On the existence and uniqueness of solutions to a nonlinear variable order time-fractional reaction–diffusion equation with delay

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Abstract

In this article, our purpose is to study the existence and uniqueness of a solution to a damped variable order fractional subdiffusion equation with time delay. Under weak assumptions on the data, we prove the uniqueness of a weak solution to the problem under consideration. The method of semi-discretization is extended to this kind of time fractional parabolic problem with delay in the case that the time delay parameter $s > 0$ satisfies $s \leqslant T$, where T denotes the final time. As a consequence, two a priori estimates are predicted based on a discrete variational formulation of the problem. The existence of the problem's weak solution on the time frame $\left[0, \lfloor \frac{T}{s} \rfloor s\right]$ is established by the aid of these derived a priori estimates. The paper is closed by introducing a fully discrete scheme based on Galerkin Legendre spectral approximation for the spatial operator and the backward Euler difference approximation for the temporal variable order operator. Accordingly, the accuracy and efficiency of the proposed scheme are justified by giving some numerical experiments for the sake of clearness.

Keywords: Variable order subdiffusion, Time delay, Uniqueness, Existence, A priori estimates, Rothe's method

1. Introduction

1.1. Problem formulation

The following nonlinear variable order time-fractional reaction-diffusion equation with delay is under consideration

$$\frac{\partial^{\beta(t)} u}{\partial t^{\beta(t)}}(\mathbf{x},t)+\lambda \frac{\partial u}{\partial t}(\mathbf{x},t) = \kappa\, \Delta u(\mathbf{x},t)+f(u(\mathbf{x},t),u(\mathbf{x},t-s))+g(x,t), \quad 0<\beta(t)<\bar{\beta}<1,\ x\in\Omega,\ t\in I, \tag{1.1}$$

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such that $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded Lipschitz domain with boundary $\partial\Omega$. The problem is endowed with initial-boundary conditions of the form

$$\begin{cases} u(\mathbf{x},t) = \psi(\mathbf{x},t), & \mathbf{x} \in \Omega, \quad t \in [-s,0], \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times I, \end{cases} \tag{1.2}$$

where $I = (0,T] \subset \mathbb{R}$ is the time domain. The diffusion coefficient κ, the damping coefficient λ and the delay parameter s are strict positive constants. The variable-order fractional integral operator ${}_0I_t^{\beta(t)}$, and the variable-order fractional Caputo operator $\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}}$ involve the memory of its order history and are defined by the following [1, 2, 3]

$$ {}_0I_t^{\beta(t)} u(t) := \int_0^t \frac{1}{\Gamma\left(\beta(t-r)\right)} \frac{u(r)}{(t-r)^{1-\beta(t-r)}} \, \mathrm{d}r,$$

$$\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} u(t) := {}_0I_t^{1-\beta(t)} u'(t) = \int_0^t \frac{1}{\Gamma\left(1-\beta(t-r)\right)} \frac{u'(r)}{(t-r)^{\beta(t-r)}} \, \mathrm{d}r. \tag{1.3}$$

1.2. Literature overview

The first time that a variable order fractional differential derivative appeared as a generalization of a constant order fractional differential operator was in an article of Samko and Ross in 1993 [4]. An employment of fractional partial differential equations (FPDEs) with variable orders have been encountered in different physical and dynamical systems [5, 6]. A successful application of variable order differential operators to a wide range of real-world problems has been verified due to the ability of constructing governing equations of evolutionary type [7]. These applications have been efficiently used in fields of science in which investigating memory properties that vary in time and space is needed, as in mechanics, transport processes, control theory and biology [6, 8]. New difficulties from mathematical, numerical and computational viewpoint raised up when dealing with FPDEs. For example, an exhibition of singularity near the initial time is noticed for the first-order time derivative of the solutions to the time-fractional diffusion equations. This weakly singularity gives an evidence that the error estimates in the literature are inappropriate if their proofs were done under full regularity assumptions of the true solutions, which is discussed in e.g. [9, 10]. Numerical analysis for nonlinear FPDEs with a fixed and functional delay has been investigated by using novel techniques of discrete fractional Grönwall inequalities and discrete energy estimates, see e.g. [11, 12].According to the works [13, 14], the root of occurrence of solutions with non-physical sense for constant order FPDEs near $t = 0$ comes out from the incompatibility between the non-locality of the governing FPDEs (of the power-law decaying tails) and the locality of the initial conditions, respectively. Elimination of that non-physical singularity can be done as in [14] if we use FPDEs in which the order varies smoothly to an integer value at $t = 0$. It can be clearly noticed in the literature that variable-order FPDEs arise in many applications, as in [2, 15, 8] in which the variable order indicates the fractal dimension of the porous media. Different numerical approximations were developed and analyzed for FPDEs, see [16, 17, 18, 19, 15, 20, 21]. Well-posedness results for time-dependent variable order problems where the derivative order has not the memory of its history can be found in e.g. [22, 23, 14, 24, 25], whilst for a variable order FPDEs with hidden memory we refer to [26]. All these contributions consider linear sources and do not consider time delay.

Rothe's method (which is also called the method of semi-discretization in time) was initially developed as a discretization method in time for partial differential equations [27, 28]. The work in [29] introduced Rothe's method as an accurate theoretical tool for solving a wide scale of evolution problems. A theoretical and numerical treatment of initial boundary value problems of parabolic type with Volterra operators was constructed in [30], endowed by an integro-differential equation and a natural boundary condition. A constructive proof of the uniqueness and existence of its variational solution under weak assumptions on the data was done by invoking Rothe's method. We also give some recent well-posedness results illustrating the importance of this method. In [31], the authors study the existence and uniqueness of a solution to anisotropic thermoelastic systems. A consideration of a weak solution for a fractional order diffusion equation with Volterra differential operator and with fractional integral condition is shown in [32]. It has been proved that the solution existed and was unique as well as its regularity by designing an appropriate Rothe scheme. Recently, the paper [33] studies analytically and numerically a Rothe-Galerkin finite element method for approximating the solutions of a Boussinesq-type system to model water wave propagation over a time-dependent variable topography. Moreover, in [34], the author studied the well-posedness of a fractional diffusion equation with space-dependent variable order with the aid of Rothe's method.

1.3. Outline

Throughout the paper, it is assumed that $\beta(t)$ is chosen such that the corresponding convolution kernel is positive definite (the structure of (1.3) makes this possible) and we take advantage of this structure. For more details, see the beginning of Section 2, where all assumptions on the data are listed, and the weak formulation is formulated. Afterwards, the uniqueness of a solution to problem (1.1-1.2) is discussed in Theorem 2.1. Note that the variable-order fractional Caputo operator involves the memory of its order history, and that the positive definiteness of the governing integral kernel is crucial here. Next, in Section 3 the *local* existence of a solution is shown by the aid of Rothe's method. We note that it is common practice in problems with delay to discretize first the interval $[-s, 0]$ by using a uniform time mesh [35]. To be able to apply Rothe's method, we need to restrict the time frame to $[0, T_0]$ with $T_0 := \lfloor \frac{T}{s} \rfloor s$ assuming $s \leqslant T$. To the best of our knowledge, it is the first time that this approach has been considered. Then, the existence of a solution to problem (1.1-1.2) on $[0, T_0]$ is established in Theorem 3.1. It is clear that the existence of a solution is global if $\frac{T}{s} \in \mathbb{N}$. This result builds on the weakly positivity of the time-discrete convolution [36], which leads to the derivation of two a priori estimates from which we can deduce the existence of a solution. Two important remarks are formulated at the end of the section. For instance, the advantage of our method is that the results can be extended to time-dependent elliptic operators. Finally, Section 4 is devoted to discretize the problem in space direction in terms of Galerkin Legendre spectral scheme and give two examples those illustrate the convergence of the proposed scheme.

2. Uniqueness of a solution

Firstly, the uniqueness of the weak solution will be shown for the problem (1.1-1.2). Before stating the variational formulation, we introduce the following notation for the integral kernel in (1.3):

$$k(t) = \frac{t^{-\beta(t)}}{\Gamma(1-\beta(t))}, \qquad t > 0.$$

Then, (1.3) can be rewritten as

$$\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} u(t) = (k * \partial_t u)(t),$$

where the symbol '$*$' stands for the convolution product defined by $(k * z)(t) = \int_0^t k(t-s)z(s)\,\mathrm{d}s$.

Further, we make the following assumptions

- (AS1): $\beta \in \mathrm{C}([0,T])$ with $0 < \beta(t) \leqslant \bar{\beta} < 1$ is to be chosen such that $k \in \mathrm{L}^1(0,T)$ with $\partial_t k, \partial_{tt} k \in \mathrm{L}^1_{\mathrm{loc}}(0,T)$ satisfies

  $$|k(t)| \leqslant C t^{-\bar{\beta}}, \qquad \forall t > 0,$$

  and

  $$(-1)^j k^{(j)}(t) \geqslant 0, \quad \forall t > 0; j = 0,1,2; k' \neq 0;$$

- (AS2): $\psi \in \mathrm{C}\left([-s,0], \mathrm{L}^2(\Omega)\right)$ with $\psi(0) \in \mathrm{H}^1_0(\Omega)$;

- (AS3): $g \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$;

- (AS4): $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, i.e. there exists a constant $L > 0$ such that

$$|f(u_1,v_1) - f(u_2,v_2)| \leqslant L\left(|u_1 - u_2| + |v_1 - v_2|\right). \tag{2.4}$$

The (AS1) clears the strongly positive definiteness of the kernel k [37, Corollary 2.2], i.e.

$$\int_0^t \langle (k * z)(r), z(r) \rangle_X \,\mathrm{d}r \geqslant 0, \qquad t > 0, \quad \forall z \in \mathrm{L}^2_{\mathrm{loc}}\left((0,\infty), X\right),$$

with X be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$ and $\mathrm{L}^2_{\mathrm{loc}}\left((0,\infty), X\right)$ the space of functions belonging to $\mathrm{L}^2\left((0,T), X\right)$ for any $T \in (0,\infty)$.

The variational formulation of (1.1-1.2) is given by

search $u \in \mathrm{C}\left([-s,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathrm{H}^1_0(\Omega)\right)$ with $\partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$
such that for a.a. $t \in (0,T)$ and $\forall \varphi \in \mathrm{H}^1_0(\Omega)$ it holds that

$$((k * \partial_t u)(t), \varphi) + \lambda\left(\partial_t u(t), \varphi\right) + \kappa\left(\nabla u(t), \nabla \varphi\right) = \left(f\left(u(t), u(t-s)\right), \varphi\right) + \left(g(t), \varphi\right), \tag{2.5}$$

where

$$u(t) = \psi(t) \quad \text{in } \mathrm{L}^2(\Omega) \text{ for all } t \in [-s,0].$$

Note that $(\cdot,\cdot)$ represents the standard inner product in $\mathrm{L}^2(\Omega)$. Its induced norm will be denoted by $\|\cdot\|$. From Young's inequality for convolutions

$$\|f_1 * f_2\|_{\mathrm{L}^r(0,T)} \leqslant \|f_1\|_{\mathrm{L}^p(0,T)} \|f_2\|_{\mathrm{L}^q(0,T)} \text{ for } \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1 \text{ with } 1\leqslant p,q\leqslant r\leqslant\infty,$$

it follows that

$$\|k*(z,\varphi)\|_{\mathrm{L}^2(0,T)} \leqslant \|k\|_{\mathrm{L}^1(0,T)} \|\varphi\| \|z\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} < \infty, \qquad \forall\{z,\varphi\}\in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)\times \mathrm{L}^2(\Omega), \quad (2.6)$$

and

$$\|k*z\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} \leqslant \|k\|_{\mathrm{L}^1(0,T)} \|z\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} < \infty, \qquad \forall z\in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right).$$

Moreover, for completeness, we state the following lemma, which is crucial in the uniqueness theorem. It is an extension of the following result obtained from e.g. [38, Lemma 2.1]: let u be absolutely continuous on $[0,T]$ and consider $g_\beta(t) := \frac{t^{-\beta}}{\Gamma(1-\beta)}$ with $\beta\in(0,1)$, then

$$\partial_t(g_\beta * u)(t) = g_\beta(t)u(0) + (g_\beta * \partial_t u)(t).$$

Lemma 2.1. *Assume that* $0\leq g\in \mathrm{L}^1(0,T)$ *satisfies* $\partial_t g\in \mathrm{L}^1_{loc}(0,T)$ *with* $\partial_t g\leqslant 0$ *on* $(0,T)$*. Let* $u\in \mathrm{H}^{1,\infty}(0,T)$*. Then, for any* $t\in(0,T)$*, it holds that*

$$\partial_t(g*u)(t) = g(t)u(0) + (g*\partial_t u)(t).$$

Proof. The function g is continuous on $(0,T)$ as $\partial_t g\in \mathrm{L}^1_{\mathrm{loc}}(0,T)$. Hence, the function g can only have a singularity at $t=0$ as it is decreasing. If $g\in \mathrm{C}([0,T])$, then the result follows immediately from the Leibniz integral rule. If g has a singularity at $t=0$, then we consider a sequence of cutoff functions ($n\in\mathbb{N}$) defined by $g_n(t):=\min\{n,g(t)\}$ for $t\in[0,T]$. Then, we have that $g_n\leqslant g$ on $(0,T)$ for all $n\in\mathbb{N}$, and $g_n(t)\to g(t)$ for $t\in(0,T)$ as $n\to\infty$. As $\partial_t u\in \mathrm{L}^\infty(0,T)\subset \mathrm{L}^1(0,T)$, we have that

$$(g_n * u)(t) = u_0\int_0^t g_n(s)\,\mathrm{d}s + \int_0^t g_n(s)\int_0^{t-s}\partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \quad (2.7)$$

Next, we pass to the limit $n\to\infty$ in (2.7). Applying the monotone convergence theorem for the integral on the LHS and the first integral on the RHS (note that $u\in \mathrm{C}\left([0,T]\right)$), we obtain that

$$(g*u)(t) = u_0\int_0^t g(s)\,\mathrm{d}s + \lim_{n\to\infty}\int_0^t g_n(s)\int_0^{t-s}\partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Then, differentiating with respect to t gives

$$\partial_t(g*u)(t) = u_0 g(t) + \partial_t\lim_{n\to\infty}\int_0^t g_n(s)\int_0^{t-s}\partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Using $\partial_t \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau = \partial_{t-s} \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau = \partial_t u(t-s)$, we notice by the Leibniz integral rule that

$$\begin{aligned}
&\partial_t \int_0^t g_n(s) \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= 1 \cdot g_n(t) \int_0^0 \partial_t u(\tau)\,\mathrm{d}\tau - 0 \cdot g_n(0) \int_0^t \partial_t u(\tau)\,\mathrm{d}\tau + \int_0^t g_n(s)\partial_t u(t-s)\,\mathrm{d}s \\
&= \int_0^t g_n(s)\partial_t u(t-s)\,\mathrm{d}s.
\end{aligned}$$

We have by [39, Theorem 7.17] that

$$\partial_t \lim_{n\to\infty} \int_0^t g_n(s) \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \lim_{n\to\infty} \partial_t \int_0^t g_n(s) \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

since

$$\left| \int_0^t g_n(s)\partial_t u(t-s)\,\mathrm{d}s - \int_0^t g(s)\partial_t u(t-s)\,\mathrm{d}s \right| \leq \|\partial_t u\|_{\mathrm{L}^\infty(0,T)} \int_0^T (g(s) - g_n(s))\,\mathrm{d}s$$

converges uniformly to 0 (independently of $t \in [0,T]$) by the monotone convergence theorem. Therefore, we have that

$$\partial_t \lim_{n\to\infty} \int_0^t g_n(s) \int_0^{t-s} \partial_t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t g(s)\partial_t u(t-s)\,\mathrm{d}s,$$

which concludes the proof. □

Now, we are ready to show the uniqueness of a weak solution to problem (1.1)-(1.2) by contradiction.

Theorem 2.1 (Uniqueness)**.**
Let the assumptions (AS1-AS4) be fulfilled. Then, the solution to problem (2.5) *satisfying* $u \in \mathrm{C}\left([-s,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right)$ *with* $\partial_t u \in \mathrm{L}^\infty\left((0,T), \mathrm{L}^2(\Omega)\right)$ *is unique.*

Proof. Assume the existence of two solutions u_1 and u_2 to solve (2.5). Then, the difference $u := u_1 - u_2$ fulfills $u(\cdot,t) = 0$ in Ω for $t \in [-s,0]$ and

$$((k * \partial_t u)(t), \varphi) + \lambda (\partial_t u(t), \varphi) + \kappa (\nabla u(t), \nabla\varphi) = (f(u_1(t), u_1(t-s)) - f(u_2(t), u_2(t-s)), \varphi),\ \forall \varphi \in \mathrm{H}_0^1(\Omega).$$

Note that $(k * \partial_t u)(t) = \partial_t (k * u)(t)$ by applying Lemma 2.1 with $g = k$. Now, we integrate with respect to time over $t \in (0,\eta) \subset (0,T)$, take $\varphi = u(\eta)$ and integrate again with respect to time over $\eta \in (0,\xi) \subset (0,T)$. We obtain that

$$\begin{aligned}
\int_0^\xi ((k * u)(\eta), u(\eta))\,\mathrm{d}\eta + \lambda \int_0^\xi \|u(\eta)\|^2\,\mathrm{d}\eta + \frac{\kappa}{2} \left\| \int_0^\xi \nabla u(t)\,\mathrm{d}t \right\|^2& \\
= \int_0^\xi \left(\int_0^\eta [f(u_1(t), u_1(t-s)) - f(u_2(t), u_2(t-s))]\,\mathrm{d}t, u(\eta) \right) \mathrm{d}\eta.& \quad (2.8)
\end{aligned}$$

The strongly positive definiteness of k implies that the first term on the left-hand side (LHS) is positive. Using the ε-Young inequality and the global Lipschitz continuity of f, we make the following deduction for the term on the right-hand side (RHS)

$$
\begin{aligned}
&\left|\int_0^\xi \left(\int_0^\eta \left[f\left(u_1(t), u_1(t-s)\right) - f\left(u_2(t), u_2(t-s)\right)\right] \mathrm{d}t, u(\eta)\right) \mathrm{d}\eta\right| \\
&\leqslant C_\varepsilon \int_0^\xi \left(\int_0^\eta \left(\|u(t)\|^2 + \|u(t-s)\|^2\right) \mathrm{d}t\right) \mathrm{d}\eta + \varepsilon \int_0^\xi \|u(\eta)\|^2 \,\mathrm{d}\eta \\
&\leqslant C_\varepsilon \int_0^\xi \left(\int_0^\eta \|u(t)\|^2 \,\mathrm{d}t\right) \mathrm{d}\eta + \varepsilon \int_0^\xi \|u(\eta)\|^2 \,\mathrm{d}\eta
\end{aligned}
$$

as $u(\cdot, t) = 0$ in Ω for $t \in [-s, 0]$. Therefore, from (2.8), we get that

$$
(\lambda - \varepsilon) \int_0^\xi \|u(\eta)\|^2 \,\mathrm{d}\eta + \frac{\kappa}{2} \left\|\int_0^\xi \nabla u(t) \,\mathrm{d}t\right\|^2 \leqslant C_\varepsilon \int_0^\xi \left(\int_0^\eta \|u(t)\|^2 \,\mathrm{d}t\right) \mathrm{d}\eta.
$$

Next, we fix ε such that $\varepsilon < \lambda$ and apply the Grönwall lemma to obtain that $u = 0$ a.e. in $\Omega \times (0, T)$. $\square$

3. Local existence of a solution if $s \leqslant T$

Rothe's method is utilized to show the existence of a solution. First, the time interval $[-s, 0]$ is discretized by a time step $\tau < \min\{1, s\}$ defined by $\tau = \frac{s}{N}$ where N is a positive integer. Next, we define

$$
T_0 := \lfloor \frac{T}{s} \rfloor s.
$$

We will show the existence of a solution on the time interval $[0, T_0]$. The time discrete points are given by $t_i = i\tau,\ \forall -N \leq i \leq M$, where $M = \frac{T_0}{\tau} = \lfloor \frac{T}{s} \rfloor N$. The u_i denotes the approximate solution at time $t = t_i$ for $-N \leq i \leq M$. The backward Euler difference $\partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}$ is used to approximate the time derivative at time $t = t_i$ for $1 \leqslant i \leqslant M$. The time-discrete convolution is defined as (see [36])

$$
(k * z)(t_i) = \sum_{l=1}^{i} \int_{t_{l-1}}^{t_l} \frac{(t_i - r)^{-\beta(t_i - r)}}{\Gamma\left(1 - \beta(t_i - r)\right)} z(r) \,\mathrm{d}r \approx (k * z)_i := \sum_{l=1}^{i} k_{i+1-l} z_l \tau, \quad 1 \leq i \leq M. \tag{3.9}
$$

Note that

$$
(k * \delta u)(t_i) \approx \sum_{l=1}^{i} a_{i-l} \left(u_l - u_{l-1}\right) = \sum_{l=0}^{i} b_{i-l} u_l, \tag{3.10}
$$

where $a_l = k_{l+1}$ and $b_0 = a_0$, $b_i = -a_{i-1}$, $b_{i-l} = a_{i-l} - a_{i-l-1}$, for $l = 1, \ldots, i-1$.

Then, the problem (2.5) is approximated at time $t = t_i$ for $1 \leqslant i \leqslant M$ as follows:

Find $u_i \in \mathrm{H}_0^1(\Omega)$ such that

$$
\left((k * \delta u)_i, \varphi\right) + \lambda \left(\delta u_i, \varphi\right) + \kappa \left(\nabla u_i, \nabla \varphi\right) = \left(f\left(u_{i-1}, u_{i-N}\right), \varphi\right) + \left(g_i, \varphi\right), \quad \forall \varphi \in \mathrm{H}_0^1(\Omega), \tag{3.11}
$$

where

$$
u_i = \psi_i \quad \text{for } -N \leqslant i \leqslant 0.
$$

Now, the existence of a unique solution at every time step comes out by Lax-Milgram lemma.

Lemma 3.1. *Let the assumptions (AS1-AS4) be fulfilled. Then, for any $i = 1, 2, \dots, M$, there exists a unique function $u_i \in \mathrm{H}_0^1(\Omega)$ solving* (3.11).

Next, we derive two a priori estimates, which will be crucial for showing the existence later.

Lemma 3.2. *Assuming the achievement of (AS1-AS4) and $s \leqslant T$. Then, the existence of constants $C > 0$ and $\tau_0 > 0$ is guaranteed such that $\forall j = 1, 2, \dots, M$ and $\tau < \tau_0$, the inequality below is fulfilled*

$$\sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C.$$

Proof. Taking $\varphi = \delta u_i \tau$ in (3.11) and summing up for $i = 1, \dots, j$ with $1 \leqslant j \leqslant M$, i.e.

$$\sum_{i=1}^{j} \left(\sum_{l=1}^{i} k_{i-l+1} \delta u_l \tau, \delta u_i \right) \tau + \lambda \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \kappa \sum_{i=1}^{j} \left(\nabla u_i, \nabla \delta u_i\right) \tau$$
$$= \sum_{i=1}^{j} \left(f\left(u_{i-1}, u_{i-N}\right), \delta u_i\right) \tau + \sum_{i=1}^{j} \left(g_i, \delta u_i\right) \tau. \quad (3.12)$$

The positivity of the first term on the left-hand side (LHS) of (3.12) is due to the positive definiteness of the kernel k, see [36, Eq. 3.2]. Form Abel's summation rule, we get that

$$\kappa \sum_{i=1}^{j} \left(\nabla u_i, \nabla \delta u_i\right) \tau = \frac{\kappa}{2} \|\nabla u_j\|^2 - \frac{\kappa}{2} \|\nabla \psi_0\|^2 + \frac{\kappa}{2} \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 .$$

Using the ε-Young inequality, we easily see that

$$\left| \sum_{i=1}^{j} \left(g_i, \delta u_i\right) \tau \right| \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

Next, we estimate the first term on the right-hand side (RHS) of (3.12). Employing (AS2), (AS4) and $u_i = u_0 + \sum_{l=1}^{i} \delta u_l \tau$, this yields

$$\begin{aligned}
\left| \sum_{i=1}^{j} \left(f\left(u_{i-1}, u_{i-N}\right), \delta u_i\right) \tau \right| &\leqslant C_\varepsilon \sum_{i=1}^{j} \left(1 + \|u_{i-1}\|^2 + \|u_{i-N}\|^2\right) \tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau \\
&\leqslant C_\varepsilon \left(1 + \sum_{i=1-N}^{0} \|\psi_i\|^2 \tau\right) + C_\varepsilon \sum_{i=1}^{j-1} \|u_i\|^2 \tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau \\
&\leqslant C_\varepsilon + C_\varepsilon \sum_{i=1}^{j-1} \left(\sum_{l=1}^{i} \|\delta u_l\|^2 \tau \right) \tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.
\end{aligned}$$

Hence, from equation (3.12), we get that

$$(\lambda - \varepsilon) \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \frac{\kappa}{2} \|\nabla u_j\|^2 + \frac{\kappa}{2} \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C_\varepsilon + C_\varepsilon \sum_{i=1}^{j-1} \left(\sum_{l=1}^{i} \|\delta u_l\|^2 \tau \right) \tau.$$

Finally, we fix $\varepsilon < \lambda$ and apply the Grönwall lemma to conclude the proof. ☐

Lemma 3.3. *Let the assumptions (AS1-AS4) be fulfilled and $s \leqslant T$. Then, there exist constants $C > 0$ and $\tau_0 > 0$ such that $\forall j = 1, 2, \ldots, M$ and $\tau < \tau_0$, we gain*

$$|(\delta u_j, \varphi)| \leqslant C \|\varphi\|_{\mathrm{H}^1(\Omega)}, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega).$$

Proof. From (3.11) and Lemma 3.2 it follows that

$$\begin{aligned} |(\delta u_i, \varphi)| &\overset{(3.11)}{=} \frac{1}{\lambda} \left| (f(u_{i-1}, u_{i-N}), \varphi) + (g_i, \varphi) - \kappa (\nabla u_i, \nabla \varphi) - \left(\sum_{l=1}^{i} k_{i-l+1} \delta u_l \tau, \varphi \right) \right| \\ &\leqslant C \|\varphi\|_{\mathrm{H}^1(\Omega)} + \frac{1}{\lambda} \sum_{l=1}^{i} k_{i-l+1} |(\delta u_l, \varphi)| \tau. \end{aligned}$$

According to (AS1), we have that $k(\tau)\tau \to 0$ as $\tau \overset{>}{\to} 0$. Therefore,

$$|(\delta u_i, \varphi)| \leqslant C \left[\|\varphi\|_{\mathrm{H}^1(\Omega)} + \sum_{l=1}^{i-1} k_{i-l+1} |(\delta u_l, \varphi)| \tau \right].$$

Moreover, from (AS1), we see that $k_{i-l+1} \leqslant k_{i-l}$ for $l = 1, \ldots, i-1$ and thus

$$|(\delta u_i, \varphi)| \leqslant C \left[\|\varphi\|_{\mathrm{H}^1(\Omega)} + \sum_{l=1}^{i-1} (t_i - t_l)^{-\bar{\beta}} |(\delta u_l, \varphi)| \tau \right].$$

Hence, the result follows from [40, Lemma 5]. ☐

Now, the solutions on the single time steps are prolonged on the whole time frame as follows

$$U_N : [0, T_0] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \psi(t) & t \in [-s, 0], \\ u_{i-1} + (t - t_{i-1}) \delta u_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant M; \end{cases}$$

$$\overline{U}_N : [0, T_0] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \psi(t) & t \in [-s, 0], \\ u_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant M. \end{cases}$$

The $\overline{K}_N$ and $\overline{G}_N$ function can be defined in the same manner. Using these functions and the notation $\lceil t \rceil_\tau = t_i$ for $t \in (t_{i-1}, t_i]$, we rewrite (3.11) on $(0, T_0]$ as

$$\begin{aligned} &\left(\int_0^{\lceil t \rceil_\tau} \overline{K}_N(\lceil t \rceil_\tau + \tau - r) \partial_r U_N(r) \, \mathrm{d}r, \varphi \right) + \lambda (\partial_t U_N(t), \varphi) + \kappa \left(\nabla \overline{U}_N(t), \nabla \varphi \right) \\ &\qquad = \left(f\left(\overline{U}_N(t - \tau), \overline{U}_N(t - s) \right), \varphi \right) + \left(\overline{G}_N(t), \varphi \right), \quad \forall \varphi \in \mathrm{H}_0^1(\Omega). \end{aligned} \tag{3.13}$$

We show the existence of a solution on $(0, T_0)$ in the next theorem by passing to the limit $N \to \infty$ (and thus $\tau \to 0$ and hence $M \to \infty$).

Theorem 3.1 (Existence)**.** *Let the assumptions (AS1-AS4) be fulfilled and $s \leqslant T$. Then, a unique solution u exists to* (2.5) *on the time frame $[0, T_0]$ satisfying*

$$u \in \mathrm{C}\left([-s, T_0], \mathrm{L}^2(\Omega) \right) \cap \mathrm{L}^\infty \left((0, T_0), \mathrm{H}_0^1(\Omega) \right), \partial_t u \in \mathrm{L}^2 \left((0, T_0), \mathrm{L}^2(\Omega) \right).$$

Proof. Lemma 3.2 implies for all $N \geqslant N_0 > 0$ that

$$\max_{t\in[0,T_0]} \left\|\overline{U}_N(t)\right\|^2_{\mathrm{H}^1_0(\Omega)} + \int_0^{T_0} \|\partial_t U_N(t)\|^2 \,\mathrm{d}t \leqslant C.$$

As $\mathrm{H}^1_0(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$ (see [41, Theorem 6.6-3]), invoking [28, Lemma 1.3.13] leads to the existence of a function $u \in \mathrm{C}\left([0,T_0],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^\infty\left((0,T_0),\mathrm{H}^1_0(\Omega)\right)$ and a subsequence $\{U_{N_l}\}_{l\in\mathbb{N}}$ of $\{U_N\}$ such that

$$\begin{cases} U_{N_l} \to u & \text{in } \ \mathrm{C}\left([0,T_0],\mathrm{L}^2(\Omega)\right), \\ U_{N_l}(t) \rightharpoonup u(t) & \text{in } \ \mathrm{H}^1_0(\Omega), \ \ \forall t\in[0,T_0], \\ \overline{U}_{N_l}(t) \rightharpoonup u(t) & \text{in } \ \mathrm{H}^1_0(\Omega), \ \ \forall t\in[0,T_0], \\ \partial_t U_{N_l} \rightharpoonup \partial_t u & \text{in } \ \mathrm{L}^2\left((0,T_0),\mathrm{L}^2(\Omega)\right). \end{cases}$$

From Lemma 3.2, we also have that (note that $\tau = s/N_l$ here and further)

$$\int_0^{T_0} \left\|\overline{U}_{N_l}(t) - U_{N_l}(t)\right\|^2 \mathrm{d}t + \int_0^{T_0} \left\|\overline{U}_{N_l}(t-\tau) - \overline{U}_{N_l}(t)\right\|^2 \mathrm{d}t \leqslant 2\tau^2 \sum_{i=1}^{M_l} \|\delta u_i\|^2 \tau \leqslant C\tau^2, \tag{3.14}$$

so $\overline{U}_{N_l} \to u$ and $\overline{U}_{N_l}(\cdot-\tau) \to u$ in $\mathrm{L}^2\left((0,T_0),\mathrm{L}^2(\Omega)\right)$ as $l\to\infty$. Moreover, as $u(0)=\psi(0)$ and $\overline{U}_{N_l}(t)=\psi(t)$ for $t\in[-s,0]$ with $\psi\in\mathrm{C}\left([-s,0],\mathrm{L}^2(\Omega)\right)$, we can extend u continuously to $[-s,T_0]$ in $\mathrm{L}^2(\Omega)$ by defining $u(t)=\psi(t)$ for $t\in[-s,0]$. Then,

$$\int_{-s}^{T_0} \left\|\overline{U}_{N_l}(t) - u(t)\right\|^2 \mathrm{d}t = \int_0^{T_0} \left\|\overline{U}_{N_l}(t) - u(t)\right\|^2 \mathrm{d}t \to 0 \quad \text{as} \quad l\to\infty. \tag{3.15}$$

Now, we integrate (3.13) for $N=N_l$ over $t\in(0,\eta)\subset(0,T_0)$ to obtain that

$$\begin{aligned} \int_0^\eta \left(\int_0^{\lceil t\rceil_\tau} \overline{K}_{N_l}(\lceil t\rceil_\tau+\tau-r)\partial_r U_{N_l}(r)\,\mathrm{d}r,\varphi\right)\mathrm{d}t + \lambda\int_0^\eta \left(\partial_t U_{N_l}(t),\varphi\right)\mathrm{d}t + \kappa\int_0^\eta \left(\nabla\overline{U}_{N_l}(t),\nabla\varphi\right)\mathrm{d}t \\ = \int_0^\eta \left(f\left(\overline{U}_{N_l}(t-\tau),\overline{U}_{N_l}(t-s)\right),\varphi\right)\mathrm{d}t + \int_0^\eta \left(\overline{G}_{N_l}(t),\varphi\right)\mathrm{d}t, \quad \forall\varphi\in\mathrm{H}^1_0(\Omega). \end{aligned} \tag{3.16}$$

For the 1.st term in the LHS of (3.16), we can write

$$L_1 := \int_0^\eta \left(\int_0^{\lceil t\rceil_\tau} \overline{K}_{N_l}(\lceil t\rceil_\tau+\tau-r)\partial_r U_{N_l}(r)\,\mathrm{d}r,\varphi\right)\mathrm{d}t = T_1+T_2+T_3+T_4,$$

where

$$\begin{aligned} T_1 &:= \int_0^\eta \left(\int_t^{\lceil t\rceil_\tau} \overline{K}_{N_l}(\lceil t\rceil_\tau+\tau-r)\partial_r U_{N_l}(r)\,\mathrm{d}r,\varphi\right)\mathrm{d}t, \\ T_2 &:= \int_0^\eta \left(\int_0^t \overline{K}_{N_l}(\lceil t\rceil_\tau+\tau-r)\partial_r U_{N_l}(r)\,\mathrm{d}r - (k*\partial_t u_{N_l})(t),\varphi\right)\mathrm{d}t, \\ T_3 &:= \int_0^\eta \left((k*\partial_t u_{N_l})(t) - (k*\partial_t u)(t),\varphi\right)\mathrm{d}t, \\ T_4 &:= \int_0^\eta \left((k*\partial_t u)(t),\varphi\right)\mathrm{d}t. \end{aligned}$$

From Lemma 3.3, we have for $t \in (t_{i-1}, t_i]$ that

$$\left| \int_t^{\lceil t \rceil_\tau} \overline{K}_{N_l}(\lceil t \rceil_\tau + \tau - r) \left(\partial_r U_{N_l}(r), \varphi\right) \mathrm{d}r \right| \leqslant C \left\| \varphi \right\|_{\mathrm{H}^1(\Omega)} \int_t^{t_i} \overline{K}_{N_l}(t_i + \tau - r) \, \mathrm{d}r$$

$$\leqslant C \left\| \varphi \right\|_{\mathrm{H}^1(\Omega)} k(\tau)\tau \overset{l \to \infty}{\longrightarrow} 0, \quad \forall \varphi \in \mathrm{H}^1_0(\Omega).$$

Hence, $|T_1| \to 0$ as $l \to \infty$. Analogously, by the Lebesgue dominated theorem (since $\overline{K}_{N_l} \to k$ point-wise on $(0, T_0)$ as $l \to \infty$), we obtain for $t \in (t_{i-1}, t_i]$ that

$$\left| \int_0^t \left[\overline{K}_{N_l}(\lceil t \rceil_\tau + \tau - r) - k(t - r) \right] \left(\partial_r U_{N_l}(r), \varphi\right) \mathrm{d}r \right|$$

$$\leqslant C \left\| \varphi \right\|_{\mathrm{H}^1(\Omega)} \int_0^t \left| \overline{K}_{N_l}(t_i + \tau - r) - k(t - r) \right| \mathrm{d}r \overset{l \to \infty}{\longrightarrow} 0,$$

i.e. $|T_2| \to 0$ as $l \to \infty$. From (2.6) and $\partial_t U_{N_l} \rightharpoonup \partial_t u$ in $\mathrm{L}^2\left((0, T_0), \mathrm{L}^2(\Omega)\right)$, this yields $T_3 \to 0$ as $l \to \infty$. Therefore, we get that $L_1 \to T_4$ as $l \to \infty$. Next, from (3.14) and (3.15), we obtain for the first term on the RHS of (3.16) that

$$\int_0^\eta \left(f\left(\overline{U}_{N_l}(t - \tau), \overline{U}_{N_l}(t - s)\right), \varphi \right) \mathrm{d}t \overset{l \to \infty}{\longrightarrow} \int_0^\eta \left(f\left(u(t), u(t - s)\right), \varphi \right) \mathrm{d}t.$$

The limit transition of the remaining terms in (3.16) follows from $\overline{U}_{N_l}(t) \rightharpoonup u(t)$ in $\mathrm{H}^1_0(\Omega)$, $\partial_t U_{N_l} \rightharpoonup \partial_t u$ in $\mathrm{L}^2\left((0, T_0), \mathrm{L}^2(\Omega)\right)$ and $\overline{G}_{N_l} \to g$ a.e. in $\partial\Omega \times (0, T_0)$. Consequently, passing to the limit $l \to \infty$, we arrive at

$$\int_0^\eta \left((k * \partial_t u)(t), \varphi\right) \mathrm{d}t + \lambda \int_0^\eta \left(\partial_t u(t), \varphi\right) \mathrm{d}t + \int_0^\eta \kappa \left(\nabla u(t), \nabla \varphi\right) \mathrm{d}t$$

$$= \int_0^\eta \left(f\left(u(t), u(t - s)\right), \varphi \right) \mathrm{d}t + \int_0^\eta \left(g(t), \varphi\right) \mathrm{d}t.$$

Differentiating this result with respect to η gives that u solves (2.5). □

Remark 3.1. *Theorem 3.1 stays valid when the second order differential operator $L = -\kappa\Delta u$ is replaced by the more general operator*

$$L(\mathbf{x}, t)u(\mathbf{x}, t) = -\nabla \cdot \left(\boldsymbol{A}(\mathbf{x}, t)\nabla u(\mathbf{x}, t) + \boldsymbol{b}(\mathbf{x}, t)u(\mathbf{x}, t)\right) + c(\mathbf{x}, t)u(\mathbf{x}, t),$$

with

- $\boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d \times d}$ *is uniformly elliptic with ellipticity constant α, $\boldsymbol{A}^\mathsf{T} = \boldsymbol{A}$ and $\partial_t \boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d \times d}$;*
- $\boldsymbol{b} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^d$ *with $\partial_t \boldsymbol{b} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^d$ and $(\nabla \cdot \boldsymbol{b})(t) \in \mathrm{L}^\infty(\Omega)$ for all $t \in (0, T)$;*
- $c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$ *such that $c \geqslant \frac{\|\boldsymbol{b}\|^2_{\mathrm{L}^\infty\left(\overline{Q_T}\right)}}{2\alpha}$.*

We refer to [42] for more details.

Remark 3.2. *The existence and uniqueness of a solution to problem (1.1-1.2) without delay (i.e. $s = 0$) can be shown as above by discretizing the time frame $[0, T]$. The results above stay valid with T_0 replaced by T.*

4. Rothe-Galerkin spectral method

The problem is semi-discretized in time direction by Rothe method as shown in (3.11). In order to define the spatial approximations based on Legendre Galerkin spectral method, we here introduce some fundamental properties of Jacobi polynomials. Their importance in spectral methods arise from the nature of Jacobi weights, which are tied to the singular kernels of time Caputo fractional derivatives of order $0 < \beta < 1$. Denote $P_q^{\gamma,\varsigma}(x),\ \gamma,\varsigma > -1$ as the q-th order Jacobi polynomial of index γ,ς defined on $[-1,1]$. As all classical orthogonal polynomials, $\{P_q^{\gamma,\varsigma}(x)\}_{q=0}^{\mathcal{N}}$ satisfies the following three-term-recurrence relation

$$\begin{cases} P_0^{\gamma,\varsigma}(x) = 1, \\ P_1^{\gamma,\varsigma}(x) = \dfrac{1}{2}(2+\gamma+\varsigma)x + \dfrac{1}{2}(\gamma-\varsigma), \\ P_{q+1}^{\gamma,\varsigma}(x) = \left(A_q^{\gamma,\varsigma}x - B_q^{\gamma,\varsigma}\right)P_q^{\gamma,\varsigma}(x) - C_q^{\gamma,\varsigma}P_{q-1}^{\gamma,\varsigma}(x), \quad \text{if } 1 \le q \le \mathcal{N}. \end{cases}$$

The recursion coefficients are given by

$$\begin{cases} A_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+1)(2q+\gamma+\varsigma+2)}{2(q+1)(q+\gamma+\varsigma+1)}, \\ B_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+1)(\varsigma^2-\gamma^2)}{2(q+1)(q+\gamma+\varsigma+1)(2q+\gamma+\varsigma)}, \\ C_q^{\gamma,\varsigma} = \dfrac{(2q+\gamma+\varsigma+2)(q+\gamma)(q+\varsigma)}{(q+1)(q+\gamma+\varsigma+1)(2q+\gamma+\varsigma)}. \end{cases}$$

Let $\omega^{\gamma,\varsigma}(x) = (1-x)^\gamma(1+x)^\varsigma$. Then, one has

$$\int_{-1}^{1} P_q^{\gamma,\varsigma}(x)P_j^{\gamma,\varsigma}(x)\omega^{\gamma,\varsigma}(t)\,\mathrm{d}x = h_q^{\gamma,\varsigma}\delta_{q,j}, \qquad \forall q = 0,1,\ldots,\mathcal{N},$$

where $\delta_{q,j}$ is the Kronecker delta function and

$$h_q^{\gamma,\varsigma} = \frac{2^{(\gamma+\varsigma+1)}\Gamma(q+\gamma+1)\Gamma(q+\varsigma+1)}{(2q+\gamma+\varsigma+1)q!\Gamma(q+\gamma+\varsigma+1)}, \qquad \forall q = 0,1,\ldots,\mathcal{N}.$$

In particular, the Legendre polynomial is defined as $L_r(t) = P_r^{0,0}(t)$. Accordingly, we define the following function space to give an appropriate base functions such that the boundary conditions are satisfied exactly as clarified in spectral methods for space-fractional differential equations [43, 44]:

$$\mathcal{W}_0^{\mathcal{N}} = \text{span}\left\{\varphi_n(x) : n = 0,1,\ldots,\mathcal{N}-2\right\}.$$

The function φ_n is defined on $[a,b]$ by

$$\varphi_n(x) = L_n(\hat{x}) - L_{n+2}(\hat{x}) = \frac{2n+3}{2(n+1)}(1-\hat{x}^2)P_n^{1,1}(\hat{x}), \qquad \hat{x} := \frac{2x-b-a}{b-a} \in [-1,1].$$

Let us introduce the parameters

$$d = \frac{\lambda}{\tau} + b_0, \quad \hat{b}_i^\ell = -b_{i-\ell} + \frac{\lambda}{\tau}\delta_{\ell,i-1}.$$

Then, the problem (1.1) can be rewritten in the following equivalent form (using (3.10)):

$$du_i - \kappa\frac{\partial^2 u_i}{\partial x^2} = \sum_{\ell=0}^{i-1}\hat{b}_i^\ell u_\ell + f(u_{i-1}, u_{i-N}) + g_i, \qquad \forall i = 1,\ldots,M.$$

The fully discrete Rothe-Galerkin spectral scheme consists of the set of approximations $u_i^{\mathcal{N}} \in \mathcal{W}_{\mathcal{N}}^0$ satisfying the system

$$\begin{cases} d\left(u_i^{\mathcal{N}}, \varphi\right) + \kappa \left(\dfrac{\partial}{\partial x} u_i^{\mathcal{N}}, \dfrac{\partial}{\partial x} \varphi \right) \\ = \displaystyle\sum_{\ell=0}^{i-1} \hat{b}_i^{\ell} \left(u_\ell^{\mathcal{N}}, \varphi\right) + \left(\pi_{\mathcal{N}}^0 f(u_{i-1}^{\mathcal{N}}, u_{i-N}^{\mathcal{N}}), \varphi\right) + \left(\pi_{\mathcal{N}}^0 g_i, \varphi\right), & \forall \varphi \in \mathcal{W}_0^{\mathcal{N}},\ \forall i = 1, \dots, M, \\ u_i^{\mathcal{N}} = \pi_{\mathcal{N}}^{1,0} \psi(t_i, x), & -N \le i \le 0, \end{cases} \tag{4.17}$$

where $\pi_{\mathcal{N}}^0$ is a projection operator having orthogonality (OP) in the following manner

- $\pi_{\mathcal{N}}^0 : L^2(\Omega) \to \mathcal{W}_0^{\mathcal{N}}$ is OP such that if $u \in L^2(\Omega)$ then $\pi_{\mathcal{N}}^0 u \in \mathcal{W}_0^{\mathcal{N}}$ satisfies

$$(\pi_{\mathcal{N}}^0 u, \varphi) = (u, \varphi), \quad \forall \varphi \in \mathcal{W}_0^{\mathcal{N}}.$$

- $\pi_{\mathcal{N}}^{1,0} : H_0^1(\Omega) \to \mathcal{W}_0^{\mathcal{N}}$ is OP such that if $u \in H_0^1(\Omega)$ then $\pi_{\mathcal{N}}^{1,0} u \in \mathcal{W}_0^{\mathcal{N}}$ satisfies

$$(\partial_x \pi_{\mathcal{N}}^{1,0} u, \partial_x \varphi) = (\partial_x u, \partial_x \varphi), \quad \forall \varphi \in \mathcal{W}_0^{\mathcal{N}}.$$

The approximate solution is expanded as

$$u_i^{\mathcal{N}} = \sum_{\ell=0}^{\mathcal{N}-2} \hat{u}_\ell^i \varphi_\ell(x).$$

Substituting the preceding formula into (4.17) and by taking $\varphi = \varphi_j$ for each $0 \le j \le \mathcal{N} - 2$, the following matrix representation is obtained:

$$\left(d\,\bar{M} + \kappa S\right) U_i = K_{i-1} + R_{i-1} + G_i.$$

The notations in the preceding equality are given by

$$\begin{cases} s_{ij} &= \displaystyle\int_\Omega \varphi_i'(x) \varphi_j'(x)\, \mathrm{d}x,\ S = (s_{ij})_{i,j=0}^{\mathcal{N}-2}, \\ m_{ij} &= \displaystyle\int_\Omega \varphi_i(x) \varphi_j(x)\, \mathrm{d}x,\ \bar{M} = (m_{ij})_{i,j=0}^{\mathcal{N}-2}, \\ g_i^j &= \displaystyle\int_\Omega \varphi_j(x) \left(\pi_{\mathcal{N}}^0 g_i\right)(x)\, \mathrm{d}x, \\ G_i &= (g_i^0, g_i^1, \dots, g_i^{\mathcal{N}-2})^\top, \\ h_i^j &= \displaystyle\int_\Omega \varphi_j(x) \left(\pi_{\mathcal{N}}^0 f(u_{i-1}^{\mathcal{N}}, u_{i-N}^{\mathcal{N}})\right)(x)\, \mathrm{d}x, \\ R_{i-1} &= (h_{i-1}^0, h_{i-1}^1, \dots, h_{i-1}^{\mathcal{N}-2})^\top, \\ U_i &= (\hat{u}_i^0, \hat{u}_i^1, \dots, \hat{u}_i^{\mathcal{N}-2})^\top, \\ K_{i-1} &= \sum_{j=0}^{i-1} \hat{b}_i^j \bar{M} U_j. \end{cases}$$

The exact form of the stiffness and mass matrix is specified in the next lemma, see [45].

Lemma 4.1. *The stiffness matrix S is a diagonal matrix with*

$$s_{ii} = 4i + 6,\ i = 0, 1, \ldots .$$

The mass matrix $\bar{M}$ is symmetric with the nonzero elements

$$m_{ij} = m_{ji} = \begin{cases} \frac{b-a}{2j+1} + \frac{b-a}{2j+5}, & i = j, \\ \\ -\frac{b-a}{2j+5}, & i = j+2. \end{cases}$$

5. Numerical results

Finally, we investigate the order of convergence of the scheme in the following examples satisfying $T_0 = T$. We first define the convergence order in time in the L^2-norm sense as

$$\text{Order} = \frac{|\log\left(e\left(\mathcal{N}, M_1\right)/e\left(\mathcal{N}, M_2\right)\right)|}{|\log\left(M_2/M_1\right)|},$$

and the spatial approximation order (AO) by:

$$\text{AO} = \frac{\log\left(e\left(\mathcal{N}, M\right)\right)}{\log\left(\mathcal{N}\right)},$$

where $M_1 \neq M_2$ and $e = e(\mathcal{N}, M) = \max\limits_{1 \leq i \leq M} \left\| u_i^{\mathcal{N}} - u \right\|$.

Example 1. *We consider the following nonlinear delay reaction-diffusion problem*

$$\begin{aligned} &\frac{\partial^{\beta(t)} u}{\partial t^{\beta(t)}}(x,t) + \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - 2u(x,t) + \frac{u(x,t-0.1)}{1+u^2(x,t-0.1)} + g(x,t), \quad x \in (0,1), \quad t \in (0,1], \\ &u(0,t) = u(1,t) = 0, \quad t \in (0,1), \\ &u(x,t) = (1+t)^{\sigma} \sin(\pi x), \quad x \in \Omega,\ t \in [-0.1, 0], \end{aligned} \tag{5.18}$$

where $g(x,t)$ is a given function such that problem (5.18) *has the exact solution $u(x,t) = (1+t)^{\sigma}\sin(\pi x)$.*

The variable order $\beta(t)$ is given by

$$\beta(t) = \beta(T) + (\beta(0) - \beta(T)) \left(1 - \frac{t}{T} - \frac{\sin\left(2\pi\left(1 - \frac{t}{T}\right)\right)}{2\pi}\right).$$

We study the behavior of the numerical solution for the following two cases:

(I) $\beta(0) = 0.4$ and $\beta(1) = 0.6$.

(II) $\beta(0) = 0.6$ and $\beta(1) = 0.8$.

The variable order cases (I) and (II) satisfy the assumption (AS1). Tables 1 and 2 show the L^2-errors and corresponding convergence orders for $\sigma = 0.8$ and 1.8 with $\mathcal{N} = 50$. Tables 3 and 4 show the spatial convergence orders for various values of $\mathcal{N}$ and σ if $\tau = 1/1600$.

Table 1: Example 1: The errors and the order of convergence orders versus τ and σ with $\mathcal{N} = 50$ for case (I).

τ	$\sigma = 0.8$		$\sigma = 1.8$	
	Error	Order	Error	Order
1/100	1.807×10^{-3}	$--$	7.875×10^{-3}	$--$
1/200	1.212×10^{-3}	0.576	5.230×10^{-3}	0.590
1/400	8.104×10^{-4}	0.581	3.468×10^{-3}	0.592
1/800	5.403×10^{-4}	0.584	2.297×10^{-3}	0.594
1/1600	3.596×10^{-4}	0.587	1.520×10^{-3}	0.595
$\tau^{1-\beta(0)}$	$--$	0.600	$--$	0.600

Table 2: Example 1: The errors and the order of convergence orders versus τ and σ with $\mathcal{N} = 50$ for case (II).

τ	$\sigma = 0.8$		$\sigma = 1.8$	
	Error	Order	Error	Order
1/100	5.656×10^{-3}	$--$	2.378×10^{-2}	$--$
1/200	4.312×10^{-3}	0.391	1.802×10^{-2}	0.400
1/400	3.281×10^{-3}	0.394	1.365×10^{-2}	0.400
1/800	2.494×10^{-3}	0.396	1.035×10^{-2}	0.400
1/1600	1.893×10^{-3}	0.397	7.843×10^{-3}	0.400
$\tau^{1-\beta(0)}$	$--$	0.400	$--$	0.400

Table 3: Example 1: The errors and the approximation orders versus $\mathcal{N}$ and σ with $\tau = 1/1600$ for Case (I).

$\mathcal{N}$	$\sigma = 0.8$		$\sigma = 1.8$	
	Error	AO	Error	AO
5	3.498×10^{-4}	$\mathcal{N}^{-4.944}$	1.497×10^{-3}	$\mathcal{N}^{-4.041}$
10	3.596×10^{-4}	$\mathcal{N}^{-3.444}$	1.520×10^{-3}	$\mathcal{N}^{-2.817}$
20	3.596×10^{-4}	$\mathcal{N}^{-2.647}$	1.520×10^{-3}	$\mathcal{N}^{-2.165}$
30	3.596×10^{-4}	$\mathcal{N}^{-2.331}$	1.520×10^{-3}	$\mathcal{N}^{-1.907}$
40	3.596×10^{-4}	$\mathcal{N}^{-2.149}$	1.520×10^{-3}	$\mathcal{N}^{-1.758}$
50	3.596×10^{-4}	$\mathcal{N}^{-2.027}$	1.520×10^{-3}	$\mathcal{N}^{-1.658}$

Table 4: Example 1: The errors and the approximation orders versus $\mathcal{N}$ and σ with $\tau = 1/1600$ for Case (II).

$\mathcal{N}$	$\sigma = 0.8$		$\sigma = 1.8$	
	Error	AO	Error	AO
5	1.882×10^{-3}	$\mathcal{N}^{-3.898}$	7.818×10^{-3}	$\mathcal{N}^{-3.014}$
10	1.893×10^{-3}	$\mathcal{N}^{-2.722}$	7.843×10^{-3}	$\mathcal{N}^{-2.105}$
20	1.893×10^{-3}	$\mathcal{N}^{-2.092}$	7.843×10^{-3}	$\mathcal{N}^{-1.618}$
30	1.893×10^{-3}	$\mathcal{N}^{-1.843}$	7.843×10^{-3}	$\mathcal{N}^{-1.425}$
40	1.893×10^{-3}	$\mathcal{N}^{-1.699}$	7.843×10^{-3}	$\mathcal{N}^{-1.314}$
50	1.893×10^{-3}	$\mathcal{N}^{-1.602}$	7.843×10^{-3}	$\mathcal{N}^{-1.239}$

Example 2. *We consider the following nonlinear delay differential equation*

$$\begin{aligned}
&\frac{\partial^{\beta(t)}u}{\partial t^{\beta(t)}}(x,t) + \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t-0.3)(1+u(x,t-0.3)) + g(x,t), \quad x \in (0,1), \quad t \in (0,1],\\
&u(0,t) = u(1,t) = 0, \quad t \in (0,1),\\
&u(x,t) = t^2x^2(1-x)^2, \quad x \in \Omega,\, t \in [-0.3, 0],
\end{aligned} \tag{5.19}$$

where $g(x,t)$ is a given function such that problem (5.19) *has the exact solution $u(x,t) = t^2x^2(1-x)^2$.*

The variable order $\beta(t)$ is given as

(I) Linear $\beta(t)$:

$\beta(t) = \beta(T) + (\beta(0) - \beta(T))(1-t),$

$\beta(0) = 0.6, \beta(T) = 0.4.$

(II) Quadratic $\beta(t)$:

$\beta(t) = \beta(T) + (\beta(0) - \beta(T))(1-t^2),$

$\beta(0) := 0.5, \beta(T) := 0.8.$

(III) Osciliating $\beta(t)$:

$\beta(t) = \beta(T) + (\beta(0) - \beta(T))(1 - \frac{1}{2\pi}\sin(2\pi(1-t))),$

$\beta(0) = 0.6, \beta(T) := 0.8.$

Tables 5, 6 and 7 show the errors and the convergence orders in temporal and spatial directions for various values of $\mathcal{N}$, τ and different variable-order functions satisfying (AS1). We observe a temporal convergence rate of order $(1 - \beta(0))$ in Example 1 and 2, and that the accuracy of the spectral method is tied with the temporal order of convergence.

Table 5: Example 2: The errors and the order of convergence orders versus τ and $\mathcal{N}$ for case (I).

τ ($\mathcal{N} = 50$)	Error	Order	$\mathcal{N}$ ($\tau = 1/1600$)	Error	AO
1/100	9.443×10^{-4}	$--$	5	3.082×10^{-4}	$\mathcal{N}^{-5.022}$
1/200	7.129×10^{-4}	0.405	10	3.084×10^{-4}	$\mathcal{N}^{-3.510}$
1/400	5.388×10^{-4}	0.403	20	3.084×10^{-4}	$\mathcal{N}^{-2.698}$
1/800	4.075×10^{-4}	0.402	30	3.084×10^{-4}	$\mathcal{N}^{-2.376}$
1/1600	3.084×10^{-4}	0.402	40	3.084×10^{-4}	$\mathcal{N}^{-2.191}$

Table 6: Example 2: The errors and the order of convergence orders versus τ and $\mathcal{N}$ for case (II).

τ ($\mathcal{N} = 50$)	Error	Order	$\mathcal{N}$ ($\tau = 1/1600$)	Error	AO
1/100	5.810×10^{-4}	$--$	5	1.417×10^{-4}	$\mathcal{N}^{-5.505}$
1/200	4.068×10^{-4}	0.514	10	1.418×10^{-4}	$\mathcal{N}^{-3.848}$
1/400	2.857×10^{-4}	0.509	20	1.418×10^{-4}	$\mathcal{N}^{-2.95772}$
1/800	2.012×10^{-4}	0.506	30	1.418×10^{-4}	$\mathcal{N}^{-2.605}$
1/1600	1.418×10^{-4}	0.504	40	1.418×10^{-4}	$\mathcal{N}^{-2.401}$

Table 7: Example 2: The errors and the order of convergence orders versus τ and $\mathcal{N}$ for case (III).

τ ($\mathcal{N} = 50$)	Error	Order	$\mathcal{N}$ ($\tau = 1/1600$)	Error	AO
1/100	9.543×10^{-4}	$--$	5	3.104×10^{-4}	$\mathcal{N}^{-5.018}$
1/200	7.194×10^{-4}	0.407	10	3.106×10^{-4}	$\mathcal{N}^{-3.507}$
1/400	5.432×10^{-4}	0.405	20	3.106×10^{-4}	$\mathcal{N}^{-2.696}$
1/800	4.106×10^{-4}	0.403	30	3.106×10^{-4}	$\mathcal{N}^{-2.374}$
1/1600	3.106×10^{-4}	0.402	40	3.106×10^{-4}	$\mathcal{N}^{-2.189}$

6. Conclusion

This paper aimed to discuss the existence and uniqueness of a weak solution for a nonlinear delay parabolic equation with temporal variable order and a drift term. A Rothe scheme is targeted to achieve that purpose under rather weak conditions on the data. A numerical implementation of the problem based on a combined scheme is finally introduced. This scheme is based on the Galerkin Legendre spectral approximation in space and a positive definite discrete convolution approximation in time. An extension of the work to different types of delay such as variable and distributed delays are of high concern in near future works. Another direction for future research can concern the investigation of the shifted convolution quadrature developed in [46] can also be applied in this setting.

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CRediT authorship contribution statement

Karel Van Bockstal: Conceptualization, Formal analysis, Methodology, Writing-original draft. **Mahmoud A. Zaky:** Formal analysis, Writing-original draft, Validation, Software. **Ahmed S. Hendy:** Formal analysis, Writing-original draft, Validation, Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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