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A two-grid temporal second-order scheme for the two-dimensional nonlinear Volterra integro-differential equation with weakly singular kernel

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Abstract In this paper, a two-grid temporal second-order scheme for the two-dimensional nonlinear Volterra integro-differential equation with weakly singular kernel is proposed to reduce the computation time and improve the accuracy of the scheme developed by Xu et al. (Applied Numerical Mathematics 152 (2020) 169-184). The proposed scheme consists of three steps: First, a small nonlinear system is solved on the coarse grid using fix-point iteration. Second, the Lagrange's linear interpolation formula is used to arrive at some auxiliary values for analysis of the fine grid. Finally, a linearized Crank-Nicolson finite difference system is solved on the fine grid. Moreover, the algorithm uses a central difference approximation for the spatial derivatives. In the time direction, the time derivative and integral term are approximated by Crank-Nicolson technique and product integral rule, respectively. With the help of the discrete energy method, the stability and space-time second-order convergence of the proposed approach are obtained in L^2-norm. Finally, the numerical results agree with the theoretical analysis and verify the effectiveness of the algorithm.

Keywords Nonlinear fractional evolution equation · time two-grid algorithm, accurate second order · stability and convergence · numerical experiments

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1 Introduction

In this paper, we consider the following two-dimensional nonlinear Volterra integro-differential equation with weakly singular kernel

$$u_t - \mu \Delta u - I^{(\alpha)} \Delta u = f(x,y,t) + g(u), \quad (x,y,t) \in \Omega \times (0,T], \tag{1.1}$$

with the initial-boundary conditions

$$\begin{aligned} &u(x,y,0) = \psi(x,y), \qquad (x,y) \in \bar{\Omega}, \\ &u(x,y,t) = 0, \qquad (x,y,t) \in \partial\Omega \times (0,T], \end{aligned} \tag{1.2}$$

where $\Omega = (0,L_x) \times (0,L_y)$ with the boundary $\partial\Omega$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator and $u_t = \partial u/\partial t$. In addition, $\alpha \in (0,1)$, $\mu \in [0,\infty)$ and $T \in (0,\infty)$ are given constants. $f(x,y,t)$ and $\psi(x,y)$ are given functions. The nonlinear term $g(u) \in C^2(\mathbf{R}) \cap L^1(0,T]$ satisfies the Lipschitz condition $|g(u_1) - g(u_2)| \le \bar{C}|u_1 - u_2|$. Furthermore, The integral term $I^{(\alpha)}\Delta u(x,y,t)$ is defined [1, 2] as follows

$$I^{(\alpha)} \Delta u(x,y,t) = \int_0^t \rho_\alpha(t-s) \Delta u(x,y,s) ds, \quad \rho_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0. \tag{1.3}$$

In addition, throughout the article, we assume that problem (1.1)-(1.2) has a unique solution such that the following regularity assumptions [3]:

(A1) u_t, u_{tyy}, u_{txx}, u_{xxxx} and u_{yyyy} are continuous in $\bar{\Omega} \times [0,T]$;
(A2) u_{tt}, u_{ttt}, u_{ttxx} and u_{ttyy} are continuous in $\bar{\Omega} \times (0,T]$, and there exists a positive constant $\bar{C}$ satisfying for $(x,y,t) \in \bar{\Omega} \times (0,T]$ that

$$\begin{aligned} &|u_{tt}(x,y,t)| \le \bar{C} t^{\alpha-1}, \qquad |u_{ttt}(x,y,t)| \le \bar{C} t^{\alpha-2}, \\ &|u_{tt\kappa\kappa}(x,y,t)| \le \bar{C} t^{\alpha-1} (\kappa = x, y). \end{aligned}$$

Such integro-differential equations with Riemann-Liouville integral operators appear frequently in various mathematical and physical models. Problem (1.1)-(1.2) is a commonly used model for studying physical phenomena related to elastic forces. This model is mainly used in the problems of heat conduction, viscoelasticity and population dynamics of materials with memory [4–6]. In viscoelastic problems, the parameter μ in this model represents the Newtonian contribution to viscosity, and the integral term represents the viscosity part of the equation.

In recent years, high-precision computational methods for 2D partial integro-differential equations with weakly singular kernel, such as equation (1.1), have been developed. The linear case of (1.1)-(1.2) has been deeply studied in the literature, e.g., see [7–12]. Furthermore, some numerical studies on the nonlinear case were introduced. Mustapha et al. [3] applied the Crank-Nicolson scheme under graded meshes to solve semilinear integro-differential equation with weakly singular kernel. Dehghan et al. [13] proposed a spectral element technique for solving nonlinear fractional evolution equation. In addition, some numerical methods for nonlinear partial differential equations have been proposed, and we can refer to the work in [14–16].

However, when solving 2D nonlinear problems, the resulting large systems of nonlinear equations require a large computational cost as the grid is continuously subdivided. In order to save the computational cost of nonlinear problems, a spatial two-grid finite element technique was proposed by Xu [17, 18]. Inspired by Xu's ideas, the two-grid method began to be intensively studied and applied to the solution of nonlinear parabolic equations. Dawson and Wheeler et al. [19] proposed a spatial two-grid finite difference method in solving nonlinear parabolic equations and analyzed the convergence of the method on coarse and fine grid. For solving the nonlinear time-fractional parabolic equation, Li et al. [20] obtained the numerical solution of this equation using the spatial two-grid block-centered finite difference scheme. For more work regarding the spatial two-grid methods, see [21–23]. In addition, some scholars, inspired by the spatial two-grid method, started to consider using the two-grid method to solve the nonlinear equations in the time direction. Liu et al. [24] proposed a new time two-grid finite element algorithm in order to solve the time fractional water wave model, and illustrated through numerical experiments that it has higher computational efficiency than the standard finite element method. In [25], a time two-grid backward Euler finite difference method is constructed to solve problem (1.1)-(1.2). However, the time convergence order of the above methods cannot reach the exact second order.

In this paper, we design an efficient temporal two-grid Crank-Nicolson (TTGCN) finite difference method for solving problem (1.1)-(1.2). In this approach, the time and space derivatives are approximated using the Crank-Nicolson technique and the central difference formula, respectively, and the Riemann-Liouville integral term is approximated by the product integration rule designed in [26]. Then, this algorithm is divided into three steps: First, a small nonlinear system is solved on a coarse grid. Second, based on the solution of the first step, the values of each node are obtained by linearization technique as the auxiliary approximate solution. Finally, we approximate the nonlinear term $g(U^n)$ by a Taylor expansion and solve the linear system on a fine grid. Furthermore, under the regularity assumptions **(A1)** and **(A2)**, we prove that this algorithm is stability and the convergence of order $O(\tau_C^4 + \tau_F^2 + h_1^2 + h_2^2)$, where τ_C and τ_F are the time steps of the coarse and fine grid, respectively. Also, the linearization technique is used on the fine grid, so the TTGCN finite difference algorithm has the advantage of both ensuring accuracy and improving computational efficiency. In addition, the numerical results in this paper show that the TTGCN finite difference algorithm is more efficient than the standard Crank-Nicolson (SCN) finite difference method without loss of accuracy. Meanwhile, our algorithm can achieve second-order convergence in time compared to the method in [25].

The remainder of this paper is structured as follows. In Section 2, we give some notations and useful lemmas. Then, the TTGCN finite difference scheme is established in Section 3. In Section 4, the stability and convergence of the TTGCN finite difference method are analyzed by the energy method. Moreover, some numerical results are given in Section 5.

The generic positive constant $\bar{C}$ is independent of the temporal step size and the spatial step size, moreover, it is not necessarily same in different situations.

2 Preliminaries

In this section, we shall provide some useful notations and lemmas which will be used for the forthcoming work. First, for a positive integer $\mathcal{N}$, we define the time-step size on the fine grid as $\tau = \tau_F = T/\mathcal{N}$ with $t_n = n\tau_F (0 \le n \le \mathcal{N})$. Similarly, for the coarse grid, the time-step size is $\tau_C = T/N$, $t_s = s\tau_C (0 \le s \le N)$ for positive integer N, where $N = \mathcal{N}/k$, $k \ge 2$ and $k \in \mathbb{Z}^+$. For any grid function $\varphi^n (1 \le n \le \mathcal{N})$ on $(0,T]$, we define

$$\delta_t \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\tau}, \qquad \varphi^{n-\frac{1}{2}} = \frac{\varphi^n + \varphi^{n-1}}{2}.$$

Then, we define the grid functions as following

$$u^n = u(x,y,t_n), \quad f^n = f(x,y,t_n), \quad 0 \le n \le \mathcal{N}.$$

We integrate the equation (1.1) from $t = t_{n-1}$ to t_n and then multiply by $\frac{1}{\tau}$, we obtain

$$\delta_t u^n - \frac{\mu}{\tau}\int_{t_{n-1}}^{t_n} \Delta u(\cdot,t)dt - \frac{1}{\tau}\int_{t_{n-1}}^{t_n} I^{(\alpha)}\Delta u(\cdot,t)dt = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(\cdot,t)dt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} g(u(\cdot,t))dt. \tag{2.1}$$

To approximate the integral term of equation (2.1), from [12, 26], we obtain the quadrature approximation with the uniform time step

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \Delta u(\cdot,t)dt = \begin{cases} \Delta u^1 + (R1)^1, \\ \Delta u^{n-\frac{1}{2}} + (R1)^n, \quad 2 \le n \le \mathcal{N}, \end{cases} \tag{2.2}$$

and

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} I^{(\alpha)}\Delta u(\cdot,t)dt = \begin{cases} \frac{1}{\tau}\int_{t_0}^{t_1}\int_{t_0}^{t_1} \rho_\alpha(t-s)\Delta u^1 ds dt + (R2)^1, \\ \\ \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_0}^{t_1} \rho_\alpha(t-s)\Delta u^1 ds dt \\ + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\sum_{m=2}^{n}\int_{t_{m-1}}^{\min\{t,t_m\}} \rho_\alpha(t-s)\Delta u^{m-\frac{1}{2}} ds dt + (R2)^n, \quad 2 \le n \le \mathcal{N}, \end{cases} \tag{2.3}$$

where $(R1)^n$ and $(R2)^n$ are the local truncation errors.

For any grid function $\varphi^n (1 \le n \le \mathcal{N})$, we define the following two operators

$$\begin{aligned} \mathfrak{L}^n_{1,\tau}(\varphi^n) &= \begin{cases} \varphi^1, \\ \varphi^{n-\frac{1}{2}}, \quad n \ge 2, \end{cases} \\ \mathfrak{L}^n_{2,\tau}(\varphi^n) &= \begin{cases} \mathfrak{w}_{1,1}\varphi^1, \\ \mathfrak{w}_{n,1}\varphi^1 + \sum_{m=2}^{n} \mathfrak{w}_{n,m}\varphi^{m-\frac{1}{2}}, \quad n \ge 2, \end{cases} \end{aligned} \tag{2.4}$$

where

$$\mathfrak{w}_{n,m} = \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_{m-1}}^{\min\{t,t_m\}} \rho_\alpha(t-s)dsdt. \tag{2.5}$$

Therefore, for $n \geq 2$ and $1 \leq m \leq n-1$, we can get that

$$\begin{aligned}\mathfrak{w}_{n,m} &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_{m-1}}^{t_m} \rho_\alpha(t-s)dsdt \\ &= \frac{\left[(t_n - t_{m-1})^{\alpha+1} - (t_n - t_m)^{\alpha+1}\right] - \left[(t_{n-1} - t_{m-1})^{\alpha+1} - (t_{n-1} - t_m)^{\alpha+1}\right]}{\tau\Gamma(2+\alpha)}\end{aligned} \tag{2.6}$$

and

$$\mathfrak{w}_{n,n} = \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \rho_\alpha(t-s)dsdt = \frac{\tau^\alpha}{\Gamma(2+\alpha)}, \qquad 1 \leq n \leq \mathcal{N}. \tag{2.7}$$

Then the equations (2.2) and (2.3) can be rewritten as follows

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \Delta u(\cdot,t)dt = \mathfrak{L}_{1,\tau}^n(\Delta u^n) + (R1)^n, \qquad 1 \leq n \leq \mathcal{N}, \tag{2.8}$$

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} I^{(\alpha)}\Delta u(\cdot,t)dt = \mathfrak{L}_{2,\tau}^n(\Delta u^n) + (R2)^n, \qquad 1 \leq n \leq \mathcal{N}. \tag{2.9}$$

For the spatial approximation, defining the space-step size $h_1 = L_x/M_x$, $h_2 = L_y/M_y$, $h = \max\{h_1, h_2\}$ for two positive integers M_x and M_y, we arrive at $x_i = ih_1$ and $y_j = jh_2$. Denote $\bar{\Omega}_h = \{(x_i, y_j)|0 \leq i \leq M_x, 0 \leq j \leq M_y\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \Omega_h \cap \partial\Omega$. Let the grid function $Z_h = \{z_{ij}|0 \leq i \leq M_x, 0 \leq j \leq M_y\}$ on Ω_h, then we denote the following notations

$$\begin{aligned}\delta_x z_{i+\frac{1}{2},j} &= \frac{z_{i+1,j} - z_{ij}}{h_1}, \qquad \delta_x^2 z_{ij} = \frac{\delta_x z_{i+\frac{1}{2},j} - \delta_x z_{i-\frac{1}{2},j}}{h_1}, \\ \delta_y z_{i,j+\frac{1}{2}} &= \frac{z_{i,j+1} - z_{ij}}{h_2}, \qquad \delta_y^2 z_{ij} = \frac{\delta_y z_{i,j+\frac{1}{2}} - \delta_y z_{i,j-\frac{1}{2}}}{h_2}.\end{aligned}$$

Also, the discrete Laplace operator is defined by $\Delta_h = \delta_x^2 + \delta_y^2$.

Then, for any grid function $z, v \in \Omega_h$, some norm and inner product are defined as follows

$$(z,v) = h_1h_2\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} z_{ij}v_{ij}, \quad \|z\| = \sqrt{(z,z)}, \quad \|z\|_\infty = \max_{\substack{1\leq i\leq M_x-1,\\ 1\leq j\leq M_y-1}} |z_{ij}|,$$

$$\|\delta_x z\| = \sqrt{h_1h_2\sum_{i=0}^{M_x-1}\sum_{j=1}^{M_y-1}(\delta_x z_{i+\frac{1}{2},j})^2}, \quad \|\delta_y z\| = \sqrt{h_1h_2\sum_{i=1}^{M_x-1}\sum_{j=0}^{M_y-1}(\delta_x z_{i,j+\frac{1}{2}})^2}.$$

Next, some auxiliary lemmas will be given.

Lemma 1 *[27] Suppose $g(u(\cdot,t)) \in C^2(\mathbf{R}) \cap L^1(0,T)$, then it holds that*

$$\left| \int_{t_{n-1}}^{t_n} g(u(\cdot,t))dt - \frac{t_n - t_{n-1}}{2}\Big[g(u(\cdot,t_n)) + g(u(\cdot,t_{n-1}))\Big] \right| \le \frac{(t_n - t_{n-1})^3}{12}\|g''\|_\infty,$$

where $\|g''\|_\infty = \sup_{\xi \in (t_{n-1},t_n)} |g''(u(\cdot,\xi))| < \infty.$

According to Taylor series expansion with integral remainder term, we can obtain the following lemma.

Lemma 2 *[28] Assume $v(x,y) \in C^{4,4}_{x,y}([0,L_x] \times [0,L_y])$, then it satisfies that*

$$\frac{\partial^2 v}{\partial x^2}(x_i,y_j) = \delta_x^2 v(x_i,y_j) - \frac{h_1^2}{6}\int_0^1 \left[\frac{\partial^4 v}{\partial x^4}(x_i + wh_1, y_j) + \frac{\partial^4 v}{\partial x^4}(x_i - wh_1, y_j)\right](1-w)^3 dw,$$

$$\frac{\partial^2 v}{\partial y^2}(x_i,y_j) = \delta_y^2 v(x_i,y_j) - \frac{h_2^2}{6}\int_0^1 \left[\frac{\partial^4 v}{\partial y^4}(x_i, y_j + wh_2) + \frac{\partial^4 v}{\partial y^4}(x_i, y_j - wh_2)\right](1-w)^3 dw.$$

For further analysis, we present the following important lemmas.

Lemma 3 *Assume that the solution u of the problem (1.1)-(1.2) satisfies the regularity assumptions **(A1)** and **(A2)**, then we obtain that*

$$\tau \sum_{m=1}^{n} \|(R1)^m\| \le \bar{C}\tau^2, \qquad 1 \le n \le \mathcal{N}.$$

Proof Through simple calculation, we yield

$$(R1)^1 = \frac{1}{\tau}\int_{t_0}^{t_1} \Big[\Delta u(\cdot,t) - \Delta u^1\Big]\, dt, \tag{2.10}$$

$$(R1)^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \left[\Delta u(\cdot,t) - \left(\frac{t_n - t}{\tau}\Delta u^{n-1} + \frac{t - t_{n-1}}{\tau}\Delta u^n\right)\right] dt. \tag{2.11}$$

Using Taylor expansion with integral remainder term, we have

$$\Delta u(\cdot,t) - \Delta u^1 = \Delta u(\cdot,t) - \Delta u(\cdot,t_1) = -\int_t^{t_1} \Delta u_s(\cdot,s)ds, \quad t_0 \le t \le t_1, \tag{2.12}$$

therefore

$$(R1)^1 = -\frac{1}{\tau}\int_{t_0}^{t_1}\int_t^{t_1} \Delta u_s(\cdot,s)dsdt = -\frac{1}{\tau}\int_{t_0}^{t_1}\int_{t_0}^{s} \Delta u_s(\cdot,s)dtds = -\frac{1}{\tau}\int_{t_0}^{t_1} s\Delta u_s(\cdot,s)ds. \tag{2.13}$$

The continuity of $u_{t\kappa\kappa}(x,y,t)(\kappa = x,y)$ in $\bar{\Omega} \times [0,T]$ implies

$$\tau\|(R1)^1\| \le \bar{C}\tau^2. \tag{2.14}$$

Similarly, from Taylor expansion with integral remainder term, we obtain

$$\left|\Delta u(\cdot,t) - \frac{t_n - t}{\tau}\Delta u^{n-1} - \frac{t-t_{n-1}}{\tau}\Delta u^n\right| \le 2\tau \int\limits_{t_{n-1}}^{t_n} |\Delta u_{ss}(\cdot,s)|ds, \quad n \ge 2, \tag{2.15}$$

then

$$|(R1)^n| \le 2\int\limits_{t_{n-1}}^{t_n}\int\limits_{t_{n-1}}^{t_n} |\Delta u_{ss}(\cdot,s)|\,dsdt \le \bar{C}\tau \int\limits_{t_{n-1}}^{t_n} s^{\alpha-1}ds = \bar{C}\tau(t_n^\alpha - t_{n-1}^\alpha). \tag{2.16}$$

This proves

$$\tau\sum_{m=2}^{n} \|(R1)^m\| \le \bar{C}\tau^2(t_n^\alpha - t_1^\alpha) \le \bar{C}\tau^2. \tag{2.17}$$

The proof is completed.

Lemma 4 *Suppose that the solution u of the problem (1.1)-(1.2) satisfies the regularity assumptions **(A1)** and **(A2)**. Then we can obtain the following*

$$\tau\sum_{m=1}^{n} \|(R2)^m\| \le \bar{C}\tau^2, \qquad 1 \le n \le \mathcal{N}. \tag{2.18}$$

Proof See the case ($\gamma = 1$) in [26], or Lemma 2.2 in [12].

Lemma 5 *[28] For any grid function $v, w \in Z_h$, then it holds as follows*

$$-h_1h_2\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}(\delta_x^2 v_{ij})w_{ij} = h_1h_2\sum_{i=0}^{M_x-1}\sum_{j=1}^{M_y-1}(\delta_x v_{i+\frac{1}{2},j})(\delta_x w_{i+\frac{1}{2},j}),$$
$$-h_1h_2\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}(\delta_y^2 v_{ij})w_{ij} = h_1h_2\sum_{i=1}^{M_x-1}\sum_{j=0}^{M_y-1}(\delta_y v_{i,j+\frac{1}{2}})(\delta_y w_{i,j+\frac{1}{2}}).$$

Lemma 6 *[2, 26] For any grid function $v^n (1 \le n \le \mathcal{N})$, it holds that*

$$\left(\nabla_h v^1, \mathfrak{L}_{2,\tau}^1(\nabla_h v^n)\right) + \sum_{n=2}^{\mathcal{N}}\left(\nabla_h v^{n-\frac{1}{2}}, \mathfrak{L}_{2,\tau}^n(\nabla_h v^n)\right) \ge 0, \tag{2.19}$$

where $\mathfrak{L}_{2,\tau}^n$ is presented via (2.4) and the operator $\nabla_h = \delta_x + \delta_y$.

Lemma 7 *[2] For $\mathcal{N} \ge 1$ and $v^n \in Z_h$, we have*

$$\tau\left(v^1, \delta_t v^1\right) + \tau\sum_{n=2}^{\mathcal{N}}\left(v^{n-\frac{1}{2}}, \delta_t v^n\right) \ge \frac{1}{2}\left(\left\|v^{\mathcal{N}}\right\|^2 - \left\|v^0\right\|^2\right). \tag{2.20}$$

Lemma 8 *[29] (Discrete Grönwall's inequality) If $\{\mathcal{Q}_m\}$ is a non-negative real sequence and satisfies*

$$\mathcal{Q}_m \le \tilde{\gamma}_m + \sum_{n=0}^{m-1}\tilde{\beta}_n\,\mathcal{Q}_n, \qquad m \ge 1,$$

where $\{\tilde{\gamma}_m\}$ is a non-negative and non-descending sequence, $\tilde{\beta}_n \ge 0$, then, we obtain

$$\mathcal{Q}_m \le \tilde{\gamma}_m\,\exp\left(\sum_{n=0}^{m-1}\tilde{\beta}_n\right), \qquad m \ge 1.$$

3 Establishment of the two-grid difference scheme

In the following, we first establish the SCN finite difference method for nonlinear problem (1.1)-(1.2).

Applying the quadrature approximations (2.2)-(2.3) and Lemmas 1-2, then (2.1) become

$$\begin{aligned}\delta_t u_{ij}^1 - \mu \Delta_h u_{ij}^1 - \mathfrak{w}_{1,1}\Delta_h u_{ij}^1 = b_{ij}^1 + \frac{g(u_{ij}^1)+g(u_{ij}^0)}{2}\\ + (R1)_{ij}^1 + (R2)_{ij}^1 + (R3)_{ij}^1 + (R4)_{ij}^1, \qquad (x_i,y_j)\in\Omega_h,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\delta_t u_{ij}^n - \mu\Delta_h u_{ij}^{n-\frac{1}{2}} - \mathfrak{w}_{n,1}\Delta_h u_{ij}^1 - \sum_{m=2}^{n}\mathfrak{w}_{n,m}\Delta_h u_{ij}^{m-\frac{1}{2}} = b_{ij}^n + \frac{g(u_{ij}^n)+g(u_{ij}^{n-1})}{2}\\ + (R1)_{ij}^n + (R2)_{ij}^n + (R3)_{ij}^n + (R4)_{ij}^n, \qquad (x_i,y_j)\in\Omega_h, \quad 2\le n\le\mathcal{N},\end{aligned} \tag{3.2}$$

$$u_{ij}^n = 0, \qquad (x_i,y_j)\in\partial\Omega_h, \qquad 1\le n\le\mathcal{N}, \tag{3.3}$$

$$u_{ij}^0 = \psi(x_i,y_j), \qquad (x_i.y_j)\in\Omega_h, \tag{3.4}$$

where

$$b_{ij}^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(x_i,y_j,t)dt,$$

$$(R3)_{ij}^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} g(u(x_i,y_j,t))dt - \frac{g(u_{ij}^n)+g(u_{ij}^{n-1})}{2} = \mathcal{O}(\tau^2),$$

$$(R4)_{ij}^n = \mathfrak{L}_{1,\tau}^n(\Delta u_{ij}^n - \Delta_h u_{ij}^n) + \mathfrak{L}_{2,\tau}^n(\Delta u_{ij}^n - \Delta_h u_{ij}^n) = \mathcal{O}(h_1^2+h_2^2).$$

Omitting the truncation errors $(Rs)_{ij}^n (s=1,2,3,4)$, $1\le n\le\mathcal{N}$, and replacing u_{ij}^n with U_{ij}^n, we obtain the following SCN finite difference scheme

$$\delta_t U_{ij}^1 - \mu\Delta_h U_{ij}^1 - \mathfrak{w}_{1,1}\Delta_h U_{ij}^1 = b_{ij}^1 + \frac{g(U_{ij}^1)+g(U_{ij}^0)}{2}, \qquad (x_i,y_j)\in\Omega_h, \tag{3.5}$$

$$\begin{aligned}\delta_t U_{ij}^n - \mu\Delta_h U_{ij}^{n-\frac{1}{2}} - \mathfrak{w}_{n,1}\Delta_h U_{ij}^1 - \sum_{m=2}^{n}\mathfrak{w}_{n,m}\Delta_h U_{ij}^{m-\frac{1}{2}} = b_{ij}^n + \frac{g(U_{ij}^n)+g(U_{ij}^{n-1})}{2},\\ (x_i,y_j)\in\Omega_h, \qquad 2\le n\le\mathcal{N},\end{aligned} \tag{3.6}$$

$$U_{ij}^n = 0, \qquad (x_i,y_j)\in\partial\Omega_h, \qquad 1\le n\le\mathcal{N}, \tag{3.7}$$

$$U_{ij}^0 = \psi(x_i,y_j), \qquad (x_i,y_j)\in\Omega_h. \tag{3.8}$$

In order to solve (3.5)-(3.8) efficiently, we develop the following TTGCN finite difference method, which is divided into three steps.

Step I. On the coarse grid, we only calculate ks-th level, $0 \le s \le N$. Similar to the establishment of equations (3.5)-(3.6), the discrete scheme on the coarse grid is constructed as follows

$$\delta_t(U_C)_{ij}^{k} - \mu\Delta_h(U_C)_{ij}^{k} - \mathfrak{w}_{1,1}\Delta_h(U_C)_{ij}^{k} = b_{ij}^{k} + \frac{g((U_C)_{ij}^{k}) + g((U_C)_{ij}^{0})}{2}, \quad (x_i, y_j) \in \Omega_h, \tag{3.9}$$

$$\begin{aligned}&\delta_t(U_C)_{ij}^{sk} - \mu\Delta_h(U_C)_{ij}^{(s-\frac{1}{2})k} - \mathfrak{w}_{s,1}\Delta_h(U_C)_{ij}^{k} - \sum_{p=2}^{s}\mathfrak{w}_{s,p}\Delta_h(U_C)_{ij}^{(p-\frac{1}{2})k}\\ &\quad = b_{ij}^{sk} + \frac{g((U_C)_{ij}^{sk}) + g((U_C)_{ij}^{(s-1)k})}{2}, \qquad (x_i, y_j) \in \Omega_h, \qquad 2 \le s \le N.\end{aligned} \tag{3.10}$$

Step II. Then, based on the solution $(U_C)_{ij}^{sk}$ obtained in the Step I, applying Lagrange linear interpolation to calculate $(U_C)_{ij}^{(s-1)k+q}$ by points $(t_{(s-1)k}, (U_C)_{ij}^{(s-1)k})$ and $(t_{sk}, (U_C)_{ij}^{sk})$ direction on the coarse grid, with $1 \le q \le k-1$, we have

$$\begin{aligned}&\mathcal{L}_{U_C}(t_{(s-1)k+q}) = U_C^{(s-1)k+q}\\ &= \frac{t_{(s-1)k+q} - t_{sk}}{t_{(s-1)k} - t_{sk}} U_C^{(s-1)k} + \frac{t_{(s-1)k+q} - t_{(s-1)k}}{t_{sk} - t_{(s-1)k}} U_C^{sk}\\ &= (1 - \frac{q}{k}) U_C^{(s-1)k} + \frac{q}{k} U_C^{sk}, \quad 1 \le s \le N, \quad 1 \le q \le k-1.\end{aligned} \tag{3.11}$$

Step III. Finally, according to $(U_C)_{ij}^{n}$ obtained in the Step II, the linear Crank-Nicolson finite difference scheme on a time fine grid is obtained by

$$\begin{aligned}&\delta_t(U_F)_{ij}^{1} - \mu\Delta_h(U_F)_{ij}^{1} - \mathfrak{w}_{1,1}\Delta_h(U_F)_{ij}^{1}\\ &= b_{ij}^{1} + \frac{1}{2}g((U_F)_{ij}^{0}) + \frac{1}{2}\Big[g((U_C)_{ij}^{1}) + g'((U_C)_{ij}^{1})\Big((U_F)_{ij}^{1} - (U_C)_{ij}^{1}\Big)\Big],\\ &(x_i, y_j) \in \Omega_h,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\delta_t(U_F)_{ij}^{n} - \mu\Delta_h(U_F)_{ij}^{n-\frac{1}{2}} - \mathfrak{w}_{n,1}\Delta_h(U_F)_{ij}^{1} - \sum_{p=2}^{n}\mathfrak{w}_{n,p}\Delta_h(U_F)_{ij}^{p-\frac{1}{2}}\\ &= b_{ij}^{n} + \frac{1}{2}g((U_F)_{ij}^{n-1}) + \frac{1}{2}\Big[g((U_C)_{ij}^{n}) + g'((U_C)_{ij}^{n})\left((U_F)_{ij}^{n} - (U_C)_{ij}^{n}\right)\Big],\\ &\quad (x_i, y_j) \in \Omega_h, \qquad 2 \le n \le \mathcal{N}.\end{aligned} \tag{3.13}$$

4 Analysis of the two-grid difference scheme

Next, based on the TTGCN finite difference scheme (3.9)-(3.13), we will analyze the stability and convergence of the scheme under the regularity assumption **(A1)** and **(A2)**.

4.1 Stability

We use the energy method to establish the stability of the TTGCN finite difference scheme. First, consider the case on the coarse grid.

Theorem 1 *The fully discrete scheme* (3.9)-(3.11) *on the coarse grid is stable.*

Proof Let $(\tilde{U}_C)_{ij}^{sk}$ be the approximation solution of (3.9)-(3.10). Thus, we get

$$\delta_t(\tilde{U}_C)_{ij}^k - \mu\Delta_h(\tilde{U}_C)_{ij}^k - \mathfrak{w}_{1,1}\Delta_h(\tilde{U}_C)_{ij}^k = b_{ij}^k + \frac{g((\tilde{U}_C)_{ij}^k) + g((\tilde{U}_C)_{ij}^0)}{2}, \quad (x_i, y_j) \in \Omega_h, \tag{4.1}$$

$$\delta_t(\tilde{U}_C)_{ij}^{sk} - \mu\Delta_h(\tilde{U}_C)_{ij}^{(s-\frac{1}{2})k} - \mathfrak{w}_{s,1}\Delta_h(\tilde{U}_C)_{ij}^k - \sum_{p=2}^{s}\mathfrak{w}_{s,p}\Delta_h(\tilde{U}_C)_{ij}^{(p-\frac{1}{2})k} = b_{ij}^{sk} + \frac{g((\tilde{U}_C)_{ij}^{sk}) + g((\tilde{U}_C)_{ij}^{(s-1)k})}{2}, \quad (x_i, y_j) \in \Omega_h, \quad 2 \le s \le N. \tag{4.2}$$

Subtracting (4.1)-(4.2) from (3.9)-(3.10) and defining $\varepsilon_C = (U_C)_{ij}^{sk} - (\tilde{U}_C)_{ij}^{sk}$, we get

$$\delta_t(\varepsilon_C)_{ij}^k - \mu\Delta_h(\varepsilon_C)_{ij}^k - \mathfrak{w}_{1,1}\Delta_h(\varepsilon_C)_{ij}^k = \frac{1}{2}\left[g((U_C)_{ij}^k) - g((\tilde{U}_C)_{ij}^k)\right] + \frac{1}{2}\left[g((U_C)_{ij}^0) - g((\tilde{U}_C)_{ij}^0)\right], \quad (x_i, y_j) \in \Omega_h, \tag{4.3}$$

$$\delta_t(\varepsilon_C)_{ij}^{sk} - \mu\Delta_h(\varepsilon_C)_{ij}^{(s-\frac{1}{2})k} - \mathfrak{w}_{s,1}\Delta_h(\varepsilon_C)_{ij}^k - \sum_{p=2}^{s}\mathfrak{w}_{s,p}\Delta_h(\varepsilon_C)_{ij}^{(p-\frac{1}{2})k} = \frac{1}{2}\left[g((U_C)_{ij}^{sk}) - g((\tilde{U}_C)_{ij}^{sk})\right] + \frac{1}{2}\left[g((U_C)_{ij}^{(s-1)k}) - g((\tilde{U}_C)_{ij}^{(s-1)k})\right], \quad (x_i, y_j) \in \Omega_h, \qquad 2 \le s \le N. \tag{4.4}$$

We will prove this theorem in two steps as follows:

(I) Taking inner product of both sides of (4.3) with ε_C^k and multiplying it by τ_C, we yield

$$\tau_C\left(\delta_t\varepsilon_C^k, \varepsilon_C^k\right) - \tau_C\mu\left(\Delta_h\varepsilon_C^k, \varepsilon_C^k\right) - \tau_C\mathfrak{w}_{1,1}\left(\Delta_h\varepsilon_C^k, \varepsilon_C^k\right) = \frac{\tau_C}{2}\left(g(U_C^k) - g(\tilde{U}_C^k), \varepsilon_C^k\right) + \frac{\tau_C}{2}\left(g(U_C^0) - g(\tilde{U}_C^0), \varepsilon_C^k\right), \quad (x_i, y_j) \in \Omega_h. \tag{4.5}$$

For (4.4), taking the inner product of both sides with $\varepsilon_C^{(s-\frac{1}{2})k}$, multiplying it by τ_C, and summing for s from 2 to N, we obtain

$$\sum_{s=2}^{N}\tau_C\left(\delta_t\varepsilon_C^{sk}, \varepsilon_C^{(s-\frac{1}{2})k}\right) - \sum_{s=2}^{N}\mu\tau_C\left(\Delta_h\varepsilon_C^{(s-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k}\right) - \sum_{s=2}^{N}\mathfrak{w}_{s,1}\tau_C\left(\Delta_h\varepsilon_C^k, \varepsilon_C^{(s-\frac{1}{2})k}\right) - \sum_{s=2}^{N}\tau_C\sum_{p=2}^{s}\mathfrak{w}_{s,p}\left(\Delta_h\varepsilon_C^{(p-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k}\right) = \sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(U_C^{sk}) - g(\tilde{U}_C^{sk}), \varepsilon_C^{(s-\frac{1}{2})k}\right) + \sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(U_C^{(s-1)k}) - g(\tilde{U}_C^{(s-1)k}), \varepsilon_C^{(s-\frac{1}{2})k}\right), \qquad (x_i, y_j) \in \Omega_h, \qquad 2 \le s \le N. \tag{4.6}$$

Then adding the above two equations together gives

$$\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3=\frac{\tau_C}{2}\left(g(U_C^k)-g(\tilde{U}_C^k),\varepsilon_C^k\right)+\frac{\tau_C}{2}\left(g(U_C^0)-g(\tilde{U}_C^0),\varepsilon_C^k\right)$$
$$+\sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(U_C^{sk})-g(\tilde{U}_C^{sk}),\varepsilon_C^{(s-\frac{1}{2})k}\right)+\sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(U_C^{(s-1)k})-g(\tilde{U}_C^{(s-1)k}),\varepsilon_C^{(s-\frac{1}{2})k}\right), \tag{4.7}$$

where

$$\mathcal{H}_1=\tau_C\left(\delta_t\varepsilon_C^k,\varepsilon_C^k\right)+\sum_{s=2}^{N}\tau_C\left(\delta_t\varepsilon_C^{sk},\varepsilon_C^{(s-\frac{1}{2})k}\right),$$
$$\mathcal{H}_2=-\tau_C\mu\left(\Delta_h\varepsilon_C^k,\varepsilon_C^k\right)-\sum_{s=2}^{N}\mu\tau_C\left(\Delta_h\varepsilon_C^{(s-\frac{1}{2})k},\varepsilon_C^{(s-\frac{1}{2})k}\right),$$
$$\mathcal{H}_3=-\tau_C\mathfrak{w}_{1,1}\left(\Delta_h\varepsilon_C^k,\varepsilon_C^k\right)-\sum_{s=2}^{N}\mathfrak{w}_{s,1}\tau_C\left(\Delta_h\varepsilon_C^k,\varepsilon_C^{(s-\frac{1}{2})k}\right)$$
$$-\sum_{s=2}^{N}\tau_C\sum_{p=2}^{s}\mathfrak{w}_{s,p}\left(\Delta_h\varepsilon_C^{(p-\frac{1}{2})k},\varepsilon_C^{(s-\frac{1}{2})k}\right).$$

Below the terms $\mathcal{H}_q(q=1,2,3)$ will be estimated one by one. First, for $\mathcal{H}_1$, we use Lemma 7 to obtain

$$\mathcal{H}_1\geq\frac{1}{2}\left(\|\varepsilon_C^{Nk}\|^2-\|\varepsilon_C^0\|^2\right). \tag{4.8}$$

Second, from Lemma 5, we obtain

$$\begin{aligned}\mathcal{H}_2&=\mu\tau_C\left(\nabla_h\varepsilon_C^k,\nabla_h\varepsilon_C^k\right)+\sum_{s=2}^{N}\mu\tau_C\left(\nabla_h\varepsilon_C^{(s-\frac{1}{2})k},\nabla_h\varepsilon_C^{(s-\frac{1}{2})k}\right)\\&=\mu\tau_C\|\nabla_h\varepsilon_C^k\|^2+\sum_{s=2}^{N}\mu\tau_C\|\nabla_h\varepsilon_C^{(s-\frac{1}{2})k}\|^2\geq 0.\end{aligned} \tag{4.9}$$

Finally, for the third term $\mathcal{H}_3$, we use Lemma 5 and Lemma 6 to get

$$\begin{aligned}\mathcal{H}_3&=\tau_C\mathfrak{w}_{1,1}\left(\nabla_h\varepsilon_C^k,\nabla_h\varepsilon_C^k\right)+\sum_{s=2}^{N}\mathfrak{w}_{s,1}\tau_C\left(\nabla_h\varepsilon_C^k,\nabla_h\varepsilon_C^{(s-\frac{1}{2})k}\right)\\&\quad+\sum_{s=2}^{N}\tau_C\sum_{p=2}^{s}\mathfrak{w}_{s,p}\left(\nabla_h\varepsilon_C^{(p-\frac{1}{2})k},\nabla_h\varepsilon_C^{(s-\frac{1}{2})k}\right)\\&=\tau_C\mathfrak{w}_{1,1}\left(\nabla_h\varepsilon_C^k,\nabla_h\varepsilon_C^k\right)+\sum_{s=2}^{N}\tau_C\left(\mathfrak{w}_{s,1}\nabla_h\varepsilon_C^k+\sum_{p=2}^{s}\mathfrak{w}_{s,p}\nabla_h\varepsilon_C^{(p-\frac{1}{2})k},\nabla_h\varepsilon_C^{(s-\frac{1}{2})k}\right)\\&=\tau_C\mathfrak{w}_{1,1}\left(\nabla_h\varepsilon_C^k,\nabla_h\varepsilon_C^k\right)+\sum_{s=2}^{N}\tau_C(\mathfrak{L}_{2,\tau}^s\nabla_h\varepsilon_C^{sk},\nabla_h\varepsilon_C^{(s-\frac{1}{2})k})\geq 0.\end{aligned} \tag{4.10}$$

Next, $g(u)$ satisfies the Lipschitz condition. For (4.7), using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|\varepsilon_C^{Nk}\|^2 - \|\varepsilon_C^0\|^2 \\
&\le \tau_C \left\|g(U_C^k) - g(\tilde{U}_C^k)\right\| \left\|\varepsilon_C^k\right\| + \tau_C \left\|g(U_C^0) - g(\tilde{U}_C^0)\right\| \left\|\varepsilon_C^k\right\| \\
&+ \sum_{s=2}^{N} \tau_C \left\|g(U_C^{sk}) - g(\tilde{U}_C^{sk})\right\| \left\|\varepsilon_C^{(s-\frac{1}{2})k}\right\| + \sum_{s=2}^{N} \tau_C \left\|g(U_C^{(s-1)k}) - g(\tilde{U}_C^{(s-1)k})\right\| \left\|\varepsilon_C^{(s-\frac{1}{2})k}\right\| \\
&\le \bar{C}\tau_C \left(\left\|\varepsilon_C^k\right\|^2 + \left\|\varepsilon_C^0\right\| \left\|\varepsilon_C^k\right\| + \sum_{s=2}^{N} \left\|\varepsilon_C^{sk}\right\| \left\|\varepsilon_C^{(s-\frac{1}{2})k}\right\| + \sum_{s=2}^{N} \left\|\varepsilon_C^{(s-1)k}\right\| \left\|\varepsilon_C^{(s-\frac{1}{2})k}\right\| \right).
\end{aligned} \tag{4.11}$$

Now, taking the positive integer $\bar{m}$ such that $\left\|\varepsilon_C^{\bar{m}k}\right\| = \max\limits_{0\le s\le N} \left\|\varepsilon_C^{sk}\right\|$, we have

$$\begin{aligned}
\|\varepsilon_C^{Nk}\| \le \|\varepsilon_C^{\bar{m}k}\| &\le \|\varepsilon_C^0\| + \bar{C}\tau_C \left(\left\|\varepsilon_C^k\right\| + \left\|\varepsilon_C^0\right\| + \sum_{s=2}^{\bar{m}} \left\|\varepsilon_C^{sk}\right\| + \sum_{s=2}^{\bar{m}} \left\|\varepsilon_C^{(s-1)k}\right\| \right) \\
&\le \|\varepsilon_C^0\| + \bar{C}\tau_C \left(\left\|\varepsilon_C^k\right\| + \left\|\varepsilon_C^0\right\| + \sum_{s=2}^{N} \left\|\varepsilon_C^{sk}\right\| + \sum_{s=2}^{N} \left\|\varepsilon_C^{(s-1)k}\right\| \right) \\
&\le \|\varepsilon_C^0\| + \bar{C}\tau_C \left(\sum_{s=0}^{N} \left\|\varepsilon_C^{sk}\right\| + \sum_{s=1}^{N-1} \left\|\varepsilon_C^{sk}\right\| \right) \\
&\le \|\varepsilon_C^0\| + \bar{C}\tau_C \left\|\varepsilon_C^{Nk}\right\| + 2\bar{C}\tau_C \sum_{s=0}^{N-1} \left\|\varepsilon_C^{sk}\right\|.
\end{aligned} \tag{4.12}$$

When $\tau_C \le \frac{1}{2\bar{C}}$, following from Lemma 8, inequality (4.12) becomes

$$\|\varepsilon_C^{Nk}\| \le \bar{C}(T)\|\varepsilon_C^0\| \exp\{N\tau_C\} \le \bar{C}\|\varepsilon_C^0\|. \tag{4.13}$$

(II) Notice that according to **(I)** we have $\|U_C^{sk}\| \le \bar{C}$ for any $1 \le s \le N$. Then we estimate the $\|U_C^{(s-1)k+q}\|$ for $1 \le s \le N$ and $1 \le q \le k-1$. Considering (3.11) and applying the triangle inequality, we obtain

$$\|U_C^{(s-1)k+q}\| = \|(1-\frac{q}{k})U_C^{(s-1)k} + \frac{q}{k}U_C^{sk}\| \le (1-\frac{q}{k})\|U_C^{(s-1)k}\| + \frac{q}{k}\|U_C^{sk}\| \le \bar{C}, \tag{4.14}$$

which completes the proof.

In addition, we shall analyse the stability on the fine grid.

Theorem 2 *For the system* (3.12) *and* (3.13) *on the fine grid, with* $1 \le n \le \mathcal{N}$, *we have* $\|U_F^n\| \le \bar{C}$.

Proof Taking the inner product of (3.12) with $\tau_F U_F^1$, we have

$$\tau_F\left(\delta_t U_F^1, U_F^1\right) - \mu\tau_F\left(\Delta_h U_F^1, U_F^1\right) - \tau_F \mathfrak{w}_{1,1}\left(\Delta_h U_F^1, U_F^1\right)$$
$$= \tau_F\left(b^1, U_F^1\right) + \frac{\tau_F}{2}\left(g(U_F^0), U_F^1\right) + \frac{\tau_F}{2}\left(g(U_C^1) + g'(U_C^1)\left(U_F^1 - U_C^1\right), U_F^1\right). \tag{4.15}$$

For (3.13), taking the inner product of both sides with $U_F^{n-\frac{1}{2}}$, multiplying it by τ_F, and summing for n from 2 to $\mathcal{N}$, we get

$$\sum_{n=2}^{\mathcal{N}} \tau_F\left(\delta_t U_F^n, U_F^{n-\frac{1}{2}}\right) - \sum_{n=2}^{\mathcal{N}} \tau_F \mu\left(\Delta_h U_F^{n-\frac{1}{2}}, U_F^{n-\frac{1}{2}}\right) - \sum_{n=2}^{\mathcal{N}} \tau_F \mathfrak{w}_{n,1}\left(\Delta_h U_F^1, U_F^{n-\frac{1}{2}}\right)$$
$$- \sum_{n=2}^{\mathcal{N}} \tau_F \sum_{p=2}^{n} \mathfrak{w}_{n,p}\left(\Delta_h U_F^{p-\frac{1}{2}}, U_F^{n-\frac{1}{2}}\right) = \sum_{n=2}^{\mathcal{N}} \tau_F\left(b^n, U_F^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\mathcal{N}} \frac{\tau_F}{2}\left(g(U_F^{n-1}), U_F^{n-\frac{1}{2}}\right)$$
$$+ \sum_{n=2}^{\mathcal{N}} \frac{\tau_F}{2}\left(g(U_C^n) + g'(U_C^n)\left(U_F^n - U_C^n\right), U_F^{n-\frac{1}{2}}\right), \qquad (x_i, y_j) \in \Omega_h, \qquad 2 \le n \le \mathcal{N}. \tag{4.16}$$

Then, adding (4.15) and (4.16), and similar to the analysis of (4.6)-(4.10), we obtain

$$\|U_F^{\mathcal{N}}\|^2 - \|U_F^0\|^2$$
$$\le 2\tau_F\|b^1\|\|U_F^1\| + 2\sum_{n=2}^{\mathcal{N}} \tau_F\|b^n\|\|U_F^{n-\frac{1}{2}}\| + \tau_F\|g(U_F^0)\|\|U_F^1\| + \sum_{n=2}^{\mathcal{N}} \tau_F\|g(U_F^{n-1})\|\|U_F^{n-\frac{1}{2}}\|$$
$$+ \tau_F\|g(U_C^1)\|\|U_F^1\| + \tau_F\|g'(U_C^1)\left(U_F^1 - U_C^1\right)\|\|U_F^1\|$$
$$+ \sum_{n=2}^{\mathcal{N}} \tau_F\|g(U_C^n)\|\|U_F^{n-\frac{1}{2}}\| + \sum_{n=2}^{\mathcal{N}} \tau_F\|g'(U_C^n)\left(U_F^n - U_C^n\right)\|\|U_F^{n-\frac{1}{2}}\|. \tag{4.17}$$

Based on the stability of the coarse grid, $\|U_C^n\| \le \bar{C}(0 \le n \le \mathcal{N})$ can be obtained. Then according to $g(u) \in C^2(\mathbf{R}) \cap L^1(0,T]$, we have $g(U_C^n) \le \bar{C}$ and $g'(U_C^n) \le \bar{C}$. Also, assuming $\|U_F^n\| \le \bar{C}$ holds for $0 \le n \le \mathcal{N}-1$, then $g(U_F^n) \le \bar{C}$ can be obtained, thus

$$\begin{aligned}\|U_F^{\mathcal{N}}\|^2 - \|U_F^0\|^2 \le\ & 2\tau_F\|b^1\|\|U_F^1\| + 2\sum_{n=2}^{\mathcal{N}} \tau_F\|b^n\|\|U_F^{n-\frac{1}{2}}\| + \bar{C}\tau_F\|U_F^1\| \\ & + \bar{C}\sum_{n=2}^{\mathcal{N}} \tau_F\|U_F^{n-\frac{1}{2}}\| + \bar{C}\tau_F\left(\|U_F^1\| + \|U_C^1\|\right)\|U_F^1\| \\ & + \bar{C}\sum_{n=2}^{\mathcal{N}} \tau_F\left(\|U_F^n\| + \|U_C^n\|\right)\|U_F^{n-\frac{1}{2}}\|.\end{aligned} \tag{4.18}$$

Denoting $\|U_F^{\tilde{m}}\| = \max_{0\le n\le \mathcal{N}} \|U_F^n\|$, we can get

$$\begin{aligned}
\|U_F^{\tilde{m}}\|^2 \le{}& \|U_F^0\|^2 + 2\tau_F\|b^1\|\|U_F^1\| + 2\sum_{n=2}^{\tilde{m}} \tau_F\|b^n\|\|U_F^{n-\frac{1}{2}}\| + \bar{C}\tau_F\|U_F^1\| + \bar{C}\sum_{n=2}^{\tilde{m}} \tau_F\|U_F^{n-\frac{1}{2}}\| \\
&+ \bar{C}\tau_F\left(\|U_F^1\| + \|U_C^1\|\right)\|U_F^1\| + \bar{C}\sum_{n=2}^{\tilde{m}} \tau_F\left(\|U_F^n\| + \|U_C^n\|\right)\|U_F^{n-\frac{1}{2}}\| \\
\le{}& \|U_F^0\|\|U_F^{\tilde{m}}\| + 2\tau_F\|b^1\|\|U_F^{\tilde{m}}\| + 2\sum_{n=2}^{\tilde{m}} \tau_F\|b^n\|\|U_F^{\tilde{m}}\| + \bar{C}\tau_F\|U_F^{\tilde{m}}\| + \bar{C}\sum_{n=2}^{\tilde{m}} \tau_F\|U_F^{\tilde{m}}\| \\
&+ \bar{C}\tau_F\left(\|U_F^1\| + \|U_C^1\|\right)\|U_F^{\tilde{m}}\| + \bar{C}\sum_{n=2}^{\tilde{m}} \tau_F\left(\|U_F^n\| + \|U_C^n\|\right)\|U_F^{\tilde{m}}\|.
\end{aligned} \tag{4.19}$$

Then

$$\begin{aligned}
\|U_F^{\mathcal{N}}\| \le \|U_F^{\tilde{m}}\| &\le \|U_F^0\| + 2\sum_{n=1}^{\tilde{m}} \tau_F\|b^n\| + \bar{C}\sum_{n=1}^{\tilde{m}} \tau_F + \bar{C}\sum_{n=1}^{\tilde{m}} \tau_F\left(\|U_F^n\| + \|U_C^n\|\right) \\
&\le \|U_F^0\| + 2\sum_{n=1}^{\mathcal{N}} \tau_F\|b^n\| + \bar{C}\sum_{n=1}^{\mathcal{N}} \tau_F + \bar{C}\sum_{n=1}^{\mathcal{N}} \tau_F\left(\|U_F^n\| + \|U_C^n\|\right).
\end{aligned} \tag{4.20}$$

When $\tau_F \le \frac{1}{4\bar{C}}$, from Lemma 8 and Theorem 1, inequality (4.20) turn into the following

$$\|U_F^{\mathcal{N}}\| \le \bar{C}(T)\exp(\mathcal{N}\tau_F)\left(\|U_F^0\| + \sum_{n=1}^{\mathcal{N}} \tau_F\|b^n\| + \sum_{n=1}^{\mathcal{N}} \tau_F + \sum_{n=1}^{\mathcal{N}} \tau_F\|U_C^n\|\right) \le \bar{C}. \tag{4.21}$$

This finishes the proof.

4.2 Convergence

The convergence of TTGCN finite difference scheme (3.9)-(3.11) on coarse grid will be analysis using the energy method. Let

$$(e_C)_{ij}^n = u_{ij}^n - (U_C)_{ij}^n, \qquad (x_i, y_j) \in \bar{\Omega}_h, \qquad 0 \le n \le \mathcal{N}.$$

Subtracting (3.9)-(3.10), (3.7)-(3.8) from (3.1)-(3.4), respectively, we obtain the following error equations

$$\begin{aligned}
&\delta_t(e_C)_{ij}^k - \mu\Delta_h(e_C)_{ij}^k - \mathfrak{w}_{1,1}\Delta_h(e_C)_{ij}^k = \frac{1}{2}\left[g(u_{ij}^k) - g((U_C)_{ij}^k)\right] \\
&+ \frac{1}{2}\left[g(u_{ij}^0) - g((U_C)_{ij}^0)\right] + (R)_{ij}^k, \quad (x_i, y_j) \in \Omega_h,
\end{aligned} \tag{4.22}$$

$$\delta_t(e_C)_{ij}^{sk} - \mu\Delta_h(e_C)_{ij}^{(s-\frac{1}{2})k} - \mathfrak{w}_{s,1}\Delta_h(e_C)_{ij}^{k} - \sum_{p=2}^{s}\mathfrak{w}_{s,p}\Delta_h(e_C)_{ij}^{(p-\frac{1}{2})k}$$
$$= \frac{1}{2}\left[g(u_{ij}^{sk}) - g((U_C)_{ij}^{sk})\right] + \frac{1}{2}\left[g(u_{ij}^{(s-1)k}) - g((U_C)_{ij}^{(s-1)k})\right] + (R)_{ij}^{sk}, \tag{4.23}$$
$$(x_i, y_j) \in \Omega_h, \qquad 2 \le s \le N,$$

$$(e_C)_{ij}^{n} = 0, \qquad (x_i, y_j) \in \partial\Omega_h, \qquad 1 \le n \le \mathcal{N}, \tag{4.24}$$

$$(e_C)_{ij}^{0} = 0, \qquad (x_i, y_j) \in \Omega_h, \tag{4.25}$$

where $(R) = (R1) + (R2) + (R3) + (R4)$.

Theorem 3 *Assume that $u(x, y, t)$ and U_C^n are solutions of* (3.1)-(3.2) *and* (3.9)-(3.10)*, respectively, and that $u(x, y, t)$ satisfies the regularity assumptions **(A1)** and **(A2)**. Then, it holds that*

$$\max_{1\le n\le\mathcal{N}} \|u^n - U_C^n\| \le \bar{C}(\tau_C^2 + h_1^2 + h_2^2), \quad 1 \le n \le \mathcal{N}.$$

Proof The proof of this theorem is divided into two steps:

(I). Taking the inner product of equations (4.22) and (4.23) with e_C^k and $e_C^{(s-\frac{1}{2})k}$ respectively, and multiplying both equations by τ_C, summing for s from 2 to N in (4.23) and adding (4.22), then we can obtain

$$\tilde{\mathcal{H}}_1 + \tilde{\mathcal{H}}_2 + \tilde{\mathcal{H}}_3 = \frac{\tau_C}{2}\left(g(u^k) - g(U_C^k), e_C^k\right) + \frac{\tau_C}{2}\left(g(u^0) - g(U_C^0), e_C^k\right)$$
$$+ \sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(u^{sk}) - g(U_C^{sk}), e_C^{(s-\frac{1}{2})k}\right) + \sum_{s=2}^{N}\frac{\tau_C}{2}\left(g(u^{(s-1)k}) - g(U_C^{(s-1)k}), e_C^{(s-\frac{1}{2})k}\right)$$
$$+ \tau_C\left((R)^k, e_C^k\right) + \sum_{s=2}^{N}\tau_C\left((R)^{sk}, e_C^{(s-\frac{1}{2})k}\right), \tag{4.26}$$

where

$$\tilde{\mathcal{H}}_1 = \tau_C\left(\delta_t e_C^k, e_C^k\right) + \sum_{s=2}^{N}\tau_C\left(\delta_t e_C^{sk}, e_C^{(s-\frac{1}{2})k}\right),$$

$$\tilde{\mathcal{H}}_1 = -\tau_C\mu\left(\Delta_h e_C^k, e_C^k\right) - \sum_{s=2}^{N}\mu\tau_C\left(\Delta_h e_C^{(s-\frac{1}{2})k}, e_C^{(s-\frac{1}{2})k}\right),$$

$$\tilde{\mathcal{H}}_3 = -\,\tau_C\mathfrak{w}_{1,1}\left(\Delta_h e_C^k, e_C^k\right) - \sum_{s=2}^{N}\mathfrak{w}_{s,1}\tau_C\left(\Delta_h e_C^k, e_C^{(s-\frac{1}{2})k}\right)$$
$$- \sum_{s=2}^{N}\tau_C\sum_{p=2}^{s}\mathfrak{w}_{s,p}\left(\Delta_h e_C^{(p-\frac{1}{2})k}, e_C^{(s-\frac{1}{2})k}\right).$$

For (4.26), applying Lemmas 5-7 and Cauchy-Schwarz inequality, we get the following inequality

$$\begin{aligned}
&\|e_C^{Nk}\|^2 - \|e_C^0\|^2 \\
&\le \tau_C\|g(u^k) - g(U_C^k)\|\|e_C^k\| + \tau_C\|g(u^0) - g(U_C^0)\|\|e_C^k\| \\
&+ \sum_{s=2}^{N} \tau_C\|g(u^{sk}) - g(U_C^{sk})\|\|e_C^{(s-\frac{1}{2})k}\| + \sum_{s=2}^{N} \tau_C\|g(u^{(s-1)k}) - g(U_C^{(s-1)k})\|\|e_C^{(s-\frac{1}{2})k}\| \\
&+ 2\tau_C\|(R)^k\|\|e_C^k\| + 2\sum_{s=2}^{N} \tau_C\|(R)^{sk}\|\|e_C^{(s-\frac{1}{2})k}\| \\
&\le \bar{C}\tau_C\left(\|e_C^k\|^2 + \|e_C^0\|\|e_C^k\| + \sum_{s=2}^{N}\|e_C^{sk}\|\|e_C^{(s-\frac{1}{2})k}\| + \sum_{s=2}^{N}\|e_C^{(s-1)k}\|\|e_C^{(s-\frac{1}{2})k}\|\right) \\
&+ 2\tau_C\|(R)^k\|\|e_C^k\| + 2\sum_{s=2}^{N} \tau_C\|(R)^{sk}\|\|e_C^{(s-\frac{1}{2})k}\|.
\end{aligned} \tag{4.27}$$

Choosing a positive integer $\bar{s}$ such that $\|e_C^{\bar{s}k}\| = \max\limits_{0\le s\le N}\|e_C^{sk}\|$ and noting that (4.24), then we have

$$\begin{aligned}
\|e_C^{Nk}\| \le \|e_C^{\bar{s}k}\| &\le \bar{C}\tau_C\left(\|e_C^k\| + \sum_{s=2}^{\bar{s}}\|e_C^{sk}\| + \sum_{s=2}^{\bar{s}}\|e_C^{(s-1)k}\|\right) + 2\sum_{s=1}^{\bar{s}}\tau_C\|(R)^{sk}\| \\
&\le \bar{C}\tau_C\left(\|e_C^k\| + \sum_{s=2}^{N}\|e_C^{sk}\| + \sum_{s=2}^{N}\|e_C^{(s-1)k}\|\right) + 2\sum_{s=1}^{N}\tau_C\|(R)^{sk}\| \\
&\le \bar{C}\tau_C\left(\sum_{s=1}^{N}\|e_C^{sk}\| + \sum_{s=1}^{N}\|(R)^{sk}\|\right).
\end{aligned} \tag{4.28}$$

Using Lemma 8, then (4.28) becomes the following

$$\|e_C^{Nk}\| \le \bar{C}(T)\exp\{N\tau_C\}\left(\tau_C\sum_{s=1}^{N}\|(R)^{sk}\|\right). \tag{4.29}$$

In addition, from Lemmas 1-4 and using triangle inequality, we can get the following estimates

$$\begin{aligned}
\tau_C\sum_{s=1}^{N}\|(R)^{sk}\| &= \tau_C\sum_{s=1}^{N}\|(R1)^{sk} + (R2)^{sk} + (R3)^{sk} + (R4)^{sk}\| \\
&\le \tau_C\sum_{s=1}^{N}\left(\|(R1)^{sk}\| + \|(R2)^{sk}\| + \|(R3)^{sk}\| + \|(R4)^{sk}\|\right) \\
&\le \bar{C}(T)(\tau_C^2 + h_1^2 + h_2^2).
\end{aligned} \tag{4.30}$$

Finally combining (4.29) and (4.30), we have

$$\|e_C^{sk}\| \le \bar{C}(T)(\tau_C^2 + h_1^2 + h_2^2), \qquad 1 \le s \le N. \tag{4.31}$$

(II). For any $1 \leq s \leq N$ and $1 \leq q \leq k-1$, we utilize the Lagrange's interpolation formula, then

$$\begin{aligned} u^{(s-1)k+q} &= (1-\frac{q}{k})u^{(s-1)k} + \frac{q}{k}u^{sk} \\ &+ \frac{u''(\xi)}{2}(t_{(s-1)k+q} - t_{(s-1)k})(t_{(s-1)k+q} - t_{sk}), \qquad \xi \in (t_{(s-1)k}, t_{sk}). \end{aligned} \tag{4.32}$$

Subtracting (3.11) from (4.32), we have

$$e_C^{(s-1)k+q} = (1-\frac{q}{k})e_C^{(s-1)k} + \frac{q}{k}e_C^{sk} + \frac{u''(\xi)}{2}(t_{(s-1)k+q} - t_{(s-1)k})(t_{(s-1)k+q} - t_{sk}),$$

then, applying the triangle inequality and (4.31), we obtain

$$\begin{aligned} \|e_C^{(s-1)k+q}\| &\leq (1-\frac{q}{k})\|e_C^{(s-1)k}\| + \frac{q}{k}\|e_C^{sk}\| + \frac{\|u''(\xi)\|_\infty}{2}\tau_C^2 \\ &\leq \bar{C}(\tau_C^2 + h_1^2 + h_2^2), \qquad 1 \leq s \leq N, \qquad 1 \leq q \leq k-1. \end{aligned} \tag{4.33}$$

The proof is finished.

Next, the convergence on the fine grid will be considered. Let

$$(e_F)_{ij}^n = u_{ij}^n - (U_F)_{ij}^n, \qquad (x_i, y_j) \in \bar{\Omega}_h, \qquad 0 \leq n \leq \mathcal{N}.$$

Subtracting (3.12)-(3.13), (3.7)-(3.8) from (3.1)-(3.4), respectively, we yield the following error equations

$$\begin{aligned} &\delta_t(e_F)_{ij}^1 - \mu\Delta_h(e_F)_{ij}^1 - \mathfrak{w}_{1,1}\Delta_h(e_F)_{ij}^1 = \frac{1}{2}\left[g(u_{ij}^0) - g((U_F)_{ij}^0)\right] \\ &+ \frac{1}{2}\left[g(u_{ij}^1) - g((U_C)_{ij}^1) - g'((U_C)_{ij}^1)\left((U_F)_{ij}^1 - (U_C)_{ij}^1\right)\right] + (R)_{ij}^1, \quad (x_i, y_j) \in \Omega_h, \end{aligned} \tag{4.34}$$

$$\begin{aligned} &\delta_t(e_F)_{ij}^n - \mu\Delta_h(e_F)_{ij}^{n-\frac{1}{2}} - \mathfrak{w}_{n,1}\Delta_h(e_F)_{ij}^1 - \sum_{p=2}^{n}\mathfrak{w}_{n,p}\Delta_h(e_F)_{ij}^{p-\frac{1}{2}} \\ &= \frac{1}{2}\left[g(u_{ij}^{n-1}) - g((U_F)_{ij}^{n-1})\right] + \frac{1}{2}\left[g(u_{ij}^n) - g((U_C)_{ij}^n) - g'((U_C)_{ij}^n)\left((U_F)_{ij}^n - (U_C)_{ij}^n\right)\right] \\ &+ (R)_{ij}^n, \qquad (x_i, y_j) \in \Omega_h, \qquad 2 \leq n \leq \mathcal{N}, \end{aligned} \tag{4.35}$$

$$(e_F)_{ij}^n = 0, \qquad (x_i, y_j) \in \partial\Omega_h, \qquad 1 \leq n \leq \mathcal{N}, \tag{4.36}$$

$$(e_F)_{ij}^0 = 0, \qquad (x_i, y_j) \in \Omega_h. \tag{4.37}$$

Theorem 4 *Assume that* $u(x,y,t)$ *and* U_F^n *are solutions of* (3.1)-(3.2) *and* (3.12)-(3.13)*, respectively, and let* $u(x,y,t)$ *satisfy the regularity assumption* ***(A1)*** *and* ***(A2)****, then we have the following*

$$\|e_F^n\| \leq \bar{C}(\tau_F^2 + \tau_C^4 + h_1^2 + h_2^2), \qquad 1 \leq n \leq \mathcal{N}.$$

Proof *Taking the inner product of (4.34) with $\tau_F e_F^1$, we obtain*

$$\tau_F\left(\delta_t e_F^1, e_F^1\right) - \mu\tau_F\left(\Delta_h e_F^1, e_F^1\right) - \mathfrak{w}_{1,1}\tau_F\left(\Delta_h e_F^1, e_F^1\right) = \frac{\tau_F}{2}\left(g(u^0) - g(U_F^0), e_F^1\right)$$
$$+ \frac{\tau_F}{2}\left(g(u^1) - g(U_C^1) - g'(U_C^1)\left(U_F^1 - U_C^1\right), e_F^1\right) + \tau_F\left((R)^1, e_F^1\right). \tag{4.38}$$

Then taking the inner product of equation (4.35) with $\tau_F e_F^{n-\frac{1}{2}}$ and summing for n from 2 to $\mathcal{N}$, we can get

$$\sum_{n=2}^{\mathcal{N}}\tau_F\left(\delta_t e_F^n, e_F^{n-\frac{1}{2}}\right) - \sum_{n=2}^{\mathcal{N}}\mu\tau_F\left(\Delta_h e_F^{n-\frac{1}{2}}, e_F^{n-\frac{1}{2}}\right) - \sum_{n=2}^{\mathcal{N}}\mathfrak{w}_{n,1}\tau_F\left(\Delta_h e_F^1, e_F^{n-\frac{1}{2}}\right)$$
$$-\sum_{n=2}^{\mathcal{N}}\tau_F\sum_{p=2}^{n}\mathfrak{w}_{n,p}\left(\Delta_h e_F^{p-\frac{1}{2}}, e_F^{n-\frac{1}{2}}\right) = \frac{\tau_F}{2}\sum_{n=2}^{\mathcal{N}}\left(g(u^{n-1}) - g(U_F^{n-1}), e_F^{n-\frac{1}{2}}\right)$$
$$+\frac{\tau_F}{2}\sum_{n=2}^{\mathcal{N}}\left(g(u^n) - g((U_C)^n) - g'(U_C^n)\left(U_F^n - U_C^n\right), e_F^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\mathcal{N}}\tau_F\left((R)^n, e_F^{n-\frac{1}{2}}\right). \tag{4.39}$$

Adding (4.38) and (4.39), then using Lemmas 5-7, Cauchy-Schwarz inequality and triangle inequality, and noting (4.36), we can get

$$\begin{aligned}\|e_F^{\mathcal{N}}\|^2 \le{}& \tau_F\|g(u^1) - g(U_C^1) - g'(U_C^1)\left(U_F^1 - U_C^1\right)\|\|e_F^1\| + 2\tau_F\|R^1\|\|e_F^1\| \\ &+ \tau_F\sum_{n=2}^{\mathcal{N}}\|g(u^n) - g(U_C^n) - g'(U_C^n)\left(U_F^n - U_C^n\right)\|\|e_F^{n-\frac{1}{2}}\| \\ &+ \bar{C}\tau_F\sum_{n=2}^{\mathcal{N}}\|e_F^{n-1}\|\|e_F^{n-\frac{1}{2}}\| + 2\sum_{n=2}^{\mathcal{N}}\tau_F\|(R)^n\|\|e_F^{n-\frac{1}{2}}\|.\end{aligned} \tag{4.40}$$

Choosing a suitable $\check{s}$ such that $\|e_F^{\check{s}}\| = \max_{0\le n\le\mathcal{N}}\|e_F^n\|$, then it holds

$$\begin{aligned}\|e_F^{\mathcal{N}}\| \le \|e_F^{\check{s}}\| \le{}& \tau_F\sum_{n=1}^{\mathcal{N}}\|g(u^n) - g(U_C^n) - g'(U_C^n)\left(U_F^n - U_C^n\right)\| \\ &+ \bar{C}\tau_F\sum_{n=2}^{\mathcal{N}}\|e_F^{n-1}\| + 2\sum_{n=1}^{\mathcal{N}}\tau_F\|(R)^n\|.\end{aligned} \tag{4.41}$$

According to Taylor expansion, we have

$$\begin{aligned}&g(u^n) - g(U_C^n) - g'(U_C^n)\left(U_F^n - U_C^n\right) \\ &= g'(U_C^n)(u^n - U_C^n) + \frac{1}{2}g''(\theta^n)(u^n - U_C^n)^2 - g'(U_C^n)(U_F^n - U_C^n) \\ &= g'(U_C^n)e_F^n + \frac{1}{2}g''(\theta^n)(e_C^n)^2, \quad \theta^n \in \left(\min\{u^n, U_C^n\}, \max\{u^n, U_C^n\}\right).\end{aligned} \tag{4.42}$$

Substituting (4.42) into (4.41) and applying the triangle inequality, we can get

$$\begin{aligned} \|e_F^{\mathcal{N}}\| &\le \bar{C}\tau_F \sum_{n=1}^{\mathcal{N}} (\|e_F^n\| + \|e_C^n\|^2) + \bar{C}\tau_F \sum_{n=2}^{\mathcal{N}} \|e_F^{n-1}\| + 2\sum_{n=1}^{\mathcal{N}} \tau_F \|(R)^n\| \\ &\le \bar{C}\tau_F \sum_{n=1}^{\mathcal{N}} \|e_F^n\| + \bar{C}\tau_F \sum_{n=1}^{\mathcal{N}} \|e_C^n\|^2 + 2\sum_{n=1}^{\mathcal{N}} \tau_F \|(R)^n\|. \end{aligned} \tag{4.43}$$

Utilizing Lemma 8 and Theorem 3, we yield

$$\begin{aligned} \|e_F^{\mathcal{N}}\| &\le \bar{C}\exp\{\mathcal{N}\tau_F\} \left(\tau_F \sum_{n=1}^{\mathcal{N}} \|e_C^n\|^2 + \sum_{n=1}^{\mathcal{N}} \tau_F \|(R)^n\| \right) \\ &\le \bar{C}\left(\tau_C^4 + \tau_F^2 + h_1^2 + h_2^2\right), \end{aligned} \tag{4.44}$$

which completes the proof.

5 Numerical experiment

In this section, we will use the TTGCN finite difference scheme (3.9)-(3.13) to solve problem (1.1)-(1.2) and apply the method to three test problems. In order to verify the validity of the method, we also compare the results obtained from proposed scheme with the existing methods, e.g., the SCN finite difference scheme (3.5)-(3.8) and the scheme [25]. We set $L_x = L_y = 1$ and $T = 1$. All experiments are performed on a Windows 11 (64 bit) PC-Inter(R) Core(TM) i5-12500H CPU 3.10 GHz, 16.0 GB of RAM using MTALAB R2021b.

The discrete L^2-norm error is defined as follows

$$E_{TTGCN}(h,\tau) = \max_{1\le n\le \mathcal{N}} \|u^n - U_F^n\|,$$

and the time-space convergence orders are defined by

$$rate^t_{TTGCN} = \log_2\left(\frac{E_{TTGCN}(h,2\tau)}{E_{TTGCN}(h,\tau)}\right), \qquad rate^x_{TTGCN} = \log_2\left(\frac{E_{TTGCN}(2h,\tau)}{E_{TTGCN}(h,\tau)}\right).$$

In addition, we can similarly define $E_{SCN}(h,\tau)$, $rate^t_{SCN}$ and $rate^x_{SCN}$.

Example 1 We consider the nonlinear term is given by $g(u) = -u^2$, $\mu = 1$ and the inhomogeneous term is

$$\begin{aligned} f(x,y,t) = &\left[(1+2\pi)\frac{t^\alpha}{\Gamma(\alpha+1)} + 2\pi^2\left(1 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)\right]\sin\pi x \sin\pi y \\ &+ \left(1 + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\sin\pi x\sin\pi y\right)^2. \end{aligned}$$

The exact solution of this problem is presented as follows

$$u(x,y,t) = \left(1 + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\right)\sin\pi x\sin\pi y.$$

Table 1 The L^2-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $k = 4$ for Example 1.

α	τ_C	τ_F	E_{TTGCN}	$rate^t_{TTGCN}$	$CPU(s)$	E_{SCN}	$rate^t_{SCN}$	$CPU(s)$
0.25	1/2	1/8	2.9293e-2	*	41.42	2.9294e-2	*	83.85
	1/4	1/16	9.9431e-3	1.5588	75.53	9.9431e-3	1.5588	159.79
	1/8	1/32	2.9743e-3	1.7412	176.76	2.9743e-3	1.7412	307.44
	1/16	1/64	7.7382e-4	1.9425	439.63	7.7382e-4	1.9425	696.57
0.5	1/2	1/8	1.5390e-2	*	35.26	1.5391e-2	*	83.48
	1/4	1/16	4.2211e-3	1.8663	77.16	4.2212e-3	1.8664	160.76
	1/8	1/32	1.0102e-3	2.0630	177.43	1.0102e-3	2.0630	304.59
	1/16	1/64	2.0588e-4	2.2948	441.57	2.0589e-4	2.2948	700.46
0.75	1/2	1/8	7.7023e-3	*	35.64	7.7034e-3	*	83.19
	1/4	1/16	1.7363e-3	2.1493	77.64	1.7364e-3	2.1494	159.14
	1/8	1/32	3.3266e-4	2.3839	176.50	3.3266e-4	2.3840	309.35
	1/16	1/64	9.3963e-5	1.8239	414.11	9.3963e-5	1.8239	681.68

Table 2 The L^2-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $\alpha = 0.5$ for Example 1.

k	τ_C	τ_F	E_{TTGCN}	$rate^t_{TTGCN}$	$CPU(s)$	E_{SCN}	$rate^t_{SCN}$	$CPU(s)$
2	1/3	1/6	2.5490e-2	*	40.93	2.5490e-2	*	61.24
	1/6	1/12	7.3298e-3	1.7981	83.19	7.3299e-3	1.7981	120.88
	1/12	1/24	1.8622e-3	1.9767	181.48	1.8623e-3	1.9767	232.67
	1/24	1/48	4.0706e-4	2.1937	391.92	4.0706e-4	2.1937	474.09
3	1/2	1/6	2.5489e-2	*	31.25	2.5490e-2	*	61.91
	1/4	1/12	7.3297e-3	1.7980	65.04	7.3299e-3	1.7981	120.44
	1/8	1/24	1.8622e-3	1.9767	142.96	1.8623e-3	1.9767	232.99
	1/16	1/48	4.0706e-4	2.1937	320.59	4.0706e-4	2.1937	479.59
5	1/2	1/10	1.0281e-2	*	40.04	1.0282e-2	*	107.69
	1/4	1/20	2.7071e-3	1.9252	89.51	2.7071e-3	1.9253	198.24
	1/8	1/40	6.1692e-4	2.1336	214.16	6.1693e-4	2.1336	389.39
	1/16	1/80	1.1915e-4	2.3723	531.10	1.1915e-4	2.3724	913.59

In Table 1, we obtain the corresponding discrete L^2-norm errors, time convergence order and CPU time by calculating Example 1 with the TTGCN finite difference scheme (3.9)-(3.13) and the SCN finite difference method (3.5)-(3.8). The numerical results show that the convergence order of the two schemes converges to 2 in the time direction, which is consistent with the theoretical analysis. Meanwhile, we compare the numerical results of the two methods in terms of temporal convergence order and computational cost (CPU time in seconds), and see that the TTGCN finite difference scheme can save computational cost significantly without losing computational accuracy.

In addition, by the results in Table 2, we can see that the TTGCN finite difference scheme will save more computational cost than the SCN finite difference scheme as the value of k increases.

When the time step $\tau_C = 1/128$ and $\tau_F = 1/512$ are fixed, in Tables 3, the convergence order of the two schemes in space is 2 according to the numerical results. Therefore, the convergence results in the space-time direction are in good agreement with the theoretical analysis.

Table 3 The L^2-errors and convergence rates with $\tau_C = 1/128$ and $\tau_F = 1/512$ for Example 1.

α	h	E_{TTGCN}	$rate^x_{TTGCN}$	E_{SCN}	$rate^x_{SCN}$
0.20	1/2	3.8785e-1	*	3.8785e-1	-
	1/4	9.1992e-2	2.0759	9.1992e-2	2.0759
	1/8	2.2621e-2	2.0239	2.2621e-2	2.0239
	1/16	5.6309e-3	2.0062	5.6309e-3	2.0062
	1/32	1.4061e-3	2.0017	1.4061e-3	2.0017
0.50	1/2	3.5774e-1	*	3.5774e-1	*
	1/4	8.4731e-2	2.0780	8.4731e-2	2.0780
	1/8	2.0830e-2	2.0242	2.0830e-2	2.0242
	1/16	5.1848e-3	2.0063	5.1848e-3	2.0063
	1/32	1.2945e-3	2.0018	1.2945e-3	2.0018
0.80	1/2	3.2854e-1	*	3.2854e-1	*
	1/4	7.7643e-2	2.0811	7.7643e-2	2.0811
	1/8	1.9081e-2	2.0248	1.9081e-2	2.0248
	1/16	4.7487e-3	2.0065	4.7487e-3	2.0065
	1/32	1.1855e-3	2.0020	1.1855e-3	2.0020

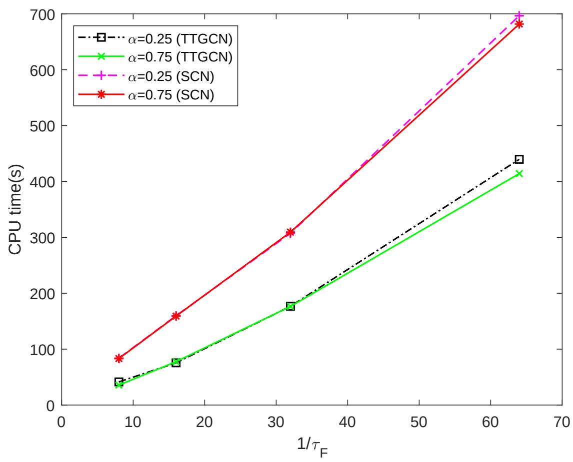

Fig. 1 The comparison of two methods for CPU time with $h = 1/100$ and $k = 4$ for Example 1.

Fig. 1 compares the computation time of the two-grid method and the standard method in the time direction for the Crank-Nicolson finite difference scheme. It can be observed that the computational cost of the TTGCN finite difference method is lower without losing the accuracy. Also, Fig. 2 gives the L^2-norm error for both methods, which can show intuitively second-order convergence for time.

Example 2 we consider $g(u) = -u - u^3$ and $\mu = 1$. The exact solution is given via

$$u(x, y, t) = \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} \sin \pi x \sin \pi y,$$

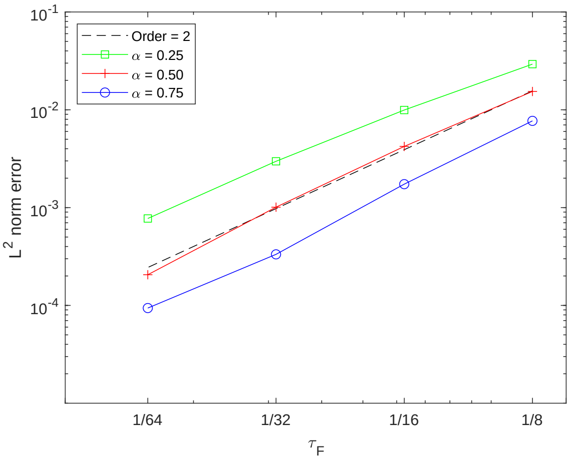

Fig. 2 The time convergence order with $h = 1/100$ and $k = 4$ for Example 1.

thus, $\psi(x, y) = 0$ and the corresponding force term can be obtained as follows

$$f(x,y,t) = \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(2\pi^2\mu+1)t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)\sin\pi x \sin\pi y + \left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\sin\pi x\sin\pi y\right)^3.$$

In Table 4, we give the numerical results with $\alpha = 0.25$, 0.5 and 0.75 calculated using the TTGCN finite difference method and the SCN finite difference method, respectively. This numerical result fully demonstrates that the computational efficiency of the TTGCN finite difference method is much higher than that of the SCN finite difference method. Also, according to the numerical results in Table 5, the order of convergence of the two methods in space ≈ 2. Therefore, the numerical results are consistent with the theoretical analysis. In addition, we also compared with the method in [25]. It is obvious from Table 6 that the TTGCN finite difference method has higher accuracy and convergence order.

When $h = 1/100$ and $k = 4$, Fig. 3 compares the CPU time of the two-grid finite difference method and the standard finite difference method for the time direction, which intuitively demonstrates the effectiveness of our method. Besides, Fig. 4 shows intuitively temporal second-order convergence of two-grid finite difference method.

Example 3 we consider

$$\begin{aligned} &u_t - \Delta u - I^{(\alpha)}\Delta u = -u^3, \quad (x,y,t) \in \Omega \times (0,T],\\ &u(x,y,t) = 0, \qquad (x,y) \in \partial\Omega, \quad t \in [0,T],\\ &u(x,y,0) = xy(1-x)(1-y), \qquad (x,y) \in \Omega. \end{aligned}$$

In this example, since the exact solution is unknown, we assume that the numerical solution with fixed spatial step $h = 1/32$ and half of the original time steps

Table 4 The L^2-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $k = 4$ for Example 2.

α	τ_C	τ_F	E_{TTGCN}	$rate^t_{TTGCN}$	$CPU(s)$	E_{SCN}	$rate^t_{SCN}$	$CPU(s)$
	1/2	1/8	2.9535e-2	*	34.31	2.9535e-2	*	64.26
	1/4	1/16	1.0043e-2	1.5563	71.10	1.0043e-2	1.5563	125.59
0.25	1/8	1/32	3.0208e-3	1.7331	161.96	3.0208e-3	1.7331	264.99
	1/16	1/64	7.9828e-4	1.9200	412.81	7.9828e-4	1.9200	561.79
	1/2	1/8	1.5532e-2	*	32.46	1.5532e-2	*	58.22
	1/4	1/16	4.2840e-3	1.8582	69.57	4.2840e-3	1.8582	120.92
0.5	1/8	1/32	1.0448e-3	2.0357	156.46	1.0448e-3	2.0357	242.71
	1/16	1/64	2.2614e-4	2.2080	391.87	2.2614e-4	2.2080	561.08
	1/2	1/8	7.7945e-3	*	30.86	7.7946e-3	*	59.13
	1/4	1/16	1.7861e-3	2.1256	67.44	1.7861e-3	2.1256	113.23
0.75	1/8	1/32	3.6423e-4	2.2939	156.47	3.6423e-4	2.2939	233.35
	1/16	1/64	6.6431e-5	2.4549	402.73	6.6431e-5	2.4549	545.91

Table 5 The L^2-errors and convergence rates with $\tau_C = 1/128$ and $\tau_F = 1/512$ for Example 2.

h	$\alpha = 0.2$				$\alpha = 0.8$			
	E_{TTGCN}	$rate^x_{TTGCN}$	E_{SCN}	$rate^x_{SCN}$	E_{TTGCN}	$rate^x_{TTGCN}$	E_{SCN}	$rate^x_{SCN}$
1/2	1.8132e-1	*	1.8132e-1	*	1.1940e-1	*	1.1941e-1	*
1/4	4.3294e-2	2.0663	4.3296e-2	2.0663	2.8125e-2	2.0859	2.8133e-2	2.0856
1/8	1.0650e-2	2.0233	1.0652e-2	2.0231	6.9065e-3	2.0259	6.9139e-2	2.0247
1/16	2.6486e-3	2.0070	2.6519e-3	2.0060	1.7133e-3	2.0112	1.7206e-3	2.0066
1/32	6.5992e-4	2.0054	6.6217e-4	2.0017	4.2551e-4	2.0095	4.2932e-4	2.0028

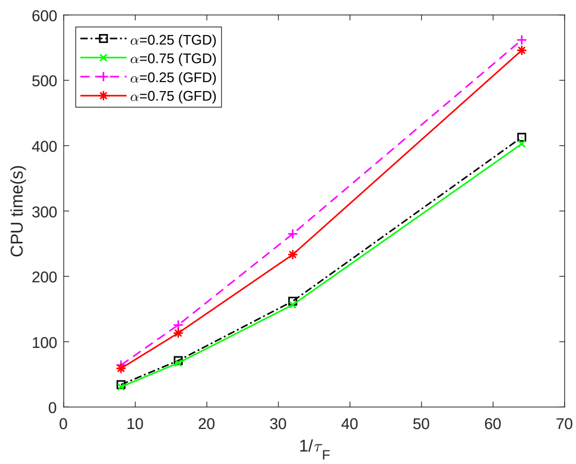

Fig. 3 The CPU time for Example 2 with $h = 1/100$ and $k = 4$.

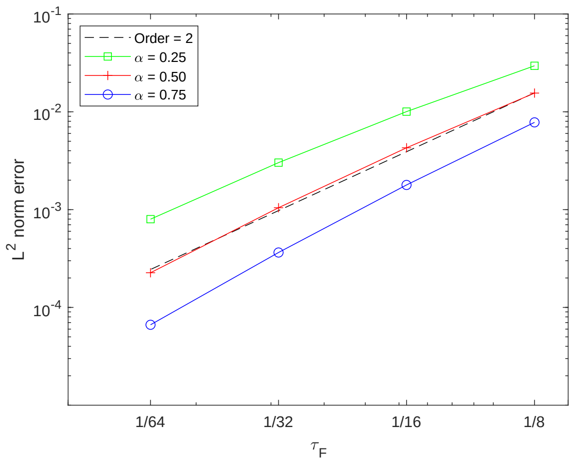

Fig. 4 The time convergence order for Example 2 with $h = 1/100$ and $k = 4$.

Table 6 The comparison between the scheme (3.9)-(3.13) and the scheme [25] whit $h = 1/100$ and $k = 4$ for Example 2.

α	τ_C	τ_F	Scheme (3.9)-(3.13)		Scheme in [25]	
			E_{TTGCN}	$rate^t_{TTGCN}$	E	$rate^t$
0.25	1/2	1/8	2.9535e-2	*	3.9266e-3	*
	1/4	1/16	1.0043e-2	1.5563	1.9639e-3	0.9996
	1/8	1/32	3.0208e-3	1.7331	9.7001e-4	1.0176
	1/16	1/64	7.9828e-4	1.9200	4.6979e-4	1.0460
0.5	1/2	1/8	1.5532e-2	*	7.6809e-3	*
	1/4	1/16	4.2840e-3	1.8582	3.8683e-3	0.9896
	1/8	1/32	1.0448e-3	2.0357	1.9311e-3	1.0023
	1/16	1/64	2.2614e-4	2.2080	9.5444e-4	1.0167
0.75	1/2	1/8	7.7945e-3	*	9.7620e-3	*
	1/4	1/16	1.7861e-3	2.1256	4.9287e-3	0.9860
	1/8	1/32	3.6423e-4	2.2939	2.2683e-3	0.9977
	1/16	1/64	6.6431e-5	2.4549	1.2266e-3	1.0088

τ_C and τ_F is the "exact" solution. From Table 7, we can see that for the time direction convergence order TTGCN and SCN finite difference methods in both can approach 2, which agrees with the theoretical analysis.

Declaration of Competing Interest

The authors declare that they have no conflict of interest.

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Table 7 The L^2-errors and convergence rates with $h = 1/32$ and $k = 4$ for Example 3

α	τ_C	τ_F	E_{TTGCN}	$rate^t_{TTGCN}$	E_{SCN}	$rate^t_{SCN}$
0.25	1/12	1/48	6.0750e-7	*	6.0563e-7	*
	1/24	1/96	1.8258e-7	1.7344	1.8214e-7	1.7334
	1/48	1/192	4.9624e-8	1.8794	4.9558e-8	1.8779
	1/96	1/384	1.2745e-8	1.9611	1.2739e-8	1.9599
0.5	1/12	1/48	1.2281e-6	*	1.2246e-6	*
	1/24	1/96	3.6180e-7	1.7661	3.6036e-7	1.7648
	1/48	1/192	9.7691e-8	1.8860	9.7595e-8	1.8846
	1/96	1/384	2.5272e-8	1.9507	2.5363e-8	1.9498
0.75	1/12	1/48	2.1532e-6	*	2.1477e-6	*
	1/24	1/96	6.3459e-7	1.7626	6.3351e-7	1.7614
	1/48	1/192	1.7308e-7	1.8744	1.7294e-7	1.8731
	1/96	1/384	4.5190e-8	1.9373	4.5177e-8	1.9366

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