

Let us consider the problem of optimal fare choice for a new route. Our approach bases on the probabilistic model of passenger preferences [10]. The carrier that sets the fare is treated as the first player and the set of passengers is treated as the second player. The second player's strategy is formalized as the probabilistic solution to the optimization problem with a random objective function.

Let a passenger have a choice between n possible alternatives (routes). The set of alternatives is denoted by $Y_0 = \{e_1,\ldots,e_n\}$, where e_i are basis vectors in R^n. The vector $y \in Y_0$ denotes the indicator of route selection (element of the set of alternatives), that is $y = e_i$ means the passenger chooses the ith route.

Let a_i are known trip costs and b_i are trip times, $i = \overline{1,n}$.

In our consideration other conditions of travel (such as the convenience of timetables, *etc.*) are not taken into account, so a passenger has two criteria for choosing between alternatives $\{e_1,\ldots,e_n\}$:

$$A(y) = a^T y, \;\; B(y) = b^T y, \tag{11}$$

where $a = \{a_1,\ldots,a_n\}$, $b = \{b_1,\ldots,b_n\}$.

We use a generalized trip cost $f(A,B,\xi)$ as the criterion of passengers' choice. In the simplest case it is the sum of two criteria

$$f(y;\xi) = A(y) + \xi B(y), \tag{12}$$

where $\xi \geq 0$ is «an individual price» of the time spent by a passenger. This parameter depends on a randomly chosen passenger and is treated as random.

Thus, the choice of a random decision maker (passenger) is described as a probabilistic solution to the optimization problem depending on the random parameter ξ

$$f(y;\xi) = a^T y + \xi b^T y \to \min_{y \in Y_0}. \tag{13}$$

The probabilistic solution to this problem is denoted by $y^*(\xi)$.

It is assumed that the random value ξ has a continuous distribution $\mathscr{P}_\xi$ on the interval $[t_1,t_2] \subseteq [0,+\infty)$. In this case the solution $y^*(\xi)$ to the problem (13) consists of a single point with probability 1 [11]. The random vector $y^*(\xi)$ has a discrete distribution that depends on the distribution of the random parameter ξ and the parameters of all routes a and b.

Consider the problem of the setting a price for a newly introduced route. Let us denote the new route by e_{n+1}, and an expanded set of alternatives by $Y_1 = Y_0 \cup e_{n+1}$. The passenger's choice of the newly entered route is indicated by $y^*(\xi) = e_{n+1}$.

Let us consider a game $\mathbb{G}(\mathscr{P})$. The first player is a carrier of a new type of transport (or a new route), his task is to choose the fare $x = a_{n+1}$ for the new route to optimize the income from ticket sales.

The proposed model is simplified:

- as a rule, there are several tariffs (first and other classes, for frequently traveling passengers, *etc.*) for each type of transport (route),
- travel time, generally speaking, is not a deterministic value, but has a dispersion,
- other conditions of travel (such as the convenience of timetables, *etc.*) are not taken into account,

- the increase in customers flow due to the provision of more convenient travel is not taken into account in this formalization.

The expected number of passengers on the new route is $N_{n+1} = N \cdot q_{n+1}(x)$, here N is the total number of passengers using this direction, $q_{n+1}(x)$ is probability of choosing a new route by a random passenger. This probability can be written as

$$q_{n+1}(x) = Pr\{y^*(\xi,x) = e_{n+1}\} = E\left(e_{n+1}^T y^*(\xi,x)\right),$$

where $y^*(\xi,x)$ is the probabilistic solution to the problem

$$f_2(x,y,\xi) = a(x)^T y + \xi b^T y \to \min_{y\in Y_1}, \tag{14}$$

with $a(x) = \{a_1,\ldots,a_n,x\}$, $b = \{b_1,\ldots,b_{n+1}\}$.

Let us denote the cost of servicing one passenger on a new route by a_0. The carrier's income on the new route received from the travel of an individual passenger along a route $y \in Y_1$ is denoted by $f_1(x,y)$, where

$$f_1(x,y) = (x-a_0)e_{n+1}^T y. \tag{15}$$

Choosing the price $x = a_{n+1} \in X = [a_0;A_0]$, the carrier (the first player) solves the problem of maximizing the mathematical expectation of his income

$$Ef_1(x,y^*(\xi,x)) \to \max_{x\in X}. \tag{16}$$

Since the function $f_1(x,y)$ is linear in y, then

$$Ef_1(x,y^*(\xi,x)) = f_1(x,Ey^*(\xi,x)) = (x-a_0)\left(e_{n+1}^T Ey^*(x,\xi)\right) = (x-a_0)q_{n+1}(x), \tag{17}$$

and $c = c_2$, where c_2 is defined by equation (10).

The obtained hierarchical game can be written in the form

$$\mathbb{G}(\mathscr{P}_\xi) = \langle X, Y_1, f_1(x,y), f_2(x,y,\xi), \mathscr{P}_\xi\rangle, \tag{18}$$

where functions $f_1(x,y)$ and $f_2(x,y,\xi)$ are defined by relations (15) and (14). It is assumed that the random value ξ has a given continuous distribution $\mathscr{P}_\xi$.

Generally speaking, the game (18) with criteria (15), (14) does not satisfy the conditions of Theorem 2, as the set $Y_1 = \{e_1,\ldots,e_{n+1}\}$ is nonconvex. This set may be replaced by its convex hull

$$\hat{Y}_1 = \{y \in R^{n+1} : y_1 + \ldots + y_{n+1} = 1,\ y_i \geq 1\}.$$

The optimal solution $\hat{y}^*(x,\xi)$ to the problem

$$f_2(x,y,\xi) = a(x)^T y + \xi b^T y \to \min_{y\in\hat{Y}_1},$$

coincides with the solution to problem (14) with probability 1 (see [11]).

The game $\hat{\mathbb{G}}(\mathscr{P}_\xi) = \langle X, \hat{Y}_1, f_1(x,y), f_2(x,y,\xi), \mathscr{P}_\xi\rangle$ satisfy the conditions of Theorem 2, and optimal strategies in the game $\hat{\mathbb{G}}(\mathscr{P}_\xi)$ are optimal strategies in the game $\mathbb{G}(\mathscr{P}_\xi)$. Thus, the optimal strategies in the game $\mathbb{G}(\mathscr{P}_\xi)$ exit.

Theorem 3. [10] The probability $q_{n+1}(x)$ of choosing $(n+1)$-th route equals

$$q_{n+1}(x) = Pr\{\xi \in [L(x), R(x)]\}, \tag{19}$$

where $L(x)$ and $R(x)$ are defined by relations

$$L(x) = \max_{b_j > b_{n+1}, j=\overline{1,n}} \left\{0, \frac{x-a_j}{b_j - b_{n+1}}\right\}, \quad R(x) = \min_{b_j < b_{n+1}, j=\overline{1,n}} \left\{\frac{x-a_j}{b_j - b_{n+1}}, +\infty\right\}. \tag{20}$$

If $(n+1)$-th route is dominated by criteria $\{a_i, b_i\}$, then $q_{n+1}(x) = 0$. In this case $L(x) > R(x)$ and the segment $[L(x), R(x)]$ is the empty set.

The solution to the problem of choosing the optimal price consists of two stages:

- finding the value $\widehat{a}$ by formula

$$\widehat{a} = \max_{j: b_j < b_{n+1}} a_j,$$

- calculation of the optimal value $x \in [a_0; \widehat{a}]$ as a solution to the problem (16), (19) using numerical methods.

Example

Let there be 3 possible routes (modes of transport) with given values of transportation prices and time $\{a_i, b_i\}$, $i = 1,2,3$ and one more route (mode of transport) with a given travel time b_4 has been added. The problem of the first player is to choose the optimal transportation price $x = a_4$ to obtain the maximal profit under the assumption that the second player (a randomly chosen passenger) chooses the route that is optimal according to the criterion of generalized cost (12).

The calculations were carried out for the following data for 3 existing routes

$$\{a_1, b_1\} = \{10;4\},\ \ \{a_2, b_2\} = \{20;3\},\ \ \{a_3, b_3\} = \{40;2\}.$$

It is assumed that the random parameter ξ has a lognormal distribution with parameters $\mu = 2$, $\sigma = 1$. The problem of determining the optimal price for a route with travel time $b_4 = 3.5$ and $a_0 = 5$ is considered.

For the data in question $\widehat{a} = 4$. The third route is obviously preferable to the newly introduced one if $x \geq 4$ and the probability that a passenger chooses a new route equals 0.

In this case the optimal price for the new route is $x^* = 10.12$, the game price (the first player's profit) is $c = 4.96$ (see Figure 1).

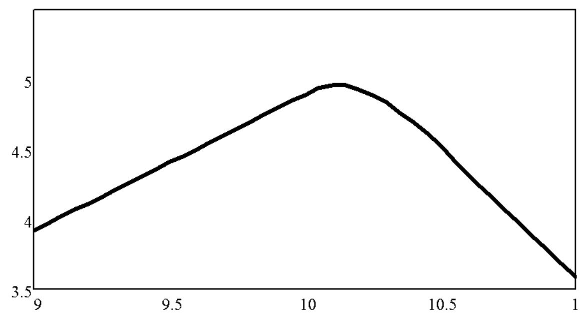

FIGURE 1. Dependence of the first player's profit on the ticket price (along the horizontal axis)

The price $c_1(z)$ for parametric game $\mathbb{G}(z)$ has been calculated too. The dependence of $c_1(z)$ on z is shown on Figure 2. If z is the random parameter $z = \xi$ having the lognormal distribution with $\mu = 2$, $\sigma = 1$, then

$$Ec_1(\xi) = \bar{c}_1 = 6.54,\ \bar{c}_1 > c.$$

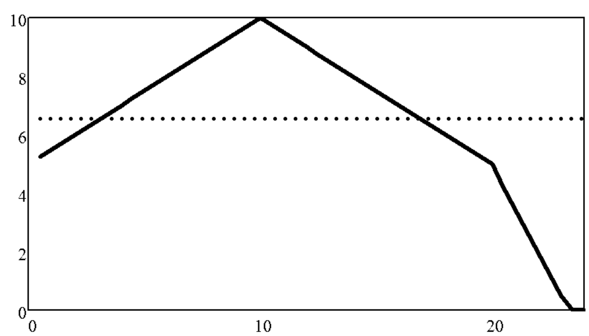

FIGURE 2. Dependence of the price $c_1(z)$ for parametric game on the parameter z

CONCLUSION

The hierarchical game with a random second player, in which the first player chooses a deterministic solution and the second player is represented by a set of decision makers, have been proposed. The strategy of the second player is formalized as the probabilistic solution to the optimization problem with the objective function depending on a continuously distributed random parameter. Some properties of the game price have been obtained.

The problem of optimal fare choice for a new route based on the probabilistic model of passenger preferences is studied as a hierarchical game with a random second player.

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