Approximation schemes for capacity vehicle routing problems: A survey

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***Abstract*—The Capacity Vehicle Routing Problem (CVRP) pertains to the combinatorial optimization problem of identifying the optimal route for vehicles with a capacity constraint of k to travel from the depots to customers, which minimizes the total distance traveled. The Capacitated Vehicle Routing Problem (CVRP), which is essential to modelling logistics networks, has drawn a lot of interest in the field of combinatorial optimization. It has been established that the CVRP (Capacitated Vehicle Routing Problem) with a value of k greater than or equal to three exhibits computational complexity that is classified as NP-hard. Furthermore, it has been established that the problem is APX-hard. It has been previously established that the solution is not approximatable in a metric space. Furthermore, this constitutes the principal challenge among the array of issues that confront Arora's approximation algorithm. The outstanding matter concerns the presence of a (1+ϵ) PTAS (polynomial time approximation scheme) for the capacity vehicle routing problem in Euclidean space, regardless of the vehicle's capacity. The objective of this manuscript is to furnish a thorough and all-encompassing survey of the research progressions in the domain, encompassing the evolution of the field from its inception to the most recent cutting-edge discoveries.**

***Keywords*—Approximation algorithms, capacity vehicle routing problems, polynomial-time approximation scheme, combinatorial optimization**

I. INTRODUCTION

The Capacitated Vehicle Routing Problem concerns the optimal distribution of vehicles to transport goods from depots to terminals in a resource-efficient manner. The current issue pertains to the identification of the most efficient path for a solitary depot and n clients, with the aim of minimizing the overall distance traversed. The prescribed itinerary is required to commence at the central hub and encompass all clients while complying with the limitation of the vehicle's capacity denoted as k.

CVRP (Capacitated Vehicle Routing Problem) is a discrete mathematical optimization problem that can be viewed as a broadening or expansion of the Traveling Salesman Problem (TSP). When there is only one depot available, it is possible to treat a CVRP (Capacitated Vehicle Routing Problem) without capacity limitations as a Traveling Salesman Problem (TSP).

There exist several variations of the capacity vehicle routing problem, as shown on the table 1.

The subsequent options are prospective resolutions for tackling the CVRP (Capacitated Vehicle Routing Problem).

Heuristic algorithms are commonly utilized in industrial contexts, notwithstanding their deficiency in providing a guarantee for worst-case performance. The effectiveness of these solutions can exhibit variability, and in specific cases, they may not possess the capability to ascertain the most advantageous resolution. Furthermore, it has been shown that the lack of a polynomial-time local search algorithm, such as those in linear programming, can be attributed to the NP-hardness of determining whether a given subproblem is suboptimal.

Linear programming can be employed to tackle problems of combinatorial optimization as well. Linear programming has been addressed through the development of polynomial algorithms, including the ellipsoids algorithm. Although these techniques may not be deemed pragmatic, the interior point algorithm has exhibited advantageous efficacy in practical applications. The constraints of combinatorial optimization can be relaxed by shifting to a continuous domain, resolving it via linear programming, and then transforming it into an integer variable.

However, it is important to acknowledge that the final stage of reduction is classified as NP-hard. The utilization of geometric topological characteristics in combinatorial optimization has the potential to facilitate the creation of diverse algorithms, like the branch and bound algorithm, branch and price algorithm, and cutting plane method.

However, it is essential to acknowledge that none of the techniques mentioned above can guarantee a polynomial time solution that is optimal or even approximately optimal. The objective of developing approximation algorithms is to attain an approximate solution for a specified problem in a reasonable time. The main objective of our article is to furnish a thorough exposition of the historical advancements achieved in the domain of CVRP approximation algorithms. Our focus is on analyzing the issues that have been tackled and the constraints that have been faced. It is advantageous for inexperienced researchers to have a thorough comprehension of the subject matter, even in the absence of an extensive literature review.

The early proponents of these algorithms harbored optimistic expectations regarding their effectiveness. However,

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recent discussions from Google suggest that the ITP algorithms may be comparatively less viable than other available alternatives.

Reference [1] documents the latest review article on approximation algorithms for geometric optimization problems of NP-hard, authored by Sanjeev Arora in 2002, which can be considered as an extension of the prior investigation documented in reference [2].

TABLE I. VARIATIONS OF CVRP

Variation	Definition
Unsplittable	Each customer is attended to by a single vehicle on a singular occasion.
Splittable	Multiple vehicles can serve a single customer on multiple occasions.
Unit demand	All customers demand an identical unit.
Non-uniform demand	Diverse clientele may necessitate varying quantities of merchandise.
One-depot	There exists a singular depot.
Multi-depot	There exist multiple depots.
Metric	The edges of a triangle adhere to the principle of the triangle inequality.
Euclidean	The computation of the distance function is achieved through the utilization of the Euclidean norm.

II. BEFORE THE MILLENNIUM

The survey is based on collecting data from leading academic search engines, including Google Scholar, ResearchGate, Arxiv and the articles received from the authors. The arrangement of the articles shall be determined by their individual years of publication. Occasionally, outcomes featuring analogous themes or techniques are amalgamated to facilitate a comparative analysis of their progress.

The Vehicle Routing problem was first presented by Dantzig in his academic publication entitled "The Truck Dispatching Problem" [3].

Christofides [4] and Serdyukov [5], working independently, proposed the initial approximation algorithm for the metric Travelling Salesman problem, which exhibits an approximation ratio of 3/2. This result has been the most advantageous for more than 50 years, although it has undergone slight improvements in recent years [6]. However, there remains a disparity from the hypothesized ratio of 4/3.

Haimovich and Rinnooy Kan introduced the ITP (Iterated Tour Partitioning) algorithm and the initial constant factor algorithms ($1 + \alpha$) for the Euclidean CVRP (Capacity Vehicle Routing Problem) in 1985 [7]. Here, α represents the approximation ratio of TSP (Travelling Salesman Problem).

Arora et al. (1992) [8] presented theoretical support for the notion that the travelling salesman problem in metric space cannot be approximated in $1 + \epsilon$ for any ϵ. The CVRP (Capacitated Vehicle Routing Problem) is a problem that can be regarded as a generation of the Traveling Salesman Problem (TSP). So CVRP is inapproximable as well.

Arora (1998) [9] proposed a PTAS (Polynomial Time Approximation Scheme) that addresses the Euclidean Traveling Salesman Problem (TSP) in fixed dimensions. The scheme is applicable when the capacity k is $k = \Omega(n)$, and it employs the ITP algorithm. Subsequently, Asano, Katoh and Kawashima [10] expanded the outcome to $k \leq c\log n/\log\log n$, where c represents a constant.

The segment labeled as "A sample constant approximation algorithm for CVRP" will furnish a concise constant approximation algorithm by using ITP, which provided the basis for most subsequent publications, and at present, there seems to be no feasible approach to bypassing ITP.

III. FROM 2000 TO 2020

Asano (2001) introduced a new approach to approximate the CVRP (Capacitated Vehicle Routing Problem) in a tree, resulting in an approximation ratio of $(\sqrt{41} - 1)/4 \approx 1.35078$ [11]. The matter under consideration pertains to a depot, which bears a resemblance to a tree-like configuration, and entails demands that can be splittable. In 2018, Becker expanded upon the ratio to reach a value of 4/3, which closely aligns with the established minimum threshold of 4/3 [12].

In 2003 [13], Refael and colleagues established that the problem of determining capacity ≥ 3 is NP-hard. Additionally, they demonstrated that the asymmetric Capacitated Vehicle Routing Problem (CVRP) with a minimum of three vehicles is not approximable within a polynomial factor of 2^poly, where "poly" represents any polynomial, unless P equals NP. The authors put forth a 4-constant approximation algorithm for a capacity of 3 and a 3-constant approximation algorithm for a capacity of 4 by using the technique of bipartite graph matching and transforming the problem into a minimal flow problem.

In his 2009 doctoral dissertation [14], Yuzhuang Hu presented various constant factor approximation schemes for different variations of the CVRP (Capacitated Vehicle Routing Problem). The aforementioned variations encompass the vehicle routing problem with multi-depot, cycle covering problem, and capacitated vehicle routing problem with pickups and deliveries. The thesis utilizes a range of methodologies, such as the come-back rule, matching and König's theorem, GW-algorithm, and dynamic programming.

In 2010, Das and Mathieu conducted an investigation into a QPTAS (Quasi-Polynomial Time Approximation Scheme) designed to address the CVRP (Capacitated Vehicle Routing Problem) in Euclidean space [15]. The research centered on a hypothetical situation in which a solitary depot caters to n clients, subject to a consistent capacity restriction of k, which applies uniformly across all vehicles. This study was based on Arora's research, and involved allocating extra costs to maintain the number of tour segments that have a unique profile compared to the Traveling Salesman Problem (TSP).

In their publication, Adamaszek, Czumaj, and Lingas [16] put forth a polynomial time algorithm that can be employed to determine an approximate solution to the CVRP (Capacitated Vehicle Routing Problem) in a two-dimensional space. The algorithm boasts an approximation ratio of (1+ε), provided that $q \leq 2log\delta n$ for some $\delta = \delta(\epsilon)$, and is developed based on the methodologies outlined in [15].

In their study, Khachay and Zaytseva (2015) proposed a PTAS (Polynomial Time Approximation Scheme) to address the CVRP (Capacitated Vehicle Routing Problem) in a 3-dimensional Euclidean space for q $\leq 2^{log^{\delta} n}$ for a given $\delta = \delta(\epsilon)$. [17].

In 2017, Becker, Klein, and Saulpic devised a polynomial-time approximation scheme (PTAS) for the Capacitated Vehicle Routing Problem (CVRP) in graphs that possess a bounded highway dimension through the use of metric embedding [18].

Khachay and Ogorodnikov presented an algorithmic solution in 2019 for the CVRPTW (capacitated vehicle routing problem with time windows) that accounts for non-uniform demand [19]. The present contribution constitutes the preliminary resolution to the CVRPTW (capacitated vehicle routing problem with time windows). The algorithm exhibits an approximation ratio of (1+ε) for a specified value of ε belongs to (0, 1) while maintaining a constant total customer demand D. The computational time complexity of the algorithm is expressed as O(nlogD). Under the stipulation that the maximum time satisfies $t_{max}\{p, q\} \leq 2^{log^{\delta} n}$ for a given $\delta = O(\epsilon)$, the algorithm could be categorized as a polynomial-time approximation scheme (PTAS). The results have been obtained from Adamaszek's study.

Khachay and Ogorodnikov (2019) proposed improvements to the PTAS (Polynomial Time Approximation Scheme) for the CVRPTW (capacitated vehicle routing problem with time windows) [20]. The current investigation presents the EPTAS (Efficient Polynomial Time Approximation Scheme), designed to achieve a (1+ε) proximate resolution for the Euclidean CVRPTW (capacitated vehicle routing problem with time windows) in a 2-dimensional space. The time complexity of the algorithm under consideration is $O(n^3 + exp(exp(1/\varepsilon))$, where n refers to the count of nodes, and p and q denote the fixed number ≥ 1 of time windows and capacity, respectively. Moreover, it can be categorized as a PTAS, provided that $p^3 q^4 = O(logn)$.

Haimovich, Khachay, and Ogorodnikov (2019) implemented additional enhancements to the operational efficiency of the CVRPTW (capacitated vehicle routing problem with time windows) [21]. The aim is to derive a solution that achieves a (1 + ε)-approximation for the Euclidean CVRPTW (Capacity Vehicle Routing Problem with Time Windows), where ε denotes a predetermined positive quantity. The algorithm in question is categorized as a PTAS (Polynomial Time Approximation Scheme) for the CVRPTW (capacitated vehicle routing problem with time windows) if its performance is determined by the constraint that $p^3 q^4 = O(logn)$. Moreover, it can be classified as an EPTAS for fixed values of p and q.

Becker, Klein, and Schild (2019) proposed a PTAS (Polynomial-Time Approximation Scheme) for the Bounded Capacity Vehicle Routing Problem (BCVRP) in planar graphs with a restricted capacity [22]. The authors embedded planar graphs into graphs that possess restricted treewidth, while preserving the anticipated inter-client distances with a minor deviation that is commensurate with the distances from the central depot to the clients.

Becker and Paul (2019) proposed a theoretical construct for an estimation method tailored to address the Vehicle Routing Problem (VRP) in tree structures that encompass a solitary depot and n customers [23]. The current research endeavors to introduce a PTAS with a bicriteria polynomial-time approximation of 1+ε for the CVRP in tree structures. The suggested methodology entails the division of the tree into distinct clusters, followed by the application of dynamic programming methodologies.

In their study, Khachay and Ogorodnikov (2020) introduced a QPTAS (Quasi-Polynomial Time Approximation Scheme) that addresses the capacitated vehicle routing problem in metric spaces with a predetermined doubling dimension [24]. There exists a constraint whereby the variable q is restricted to a maximum value of O(loglog(n)). The algorithms were formulated utilizing the Das-Mathieu methodology.

Tarnawski, Ola, and Végh have achieved a significant advancement in the field of the asymmetric travelling salesman problem [25]. They have introduced the first constant approximation algorithm for this problem.

IV. AFTER 2020

Khachay and Ogorodnikov (2021) introduced a PTAS (Polynomial Time Approximation Scheme) that features bounded routes in a metric space with a fixed doubling dimension for the CVRP (Capacitated Vehicle Routing Problem) [26]. In this particular context, it can be observed that the CVRP (Capacitated Vehicle Routing Problem) demonstrates a QPTAS (Quasi-Polynomial Time Approximation Scheme) when the quantity of routes does not exceed a polynomial function of the logarithm of the input size.

Friggstad et al. (2021) conducted a study wherein they improved the approximation algorithms employed in solving the CVRP (Capacitated Vehicle Routing Problem) with Unsplittable Demands [27]. The study's results indicate that the unsplittable CVRP (Capacitated Vehicle Routing Problem) can be approximated up to 3.25, while the utilization of linear programming rounding can yield an approximation of 3.194.

Article [28] documents the probabilistic analysis of Euclidean capacitated vehicle routing with unit demand and capacity k in a random setting in a metric space, as conducted by Mathieu and Zhou. The results of their study produced a lower limit of 0.915 + α with a significant level of confidence.

In the year 2022, Zhao and Xiao [29] implemented improvements to the constant approximation ratios for the k-Capacitated Vehicle Routing Problem (k-CVRP) in a metric space that has a fixed capacity k. The aforementioned enhancements were implemented utilizing the ITP algorithm. In the context of unit demand and splittable case, the ratio exhibited improvement to 1.500 when k equals 3 and 1.667 when k equals 4. In situations where splitability is not feasible, the proportion increased to 1.500 for k=3, 1.750 for k=4, and 2.157 for k=5.

The researchers Grandoni, Mathieu, and Zhou have achieved notable progress in the domain of Euclidean CVRP (Capacitated Vehicle Routing Problem) involving unsplittable demands. The researchers have developed several algorithms for optimization problems. Specifically, they have proposed a polynomial time approximation algorithm with a performance ratio of (2+ε) for the 2-D Euclidean plane. Additionally, they have presented a gap-tight approximation algorithm with a performance ratio of (1.5+ε) for the CVRP (Capacitated Vehicle Routing Problem) with arbitrary unsplittable demand in trees. They have also proposed a PTAS (polynomial time approximation scheme) for CVRP in trees with unit demand for all capacity and have extended this result to splittable tree CVRP. These findings are documented in references [30], [31], and [32].

Heine, Demleitner and Matuschke (2023) developed a four-constant bifactor algorithm to approximate the optimal placement of facilities with limited capacities [33]. The authors conducted a comprehensive analysis of the approximation, including a worst-case scenario evaluation, and assessed its practical efficacy through computational experiments that utilized minimal spanning trees and linear programming.

Jayaprakash and Salavatipour [34] introduced a QPTAS (QuasiPolynomial time Approximation Scheme) in 2023, which is applicable to graphs with limited treewidth, highway dimensions, and doubling dimensions.

Blauth, Traub, and Vygen (2023) improved the Approximation Ratio for Capacitated Vehicle Routing in their research [35]. The mathematical expression for the ratio pertaining to the CVRP (Capacitated Vehicle Routing Problem) can be denoted as α+2*(1-ε), where α has a value of 3/2, and ε is a positive number greater than 1/3000. It is noteworthy that both α and ε are positive numbers. The unit-demand and splittable CVRP (Capacitated Vehicle Routing Problem) ratio can be mathematically represented as α + 1 - ε, where α is equivalent to 3/2, and ε exceeds 1/3000. It is noteworthy that the values of both α and ε are positive. The variable α is widely regarded as the optimal approximation ratio for the Traveling Salesman Problem (TSP). The algorithms employed are based on ITP (Iterated tour partition) algorithm.

Nie and Zhou [36] demonstrated in 2023 that they were able to achieve a 1.55 approximation ratio with high probability for the CVRP (Capacitated Vehicle Routing Problem) in the Euclidean space. The problem involved unit demand and was situated in a random environment with a depot, n terminals, and a capacity of k. The research methodology utilized by the authors is based on integrating sweep heuristic and Arora's framework.

In their study, Mömke and Zhou (2023) developed an algorithm with a 1.95 approximation for the Capacity Vehicle Routing Problem (CVRP) in graphic metrics [37].

Vincent et al. presented a quasi-polynomial time approximation scheme (QPTAS) for the Capacitated Vehicle Routing Problem (CVRP) in minor-free metrics, which is applicable for all capacity values [38] [39]. This scheme is based on the metric embedding results obtained by Berker, Klein and Schild. [22].

V. A SAMPLE CONSTANT APPROXIMATION ALGORITHM FOR CVRP

A. *Christofides–Serdyukov algorithm [40]*

Let G = (V, w) be a complete graph, where V denotes the set of vertices and w denotes a weight function that assigns non-negative weights to the edges of G. In the realm of metric spaces, the triangle inequality postulates that the aggregate of the distances connecting any three vertices, denoted by w(i, j), w(j, k), and w(i, k) correspondingly, must be no less than the distance linking vertices i and k.

Following that, the algorithm can be elucidated in pseudocode in the following manner.

a) Create a subgraph of B representing the minimum spanning tree (MST) of graph G.

b) The set of vertices in T that possess an odd degree can be denoted as O. As per the Handshaking Lemma, the cardinality of the set of vertices in graph O is necessarily even.

c) Determine a perfect matching M with the lowest possible weight in the subgraph induced by the vertices belonging to set O.

d) Amalgamate the edges of the multigraphs M and T to create a connected multigraph H, where the vertices exhibit even degrees.

e) Generate an Eulerian circuit for graph H.

f) In order to acquire a Hamiltonian circuit, it is imperative to exclude duplicated vertices from the circuit that was obtained in part e).

B. *ITP (Iterated tour partitioning) algorithm [41]*

a) Determine an itinerary for the Traveling Salesman Problem encompassing all terminals by algorithm A.

b) Divide the tour into multiple segments in such a way that the cumulative demand for each segment does not exceed 1.

c) Establish a correlation between the depot and the endpoints of each segment.

Overall, if the upper bound of n/q is close to n/q, the worst-case ratio is 5/2 - 3/2q. So for CVRP, it is a 5/2–3/2 k approximation approach.

VI. TIMELINES AND ACTIVE RESEARCH TEAMS:

The subsequent tables present the timelines and active research groups pertaining to the Capacitated Vehicle Routing Problem (CVRP).

TABLE II. TIMELINES FOR CVRP

Year and Contributor representative	
For general metric	***For Euclidean metric***
1985 Hernan Haimovich	1997 Takashi Asano
1990 Kemal Altinkemer	2010 Michal Adamaszek
2006 Agustin Bompadre	2010 Claire Mathieu
2020 Michael Khachay	2015 Michael Khachay
2021 Jannis Blauth	2021 Claire Mathieu

TABLE III. ACTIVE RESEARCH TEAMS

Country	Researcher	University	Starting year
France	Claire Mathieu	CNRS	2010
	Hang Zhou	Ecole Polytechnique	2020
	Vincent Cohen-Addad	Google Research	2020
Russia	Michael Kchachay	N.N. Krasovskii Institute of Mathematics and Mechanics	2015
	Yuri Ogorodnikov		2019
Germany	Jens Vygen	University of Bonn	2019
	Jannis Blauth		2021
China	Mingyu Xiao	University of Electronic Science and Technology of China	2020
Canada	Mohammad Salavatipour	University of Alberta	2010
	Zachary Friggstad		2010
U.S.A	Philip Klein	Brown University	2017
	Amariah Becker		2017

VII. CONCLUSIONS

The development of the CVRP (Capacitated Vehicle Routing Problem) can be traced back to its initial iteration in a metric space. The following research endeavors have integrated pragmatic limitations such as temporal constraints (time windows), multiple depots, asymmetry, and unsplittable demands. The above additions have the potential to reduce the complexity of the problem at hand. Scholars have tried to figure out the complexity of the problem and come up with helpful techniques using different types of data structures and metrics.

In contemporary times, innovative mathematical methodologies have surfaced, which exhibit potential for tackling these concerns. It is imperative to acknowledge that these tools can solely provide resolutions within specific limitations. Moreover, additional research is necessary regarding this issue. Due to constraints in time, they were not included in this survey.

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