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Influence of the Dufour Effect on Shear Thermal Diffusion Flows

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Abstract: The article considers thermal diffusion shear flows of a viscous incompressible fluid with spatial acceleration. The simulation uses a system of thermal diffusion equations (in the Boussinesq approximation), taking into account the Dufour effect. This system makes it possible to describe incompressible gases, for which this effect prevails, from a unified standpoint. It is shown that for shear flows, the system of equations under study is nonlinear and overdetermined. In view of the absence of a theorem on the existence and smoothness of the solution of the Navier–Stokes equation, the integration of the existing system seems to be an extremely difficult task. The article studies the question of the existence of a solution in the class of functions represented as complete linear forms in two Cartesian coordinates with non-linear (with respect to the third Cartesian coordinate) coefficients. It is shown that the system is non-trivially solvable under a certain condition (compatibility condition) constructed by the authors. The corresponding theorem is formulated and proven. These conclusions are illustrated by a comparison with the previously obtained results.

Keywords: exact solution; thermal diffusion; Dufour effect; shear flow; overdetermined system; compatibility condition

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1. Introduction

The inhomogeneity of the density and viscosity properties of real fluids is caused by their different chemical composition. It leads to the fact that the distribution of thermal force fields in the fluid and the characteristics of its flow are influenced not only by the edge thermal effect on the flow area, but also by the parameters of the distribution of impurities (salinity) in it. As a result, it becomes necessary to take into account the concentration factors in the constitutive relations themselves (the law of motion of a viscous fluid and various energy laws and their analogues) [1–12].

Accounting for such factors (the temperature, concentration, magnetization, etc.) affects the number of equations of the viscous fluid model used and, as mentioned above, the form of the energy and other laws themselves. One thing remains unchanged: the quadratic nonlinearity of these equations with respect to the components of the fields to be determined, which makes it difficult to find solutions to the system of equations under consideration. The exception is intentionally linearized models (the Stokes approximation [13–15], the Oseen approximation [16,17], etc.).

Despite the three-dimensional nature of most real-fluid flows, in certain situations, with some accuracy, it is permissible to use models of the so-called shear flows [9,11,18–32]. The shear flow takes place when one of the projections of the velocity vector is assumed to be zero (while maintaining the dependencies of an arbitrary form of the remaining two projections on the coordinates of the selected system). More generally, one of the velocity field components is assumed to be constant, as in problems problems with permeable boundaries [33–37].

With a formal simplification of the velocity field structure, the problem of redefining [11,29–32,38–41] the system of basic relations of the model arises. Hence, it leads to the need to search for compatibility conditions for non-trivial solutions of individual equations of this system.

This paper attempts to construct such a condition for shear thermal diffusion flows, taking into account only the Dufour [1,12,42–45] effect. This affect is traditionally ignored when describing thermal diffusion in viscous fluids. Attention is paid to (1) the analysis of the solvability of the system in the Lin–Sidorov–Aristov [46–48] class of functions, which are linear in the part of the coordinates, and (2) to the study of the influence of the Dufour effect on the constructed exact solution. Additionally, a comparison with similar solutions for the purely thermal convection and thermal diffusion with the Soret effect [1,4,10–12,45] in the same class of functions is made.

2. Problem Statement

A model of steady thermal diffusion flows of viscous incompressible fluids [49,50] is considered. It consists of:

- The basic equation of a motion

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \nu \Delta \mathbf{V} + \mathbf{F}, \tag{1}$$

- Incompressibility equation

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

- Heat equation

$$(\mathbf{V} \cdot \nabla)T = \chi \Delta T + \chi \delta \Delta C \tag{3}$$

and the law of the impurity distribution in the considered fluid volume

$$(\mathbf{V} \cdot \nabla)C = d \Delta C. \tag{4}$$

Here, $\mathbf{V}(x,y,z) = (Vx, Vy, Vz)$ is a fluid velocity vector; **F** is a vector of volumetric (mass) forces; $T(x,y,z)$ is a temperature deviation from equilibrium state; $C\ (x,y,z)$ is a deviation of the concentration of the light phase (impurities) of a binary fluid mixture from the equilibrium value; $P(x,y,z)$ is a deviation of the pressure from the hydrostatic, normalized to the average (constant) density of the fluid r; n is a kinematic (molecular) viscosity; c is a thermal diffusivity coefficient; d is a diffusion coefficient; d is the Dufour parameter;

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

are the Hamilton and Laplace operators accordantly;

$$(\mathbf{V} \cdot \nabla) = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}$$

is a convective derivative.

Let us pay a special attention to the form of Equations (3) and (4). Usually, when studying thermal diffusion flows, the contribution of the Dufour effect, characterized by the parameter δ, is ignored, and the counter process (the Ludwig–Soret effect) is introduced into the consideration. This effect consists of the appearance of an impurity concentration gradient in the binary fluid mixture due to the temperature difference. Such a "replacement" is traditionally explained by the smallness of the contribution of the Dufour effect, the special influence of which is large mainly for gases. It is for this reason that in this paper, an attempt is made to study the degree of influence of the Dufour effect on the properties of exact shear solutions of the considered binary mixtures. Additionally, an attempt to compare the obtained results with similar results for thermal convection [32] and for thermal diffusion flows, taking into account the Soret effect [11], is made.

We also note that the system of Equations (1)–(4) is applicable to describe gas flows, despite the use of the incompressibility Equation (2) in the model and the large intermolecular distance between the particles of real gases. The fact is that at low gas flow rates, the change in pressure in the flow is insignificant. This leads to a small change in density, and therefore, the effect of the compressibility is small. It turns out that the error from the application of the incompressibility model in determining the gas density at speeds of up to 70–80 m/s is about 2% [51]. With a significant increase in velocity, the error in the calculations using the formulas for the incompressible fluid, of course, sharply increases. Similar facts are given, for example, in [52–56], where it is shown that the gas flow with an undersonic velocity behaves similarly to the incompressible fluid flow. Thus, we have grounds to consider from the same point of view regarding the flows of both real gases, for which the Dufour effect is prevailing, and incompressible fluids.

Equations (1)–(4) form a closed system of quadratically nonlinear partial differential equations with respect to six unknown functions (three projections of the velocity vector $\mathbf{V}$, the pressure P, the temperature T and the concentration C). The last two characteristics, in view of the acceptance of the Boussinesq hypothesis [49,50], are included in the Navier–Stokes Equation (1):

$$\mathbf{F} = (0, 0, g(\beta_1 T + \beta_2 C)),$$

where g is the gravitational acceleration, β_1, β_2 are the coefficients of thermal and concentration expansion, respectively.

Note that there are no isolated equations in the systems (1)–(4). Formally, it is solvable (because the number of equations coincides with the number of unknown functions), although the method of integrating it in general form is not known. Finding this method is significantly complicated by the quadratic nonlinearity of Equations (1), (3) and (4).

Until now, the question of the existence and smoothness regarding the solution of the system of the Navier–Stokes Equation (1) remains open. The formulated problem belongs to the list of seven millennium problems—seven mathematical problems identified by the Clay Mathematical Institute in 2000. For this reason, various approaches are being taken to find solutions for the systems of a hydrodynamic-type (systems of the form (1)–(4) and the like). There are several such approaches. The first one is based on simplifying the system of equations of the form (1)–(4) by discarding non-linear terms (ignoring the convective derivative $((\mathbf{V} \mid \nabla)\mathbf{V})$ and replacing the indicated terms with their linearized counterpart. Examples of such replacements are the Stokes [13–15] approximation and Oseen [16,17] approximation. After simplification, the system is more easily integrated. However, its solution largely loses the ability to describe the nonlinear effects observed in real fluids. The second most widely used approach is a numerical simulation. The discretization of the problem space (i.e., the area of a fluid or gas flow, function-characteristics of the medium under consideration, etc.) and the reduction of initial equations to the corresponding difference equations entail the need for additional research. It becomes necessary to answer on some questions. Firstly, we need to investigate the convergence of the applied method. Secondly, we should clarify whether the numerical method converges to a real solution (in the case that the convergence of the applied method is proven) or to something else.

This question can be answered by the exact solutions of the hydrodynamic-type system under consideration, constructed within the framework of various classes. It is their construction that constitutes the essence of the third approach when solving nonlinear systems of equations of the type of systems (1)–(4). The exact solutions make it possible to test already known numerical methods for the convenience of their application to problems of hydrodynamics. They also can clarify the main questions of the convergence of calculated discrete schemes: does the used method actually convergence to the exact solution? What is the rate of convergence of this method? Finding (constructing) a new exact solution is indisputably important. However, the question of how to construct this solution also remains difficult. As a result, one has to agree to some simplifications made along the way. In particular, the transition to steady flows, to the solution of which the solution of the hydrodynamic equations converges at large times, taking into account the time factor. Another such "classical" assumption is the consideration of shear flows.

Shear flows play a huge role in cases where the flow velocity in two directions prevails over the flow velocity in the third direction. Such models take place in the analysis of flows in thin layers [57–60]. In the experimental works of Petrov [61], the hydrodynamic nature of the lubrication process was established. The basis of the hydrodynamic theory of lubrication is the assumption of the Newtonian nature of a viscous fluid. One of the assumptions used in this theory is that the flow occurs through a slot with a slowly changing gap width. As a result, a characteristic approximation is obtained when the velocity profile for any value of the horizontal coordinate is considered identical to the velocity profile between infinite parallel plates.

A similar situation arises in biomechanics. For example, it is known that blood is a viscous, incompressible fluid. At relatively low linear flow velocities, blood particles are displaced parallel to each other and to the axis of the vessel. In this case, the blood flow has a layered character. If the linear velocity increases and exceeds a certain value, which is different for each vessel, then the shear flow turns into a random, vortex one. The velocity of blood movement at which the shear flow becomes turbulent is determined using the Reynolds number. For blood vessels, it is approximately 1160 [62].

Thus, flows with a "simplified" structure of the flow velocity field, when the third component of this vector is assumed to be zero, take place in reality. It means that the corresponding model of shear flows becomes acceptably admissible.

Thus, let us further consider shear flows, fixing the vertical component of the velocity vector:

$$V_z = 0.$$

Then, the systems (1)–(4) in the coordinate notation will be reduced to the system:

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V_x,$$

$$V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V_y,$$

$$\frac{\partial P}{\partial z} = g(\beta_1 T + \beta_2 C),$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0,$$

$$V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = \chi \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \chi \delta \Delta C,$$

$$V_x \frac{\partial C}{\partial x} + V_y \frac{\partial C}{\partial y} = d\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right). \tag{5}$$

Despite the simplification of the velocity field representation (one of the three components is assumed to be zero), the system (5) inherits the nonlinearity and absence of isolated equations. However, at the same time, it becomes overdetermined: the number of functions to be determined is now less than the number of system equations. Thus, the situation is likely that the solutions to the equations of the system (5) will not be compatible. Therefore, it is necessary to look for a condition regarding the solvability of this system (a condition for the compatibility of solutions to individual equations of this system).

The solution of the overdetermined equation system (5), which describes the shear flow, will be sought in the exact formulation. As a base class, let us consider the Lin–Sidorov–Aristov class of functions that are linear in the part of coordinates [6,11,29–31,46–48]:

$$V_x(x, y, z) = U(z) + u_1(z)x + u_2(z)y,\ V_y(x, y, z) = V(z) + v_1(z)x + v_2(z)y,$$

$$P(x, y, z) = P_0(z) + P_1(z)x + P_2(z)y,\ T(x, y, z) = T_0(z) + T_1(z)x + T_2(z)y,$$

$$C(x, y, z) = C_0(z) + C_1(z)x + C_2(z)y. \tag{6}$$

The peculiarity of the chosen class lies in its external simplicity. Additionally, at the same time (due to the dependence of the arbitrary form of the coefficients of forms (6) on the vertical coordinate z), this class can describe complex nonlinear effects observed in binary fluids.

It is this deceptively simple structure that can give the reader a false impression. At first sight, the further research appears to be like solving a mathematical problem for students regarding the basics of algebra and differential equations. However, we note that this impression is precisely false.

The whole point is in the essential nonlinearity of the coefficients of forms (6) relatively the third Cartesian coordinate, on which no additional restrictions are made. Expressions (6) describe the so-called solutions of dimension 2.5 (in other words, solutions of dimension (2 + 1)). This solutions describe flat flows, but depend on all three Cartesian coordinates [63].

Moreover, the exact solutions with structure (6) generalize all known classical solutions for viscous fluids—the Couette flow [64], Poiseuille flow [65–67], Karman flow [68], Ekman flow [69], Ostroumov–Birich solution [70,71], Sidorov–Aristov solution [47,48] and many others.

Using the exact solutions of class (6) [11,29–32] allows us to pass to a system of ordinary differential equations. The solution of this system is a non-trivial exact solution of the system (5):

$$u_1{}^2 + u_2 v_1 = \nu u_1{}'',\ u_1 u_2 + u_2 v_2 = \nu u_2{}'',$$

$$u_1 v_1 + v_1 v_2 = \nu v_1{}'',\ u_2 v_1 + v_2{}^2 = \nu v_2{}'',$$

$$u_1 + v_2 = 0; \tag{7}$$

$$u_1 C_1 + v_1 C_2 = d C_1{}'',\ u_2 C_1 + v_2 C_2 = d C_2{}''; \tag{8}$$

$$u_1 T_1 + v_1 T_2 = \chi T_1{}'' + \chi\delta C_1{}'',\ u_2 T_1 + v_2 T_2 = \chi T_2{}'' + \chi\delta C_2{}''; \tag{9}$$

$$P_1{}' = g(\beta_1 T_1 + \beta_2 C_1),\ P_2{}' = g(\beta_1 T_2 + \beta_2 C_2); \tag{10}$$

$$U u_1 + V u_2 = -P_1 + \nu U'',\ U v_1 + V v_2 = -P_2 + \nu V''; \tag{11}$$

$$UT_1 + VT_2 = \chi T_0{}'' + \chi\delta C_0{}'',\ UC_1 + VC_2 = dC_0{}''; \tag{12}$$

$$P_0{}' = g(\beta_1 T_0 + \beta_2 C_0). \tag{13}$$

The differentiation in the system of Equations (7)–(13) is carried out with respect to the variable z. The overdetermination of the obtained ODE system is removed if the spatial accelerations satisfy the following relations [32]:

$$u_1 = u\cos\theta\sin\theta = -v_2,\ u_2 = u\cos^2\theta,\ v_1 = -u\sin^2\theta, \tag{14}$$

where u is a function satisfying the equation $u'' = 0$, θ is some number.

Under conditions (14), the overdetermined system (7) has a nontrivial joint solution. The integration of the systems (8)–(13) occurs sequentially, according to the given order.

3. Construction of the Exact Solution

Let us first consider Equations (8) and (9), substituting expressions (14) into them:

$$\chi T_1{}'' + \chi\delta C_1{}'' = u\cos\theta\sin\theta T_1 - u\sin^2\theta T_2,\ \chi T_2{}'' + \chi\delta C_2{}'' = u\cos^2\theta T_1 - u\cos\theta\sin\theta T_2,$$

$$dC_1{}'' = u\cos\theta\sin\vartheta C_1 - u\sin^2\theta C_2,\ dC_2{}'' = u\cos^2\theta C_1 - u\cos\theta\sin\theta C_2. \tag{15}$$

We multiply the first equation of the system (15) by cosθ, and the second by sinθ:

$$\chi\cos\theta T_1{}'' + \chi\delta\cos\theta C_1{}'' = u\cos^2\theta\sin\theta T_1 - u\sin^2\theta\cos\theta T_2,$$

$$\chi\sin\theta T_2{}'' + \chi\delta\sin\theta C_2{}'' = u\cos^2\theta\sin\theta T_1 - u\cos\theta\sin^2\theta T_2.$$

whence, as a result of the double integration, the next relation follows:

$$\cos\theta T_1 + \delta\cos\theta C_1 - \sin\theta T_2 - \delta\sin\theta C_2 = \gamma_1 z + \gamma_2. \tag{16}$$

From the third and fourth equations of the system (15), we similarly obtain

$$\cos\theta C_1 - \sin\theta C_2 = \gamma_3 z + \gamma_4. \tag{17}$$

Note that in a particular case sinq = 0, from the relation (17), we directly find:

$$C_1 = \pm(\gamma_3 z + \gamma_4).$$

Further, by substituting this connection into Equation (16), we find:

$$\pm T_1 \pm \delta C_1 = \gamma_1 z + \gamma_2$$

or

$$\pm T_1 = \mp\delta C_1 + \gamma_1 z + \gamma_2 = \mp\delta(\gamma_3 z + \gamma_4) + \gamma_1 z + \gamma_2,$$

$$T_1 = -\delta(\gamma_3 z + \gamma_4) + \gamma_1 z + \gamma_2.$$

Taking into account the fact that the components T_1, C_1 are linear functions of the argument z, from the last equation of the system (15), we obtain:

$$C_2{}'' = \frac{u}{d}C_1.$$

As a result of integrating the last equation, we obtain a polynomial of the fourth degree in z. Similarly, from the second equation of the system (15), we come to the equation for determining the gradient T_2:

$$T_2'' = \frac{u}{\chi d}(dT_1 - \chi\delta C_1).$$

From this equation, it directly follows that the required function is also a polynomial of the fourth degree. The result of integrating the last two equations is not given because of two reasons. Firstly, it is not difficult in view of the polynomial inhomogeneity. Secondly, it is not the goal of the research task.

Let us return to the system (15) and discuss the algorithm for the finding its solution in the general case $\sin\theta \neq 0$. Let us express from relations (16) and (17), the gradients C_2, T_2:

$$\begin{pmatrix} -\sin\theta & -\delta\sin\theta \\ 0 & -\sin\theta \end{pmatrix}\begin{pmatrix} T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\cos\theta T_1 - \delta\cos\theta C_1 + \gamma_1 z + \gamma_2 \\ -\cos\theta C_1 + \gamma_3 z + \gamma_4 \end{pmatrix}. \tag{18}$$

Note that the determinant of this system is equal to $\sin^2\theta = 0$. It means that it is different from zero. Accordingly, the matrix will be non-singular, and the system (18) is uniquely solvable:

$$T_2 = \frac{\cos\theta}{\sin\theta}T_1 + \frac{(-\gamma_1 + \gamma_3\delta)z + (-\gamma_2 + \gamma_4\delta)}{\sin\theta},\ C_2 = \frac{\cos\theta}{\sin\theta}C_1 - \frac{\gamma_3 z + \gamma_4}{\sin\theta}. \tag{19}$$

Let us substitute relations (19) into the first and third equations of the system (15). After simple algebraic transformations, we obtain the following equations:

$$\chi T_1'' + \chi\delta C_1'' = \sin\theta((\gamma_1 - \gamma_3\delta)z + (\gamma_2 - \gamma_4\delta))u,\ dC_1'' = u\cos\theta\sin\theta C_1 - u\ \sin^2\theta C_2. \tag{20}$$

Resolving the system (51) with respect to the second derivatives of the gradients C_1, T_1, we obtain:

$$\begin{aligned} T_1'' &= \tfrac{\sin\theta((d(\gamma_1-\gamma_3\delta)-\gamma_3\delta\chi)z+d(\gamma_2-\gamma_4\delta)-\gamma_4\delta\chi)u}{d\chi}, \\ C_1'' &= \tfrac{\sin\theta(\chi\gamma_3 z+\chi\gamma_4)u}{d\chi}. \end{aligned} \tag{21}$$

The right-hand sides of the system (21) contain polynomials of the second degree, so the system (21) can be easily integrated. The resulting expressions are not presented here due to their cumbersomeness. After the exact solutions for the gradients C_1, T_1 are found, according to Formula (19), expressions for the gradients C_2, T_2 are easily written out. Thus, the system of Equations (8) and (9) is completely solved.

Further, by a single integration of Equation (10), the solutions that describe the behavior of the longitudinal pressure gradients P_1, P_2 are easily found. The further course of the solution (the integration of systems (11)–(13)) completely repeats the algorithm described in [11,32]. The difference from [32] is only in the presence of an additional term in the first equation of the system (12). However, in view of its additivity, the addition of this term does not change the algorithm for integrating the ODE systems (11)–(13) [11].

4. Analysis of Results

In summary, the following can be noted.

- For shear flows taking into account the terms describing the Dufour effect (as well as the Soret effect [11]) does not affect the solvability condition form. The compatibility condition is obtained exactly the same way as for the thermal convection [32], i.e., without taking into account the features introduced by the presence of impurities in the fluid.

- Taking into account the terms describing the Dufour effect (as well as the Soret effect [11]) does not increase the degree of polynomials on the right-hand sides of the homogeneities of used ODE. The degree of polynomials is the same for the similar equations as for the purely thermal convection [32].
- Taking into account the terms describing the Dufour effect (as well as the Soret effect [11]) does not change the algorithm for finding the exact solution of the problem. The branching of the algorithm (as in [11,32]) depends only on the value of the parameter q, which determines (along with other parameters) the type of exact solution for spatial gradients u_1, u_2, v_1, v_2.

However, despite the remarks made, there is no reason to believe that all the conclusions will be valid, even when taking into account both cross effects. It is likely that their superposition will lead to additional difficulties in determining the compatibility conditions for the solutions for shear flows. It may lead to a fundamentally different approach to the algorithm for integrating model equations, or to the emergence of additional branches of the solution construction algorithm.

5. Interpretation of Results

For a visual interpretation of the influence of the Dufour effect, let us consider a special case of class (6):

$$V_x = U + u_2 y,\ V_y = V, \tag{22}$$
$$P = P_0 + P_1 x + P_2 y,\ T = T_0 + T_1 x + T_2 y,\ C = C_0 + C_1 x + C_2 y.$$

This class for the convection problem was considered in [72], where the corresponding flow profiles and distributions of hydrodynamic fields are given.

Note that for expressions (22), the incompressibility Equation (2) is fulfilled identically, which removes the problem of the overdetermination from the system (7). For the class (22), the system of Equations (7)–(13) takes the form:

$$u_2{}'' = 0,\ C_1{}'' = 0,\ dC_2{}'' = u_2 C_1,\ T_1{}'' = 0,\ \chi T_2{}'' = u_2 T_1 - \frac{\chi\delta}{d} u_2 C_1,$$
$$P_1{}' = g(\beta_1 T_1 + \beta_2 C_1),\ P_2{}' = g(\beta_1 T_2 + \beta_2 C_2),$$
$$\nu U'' = P_1 + V u_2,\ \nu V'' = P_2,$$
$$U T_1 + V T_2 = \chi T_0{}'' + \chi\delta C_0{}'',\ U C_1 + V C_2 = d C_0{}'',$$
$$P_0' = g(\beta_1 T_0 + \beta_2 C_0). \tag{23}$$

To determine the integration constants that will appear upon a successive integration of System (23), we choose the same system of boundary conditions as in [72]. We consider the flow in an extended horizontal layer, assuming its lower surface ($z = 0$) to be absolutely solid, and the upper surface ($z = h$) to be free and non-deformable.

Thermocapillary forces act on the upper boundary $z = h$ of the fluid layer, generating convective flows of a vertically swirling fluid:

$$\eta V'_x = -\sigma\frac{\partial T}{\partial x},\ \eta V'_y = -\sigma\frac{\partial T}{\partial y}.$$

Thermal perturbations are set on both boundaries of the fluid layer:

$$T(x, y, 0) = Ax + By,\ T(x, y, h) = \vartheta + Ex + Dy.$$

We consider the upper boundary of the layer to be the base pressure reference level:

$$P(x, y, h) = 0.$$

In addition, we assume that a non-uniform parabolic distribution of velocities is specified on the upper boundary of the fluid layer:

$$V_x = W\cos\alpha + \Omega y, \quad V_y = W\sin\alpha.$$

Let us supplement the indicated system of boundary conditions with conditions for the concentration field, similar to the conditions for the temperature field:

$$C(x, y, 0) = Fx + Gy,\ C(x, y, h) = \gamma + Kx + Ly.$$

The exact solution of the formulated boundary value problem has the form:

$$T_1 = A(1 - Z) + EZ,$$

$$T_2 = B(1 - Z) + DZ + \left(\frac{E}{\chi} - \frac{\delta K}{d}\right)\frac{h^2\Omega}{6}Z\left(Z^2 - 1\right) + \left(\frac{A}{\chi} - \frac{\delta F}{d}\right)\frac{h^2\Omega}{6}\left(-2Z + 3Z^2 - Z^3\right). \quad (24)$$

Here, $Z = z/h$, where h is the thickness of the considered layer.

Note that the solution is given partially. This selectivity is due to two factors. First, the expression for the background components T_0, C_0, P_0 of linear forms (6) is described by too cumbersome expressions. Secondly, this paper does not aim to investigate the obtained exact solution for the possibility of the counterflows simulation and the stratification of hydrodynamic fields. The solution is given partially (only in part of the temperature gradients T_1, T_2) in order to compare this solution and the solution of a similar convective problem, i.e., to illustrate the influence of the Dufour coefficient on the profiles of the components of hydrodynamic fields.

Moreover, for the purity of the comparison, we put in the solution (24) $A = D = 0$, as it was carried out in [72], and as a result, we obtain the following expressions:

$$T_1 = EZ,$$

$$T_2 = B(1 - Z) + \left(\frac{E}{\chi} - \frac{\delta K}{d}\right)\frac{h^2\Omega}{6}Z\left(Z^2 - 1\right) - \delta\frac{Fh^2\Omega}{6d}\left(-2Z + 3Z^2 - Z^3\right). \quad (25)$$

The difference from the corresponding expressions given in [72] lies in taking into account the terms containing the Dufour coefficient. It is the last term that makes the most significant contribution to the second expression in (25). The polynomial ($-2Z + 3Z^2 - Z^3$) on the interval [0;1] corresponding to the considered layer is not a monotonic function. This means that taking into account such a term can lead to an increase in the number of zero points, i.e., to the complication of the topology of the T_2 component (Figure 1).

Thus, Figure 1 clearly illustrates the possibility of the appearance of a T_2-component profile with two zero points, i.e., with three stratification regions, in each of which the T_2-component retains its sign.

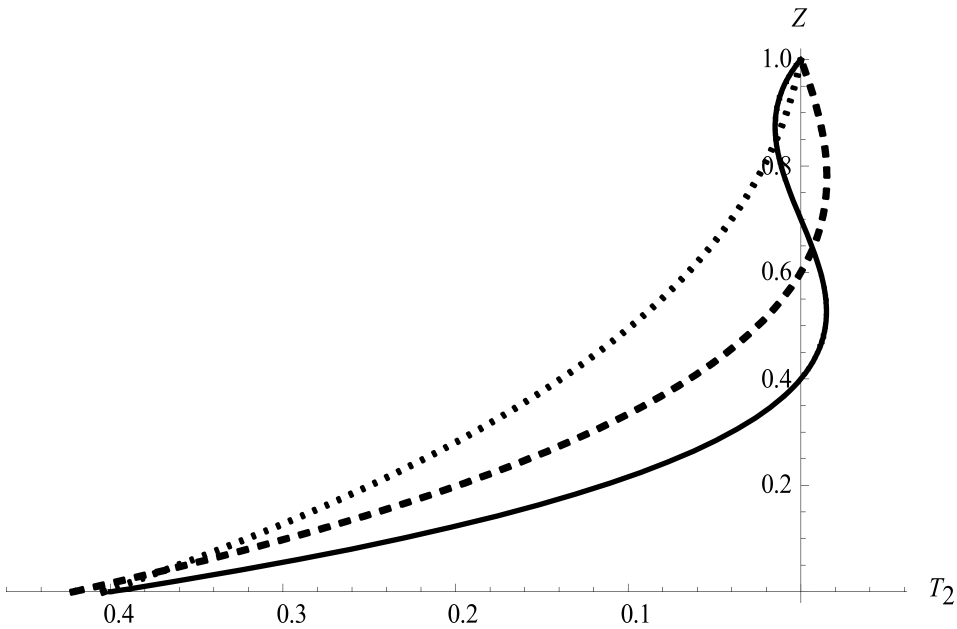

Figure 1. The profiles of the temperature gradient T_2 for different values of the problem parameters.

6. Conclusions

In the article, the problem of describing thermal diffusion flows is studied, taking into account the Dufour effect. This effect plays a special role in the gas flow formation. At low velocities, we can consider real gases as viscous incompressible gases. The system of equations for describing the viscous incompressible gas flows and for describing the flows of viscous incompressible fluids are in complete formal correspondence. This is why, when analyzing the influence of the Dufour effect, we retain the vocabulary most familiar to the authors—we continue talking about the fluid flows.

It is shown that when simulating shear flows, the system of thermal diffusion equations becomes overdetermined. For its solvability, the corresponding theorem is formulated. The proof of the theorem is carried out constructively in the class of functions linear in part of the coordinates. This means that the paper presents a new class of exact solutions that satisfy the overdetermined system of equations.

In addition, for a visual illustration of the findings, a comparison of the results previously obtained by the authors with the current results was made. It is shown that taking into account the Dufour effect leads to a complication of the topology of the spatial gradient T_2 of the temperature field. Remind that the equations of the original overdetermined reduced system are integrated sequentially. Hence, the same result means that taking into account the Dufour effect will affect the structure of other hydrodynamic fields, thus, significantly complicating it.

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