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New Criteria of Oscillation for Linear Sturm–Liouville Delay Noncanonical Dynamic Equations

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Abstract: In this work, we deduce a new criterion that guarantees the oscillation of solutions to linear Sturm–Liouville delay noncanonical dynamic equations; these results emulate the criteria of the Hille and Ohriska types for canonical dynamic equations, and these results also solve an open problem in many works in the literature. Several examples are offered, demonstrating that the findings achieved are precise, practical, and adaptable.

Keywords: oscillation behavior; linear; second order; dynamic equations; time scales

MSC: 39A21; 39A99; 34C10; 34K11; 34K42; 34N05

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1. Introduction

Various models from real-world applications include oscillation phenomena; for mathematical biology models in which oscillation and/or delay behaviors can be described with cross-diffusion expressions, see papers [1–3]. The study of dynamic equations is addressed in this work because it involves a variety of real-world issues, such as the turbulent flow of a polytrophic gas in a porous medium and non-Newtonian fluid theory; see, e.g., [4–7] for further information. In consequence, we are concerned with the oscillatory behavior of a class of Sturm–Liouville noncanonical delay dynamic equations

$$\left(px^{\Delta}\right)^{\Delta}(s)+q(s)x(\tau(s))=0 \tag{1}$$

on an arbitrary time scale $\mathbb{T}$ that is presumed above to be unbounded, where $s\in[s_0,\infty)_{\mathbb{T}}:=[s_0,\infty)\cap\mathbb{T}$, $s_0\geq 0$, $s_0\in\mathbb{T}$, and $p,q:\mathbb{T}\to\mathbb{R}^{+}$ are rd-continuous functions, and $\tau:\mathbb{T}\to\mathbb{T}$ is a nondecreasing rd-continuous function satisfying $\tau(s)\leq s$ on $[s_0,\infty)_{\mathbb{T}}$ and $\lim_{s\to\infty}\tau(s)=\infty$. A time scale $\mathbb{T}$ is any closed real set. Define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ as

$$\sigma(s)=\inf\{\xi\in\mathbb{T}:\xi>s\},$$

and it is seen that $g:\mathbb{T}\to\mathbb{R}$ is differentiable at $s\in\mathbb{T}$ given that

$$g^{\Delta}(s):=\lim_{\xi\to s}\frac{g(s)-g(\xi)}{s-\xi}$$

exists when $\sigma(s) = s$ and when g is continuous at s and $\sigma(s) > s$,

$$g^{\Delta}(s) := \frac{g(\sigma(s)) - g(s)}{\sigma(s) - s}.$$

The classical theories of differential and difference equations are notably represented when this time scale is equal to the reals or integers. There are numerous additional time scales that are intriguing; this leads to the emergence of many applications. Beyond merely unifying the corresponding theories for differential and difference equations, this novel theory of these so-called "dynamic equations" also encompasses "in-between" cases. In other words, we permitted the consideration of q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^{\gamma} : \gamma \in \mathbb{N}_0 \text{ for } q > 1\}$. These equations possess significant practical implications in quantum theory (refer to [8]). Additionally, we permitted the consideration of different time scales, including $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ represents the set of harmonic numbers. For an introduction to the calculus of time scales, see Hilger [9] and Bohner and Peterson [10]. Using a solution of Equation (1), we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [s_0, \infty)_{\mathbb{T}}$ such that $px^{\Delta} \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$ and x satisfies (1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} represents the set of rd-continuous functions. According to Trench [11], (1) is considered to be in noncanonical form if

$$\int_{s_0}^{\infty} \frac{\Delta\xi}{p(\xi)} < \infty, \tag{2}$$

and in canonical form if

$$\int_{s_0}^{\infty} \frac{\Delta\xi}{p(\xi)} = \infty. \tag{3}$$

If a solution x of (1) is neither eventually positive nor eventually negative, we refer to it as oscillatory; otherwise, we refer to it as nonoscillatory. The solutions that vanish in some neighborhood of infinity will be excluded from consideration. If all solutions of (1) oscillate, then (1) is said to oscillate.

The following is a showing of oscillation results for differential equations that are related to our oscillation results for (1), as well as an explanation of the significant contributions made by this paper. Fite [12] studied the oscillation of the differential equation

$$x''(s) + q(s)x(s) = 0, \tag{4}$$

and demonstrated that if

$$\int_{s_0}^{\infty} q(\xi)\mathrm{d}\xi = \infty, \tag{5}$$

then (4) oscillates. Hille [13] enhanced (5) and proved that if

$$\liminf_{s\to\infty} \left\{ s \int_{s}^{\infty} q(\xi)\mathrm{d}\xi \right\} > \frac{1}{4}, \tag{6}$$

then (4) oscillates. Erbe [14] extended (6) to the delay differential equation

$$x''(s) + q(s)x(\tau(s)) = 0, \tag{7}$$

and saw that if

$$\liminf_{s\to\infty} \left\{ s \int_{s}^{\infty} \frac{\tau(\xi)}{\xi} q(\xi)\mathrm{d}\xi \right\} > \frac{1}{4}, \tag{8}$$

then (7) oscillates. Ohriska [15] established another oscillation criterion of (7) and obtained that if

$$\limsup_{s\to\infty} \left\{ s \int_{s}^{\infty} \frac{\tau(\xi)}{\xi} q(\xi)\mathrm{d}\xi \right\} > 1, \tag{9}$$

then (7) oscillates.

Concerning second-order dynamic equations, Erbe et al. [16] made the Hille-type criterion extended to the delay dynamic equation

$$x^{\Delta\Delta}(s) + q(s)x(\tau(s)) = 0, \tag{10}$$

where

$$\int_{s_0}^{\infty} \tau(\xi)q(\xi)\Delta\xi = \infty, \tag{11}$$

and demonstrated that if

$$\liminf_{s\to\infty} s\int_{\sigma(s)}^{\infty} \left(\frac{\tau(\xi)}{\sigma(\xi)}\right) q(\xi)\Delta\xi > \frac{1}{4l},$$

where $l := \liminf_{s\to\infty} \frac{s}{\sigma(s)} > 0$, then (10) oscillates. Karpuz [17] considered the linear dynamic equation in the canonical form

$$\left(px^{\Delta}\right)^{\Delta}(s) + q(s)x(s) = 0, \tag{12}$$

and obtained that if

$$\liminf_{s\to\infty}\left\{R(s)\int_{s}^{\infty} q(\xi)\Delta\xi\right\} > \frac{1}{4},$$

where

$$R(s) := \int_{s_0}^{s} \frac{\Delta\xi}{p(\xi)} \to \infty \text{ as } s \to \infty, \tag{13}$$

then (12) oscillates. For the delay dynamic equation in the canonical form

$$\left(px^{\Delta}\right)^{\Delta}(s) + q(s)x(\tau(s)) = 0, \tag{14}$$

where (13) holds, Hassan et al. [18] proved that if

$$\liminf_{s\to\infty}\left\{R(s)\int_{\sigma(s)}^{\infty} \frac{R(\tau(\xi))}{R(\xi)} q(\xi)\Delta\xi\right\} > \frac{1}{4l}, \tag{15}$$

where (13) holds and $l := \liminf_{s\to\infty} \dfrac{R(s)}{R(\sigma(s))} > 0$, then (14) oscillates. Hassan et al. [19] improved condition (15) for (14) and showed that if

$$\liminf_{s\to\infty}\left\{R(s)\int_{s}^{\infty} \frac{R(\tau(\xi))}{R(\xi)} q(\xi)\Delta\xi\right\} > \frac{1}{4},$$

then (14) oscillates. For further Hille-type criteria, see papers [20–22].

It is essential to emphasize that all of the aforementioned works concerning the derivation of Hille- and Ohriska-type criteria for numerous differential and dynamical equations share the canonical case as a unifying characteristic. Therefore, the focus of this paper will be on emulating the criteria of the Hille and Ohriska types in the noncanonical case (i.e., (2) holds). This result solves an open problem presented in many papers, e.g., [19]. The reader is pointed to related papers [23–30] and the sources listed therein.

2. Main Results

In this section, we will discuss the most significant findings of this paper and provide examples to illustrate their significance.

Theorem 1. *If* (2) *holds and for sufficiently large* $T \in [s_0, \infty)_{\mathbb{T}}$,

$$\liminf_{s\to\infty}\left\{\left(\int_{s}^{\infty} \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{T}^{s} q(\xi)\Delta\xi\right)\right\} > \frac{1}{4}, \tag{16}$$

then (1) *oscillates.*

Proof. Assume that x is a nonoscillatory solution of (1) on $[s_0,\infty)_{\mathbb{T}}$. Let $x(s) > 0$ and $x(\tau(s)) > 0$ hold on $[s_0,\infty)_{\mathbb{T}}$, without a loss of generality. We have from (1) that, for $s \in [s_0,\infty)_{\mathbb{T}}$,

$$\left(px^{\Delta}\right)^{\Delta}(s) = -q(s)x(\tau(s)) < 0. \tag{17}$$

This yields that px^{Δ} is decreasing on $[s_0,\infty)_{\mathbb{T}}$. Hence, there are two possibilities:

(a) $x^{\Delta}(s) > 0$ for all $s \geq s_0$;
(b) there is $s_1 \in [s_0,\infty)_{\mathbb{T}}$ such that $x^{\Delta}(s) < 0$ for all $s \geq s_1$.

First, we assume that (a) holds. Integrating (1) from $s \geq s_0$ to $t \in [s,\infty)_{\mathbb{T}}$, we see

$$\begin{aligned} p(s)x^{\Delta}(s) &> -p(t)x^{\Delta}(t) + p(s)x^{\Delta}(s) = \int_s^t q(\xi)x(\tau(\xi))\Delta\xi \\ &\geq x(\tau(s))\int_s^t q(\xi)\Delta\xi. \end{aligned}$$

By dividing by $x(\tau(s)) > 0$ and letting $t \to \infty$, we obtain

$$\int_s^{\infty} q(\xi)\Delta\xi \leq \frac{p(s)x^{\Delta}(s)}{x(\tau(s))} < \infty,$$

which is a contradiction with the assumption in (16).

Second, we suppose that (b) holds. Let $s \in [s_1,\infty)_{\mathbb{T}}$. Define

$$w(s) := -\frac{x(s)}{p(s)x^{\Delta}(s)} > 0. \tag{18}$$

According to product and quotient rules,

$$\begin{aligned} w^{\Delta}(s) &= -\frac{1}{p(s)x^{\Delta}(s)}x^{\Delta}(s) - \left(\frac{1}{px^{\Delta}}\right)^{\Delta}(s)x^{\sigma}(s) \\ &= -\frac{1}{p(s)} + \frac{(px^{\Delta})^{\Delta}(s)}{p(s)x^{\Delta}(s)(px^{\Delta})^{\sigma}(s)}x^{\sigma}(s) \\ &\overset{(1)}{=} -\frac{1}{p(s)} - q(s)\frac{x(\tau(s))}{p(s)x^{\Delta}(s)}\left(\frac{x}{px^{\Delta}}\right)^{\sigma}(s) \\ &\leq -\frac{1}{p(s)} - q(s)\frac{x(s)}{p(s)x^{\Delta}(s)}\left(\frac{x}{px^{\Delta}}\right)^{\sigma}(s) \\ &= -\frac{1}{p(s)} - q(s)w(s)w^{\sigma}(s), \end{aligned} \tag{19}$$

which gives that $w^{\Delta}(s) < 0$. By integrating (19) from s to t, we have

$$w(t) - w(s) \leq -\int_s^t \frac{\Delta\xi}{p(\xi)} - \int_s^t q(\xi)w(\xi)w^{\sigma}(\xi)\Delta\xi.$$

Due to $w > 0$ and $w^{\Delta} < 0$ and assuming $t \to \infty$, we have

$$\int_s^{\infty} \frac{\Delta\xi}{p(\xi)} \leq w(s) - \int_s^{\infty} q(\xi)w(\xi)w^{\sigma}(\xi)\Delta\xi. \tag{20}$$

By multiplying (20) by $\int_{s_1}^{s} q(\xi)\Delta\xi$, we obtain

$$\begin{aligned}\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \quad &\leq \quad w(s)\int_{s_1}^s q(\xi)\Delta\xi \\ &\quad -\left(\int_{s_1}^s q(\xi)\Delta\xi\right)\left(\int_s^\infty q(\xi)w(\xi)w^\sigma(\xi)\Delta\xi\right). \qquad (21)\end{aligned}$$

By integrating (1) and using the facts that $\left(px^\Delta\right)^\Delta(s) < 0$ and $x^\Delta(s) < 0$, we achieve that

$$\begin{aligned} p(s)x^\Delta(s) \quad &\leq \quad p(s)x^\Delta(s) - p(s_1)x^\Delta(s_1) \\ &= \quad -\int_{s_1}^s q(\xi)x(\xi)\Delta\xi \\ &\leq \quad -x(s)\int_{s_1}^s q(\xi)\Delta\xi, \end{aligned}$$

which implies

$$0 \leq W := \liminf_{s\to\infty}\left\{w(s)\int_{s_1}^s q(\xi)\Delta\xi\right\} \leq 1.$$

Therefore, for any $\varepsilon > 0$, there is $s_2 \in [s_1,\infty)_{\mathbb{T}}$ such that, for $s \in [s_2,\infty)_{\mathbb{T}}$,

$$\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \geq P - \varepsilon \quad \text{and} \quad w(s)\int_{s_1}^s q(\xi)\Delta\xi \geq W - \varepsilon, \qquad (22)$$

where

$$P := \liminf_{s\to\infty}\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right).$$

According to (21) and (22), it follows that

$$\begin{aligned} P \quad &\leq \quad \varepsilon + w(s)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \\ &\quad -(W-\varepsilon)^2\left(\int_{s_1}^s q(\xi)\Delta\xi\right)\int_s^\infty \frac{q(\xi)}{\left(\int_{s_1}^\xi q(\zeta)\Delta\zeta\right)\left(\int_{s_1}^{\sigma(\xi)} q(\zeta)\Delta\zeta\right)}\Delta\xi \\ &= \quad \varepsilon + w(s)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \\ &\quad -(W-\varepsilon)^2\left(\int_{s_1}^s q(\xi)\Delta\xi\right)\int_s^\infty \left(\frac{-1}{\int_{s_1}^{\cdot} q(\zeta)\Delta\zeta}\right)^\Delta(\xi)\Delta\xi \\ &= \quad \varepsilon + w(s)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) - (W-\varepsilon)^2, \qquad (23)\end{aligned}$$

due to $\int_{s_1}^s q(\xi)\Delta\xi \to \infty$ as $s \to \infty$. Take the lim inf of both sides of (23) as $s \to \infty$, yielding

$$P \leq \varepsilon + W - (W-\varepsilon)^2.$$

By means of $\varepsilon > 0$ being arbitrary, we see that

$$P \leq W - W^2 \leq \frac{1}{4}.$$

This is a contradiction with (16). This completes the proof. □

Theorem 2. *If* (2) *holds and for sufficiently large* $T \in [s_0, \infty)_{\mathbb{T}}$,

$$\limsup_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_T^s q(\xi)\Delta\xi\right)\right\} > 1, \tag{24}$$

then (1) *oscillates.*

Proof. Assume that x is a nonoscillatory solution of (1) on $[s_0, \infty)_{\mathbb{T}}$. Let $x(s) > 0$ and $x(\tau(s)) > 0$ hold on $[s_0, \infty)_{\mathbb{T}}$, without a loss of generality. By (17), px^Δ is strictly decreasing on $[s_0, \infty)_{\mathbb{T}}$. This yields that $x^\Delta(s)$ is eventually of one sign. Hence, there are two possibilities:

(a) $x^\Delta(s) > 0$ eventually;
(b) $x^\Delta(s) < 0$ eventually.

If (a) is satisfied, then the proof is identical to Case (a) in Theorem 1, so it is eliminated.

If (b) is satisfied, there is $s_1 \in [s_0, \infty)$ such that $x^\Delta(s) < 0$ on $[s_1, \infty)$. By integrating (1) from s_1 to s, we obtain

$$\begin{aligned} p(s)x^\Delta(s) &\leq p(s)x^\Delta(s) - p(s_1)x^\Delta(s_1) = -\int_{s_1}^s q(\xi)x(\tau(\xi))\Delta\xi \\ &\leq -x(s)\int_{s_1}^s q(\xi)\Delta\xi. \end{aligned}$$

It follows that

$$p(\xi)x^\Delta(\xi) \leq p(s)x^\Delta(s) \leq -x(s)\int_{s_1}^s q(\xi)\Delta\xi \tag{25}$$

for $\xi \in [s, \infty)_{\mathbb{T}}$ and $s \in [s_1, \infty)_{\mathbb{T}}$. For $t \in [s, \infty)_{\mathbb{T}}$, we see

$$-x(s) \leq x(t) - x(s) = \int_s^t \frac{p(\xi)x^\Delta(\xi)}{p(\xi)}\Delta\xi. \tag{26}$$

Substituting (25) into (26), we arrive at

$$-x(s) \leq -x(s)\left(\int_{s_1}^s q(\xi)\Delta\xi\right)\left(\int_s^t \frac{\Delta\xi}{p(\xi)}\right),$$

so

$$\left(\int_s^t \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \leq 1.$$

Assuming $t \to \infty$, we obtain

$$\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right) \leq 1.$$

Consequently, we achieve that

$$\limsup_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\xi}{p(\xi)}\right)\left(\int_{s_1}^s q(\xi)\Delta\xi\right)\right\} \leq 1,$$

which is a contradiction with (24). This completes the proof. □

The examples that follow exemplify applications of the theoretical findings presented in this paper.

Example 1. *Consider the linear delay second-order dynamic equation*

$$\left[s^{\beta+1}x^\Delta(s)\right]^\Delta + \alpha\sigma^{\beta-1}(s)x(\tau(s)) = 0 \quad \text{for } s \in [s_0, \infty)_{\mathbb{T}}, \tag{27}$$

where $\beta \geq 1$ and $\alpha > 0$ are constants. It is obvious that (2) holds since

$$\int_{s_0}^{\infty} \frac{\Delta\xi}{\xi^{\beta+1}} < \infty,$$

for those time scales $[s_0, \infty)_{\mathbb{T}}$, where $\int_{s_0}^{\infty} \frac{\Delta\xi}{\xi^{\lambda}} < \infty$ when $\lambda > 1$. This is satisfied for several time scales (see [10], Theorems 5.64 and 5.65 and see [10], Example 5.63, where this result is not satisfied). Note that

$$\begin{aligned} &\liminf_{s\to\infty}\left(\int_s^{\infty} \frac{\Delta\xi}{p(\xi)}\right)\left(\int_T^s q(\xi)\Delta\xi\right) \\ = \;& \alpha \liminf_{s\to\infty}\left(\int_s^{\infty} \frac{\Delta\xi}{\xi^{\beta+1}}\right)\left(\int_T^s \sigma^{\beta-1}(\xi)\Delta\xi\right) \\ \geq \;& \frac{\alpha}{\beta^2} \liminf_{s\to\infty}\left(\int_s^{\infty} \left(\frac{-1}{\xi^{\beta}}\right)^{\Delta}\Delta\xi\right)\left(\int_T^s \left(\xi^{\beta}\right)^{\Delta}\Delta\xi\right) = \frac{\alpha}{\beta^2}. \end{aligned}$$

We conclude that if $[s_0, \infty)_{\mathbb{T}}$ is a time scale where $\int_{s_0}^{\infty} \frac{\Delta\xi}{\xi^{\lambda}} < \infty$ when $\lambda > 1$, then, according to Theorem 1, (27) oscillates if $\alpha > \frac{\beta^2}{4}$.

Example 2. *Consider the linear delay second-order dynamic equation*

$$\left[s\sigma(s)\, x^{\Delta}(s)\right]^{\Delta} + \alpha x(\tau(s)) = 0 \quad \text{for } s \in [s_0, \infty)_{\mathbb{T}}, \tag{28}$$

where $\alpha > 0$ is a constant. It is evident that (2) is satisfied since

$$\int_{s_0}^{\infty} \frac{\Delta\xi}{\xi\sigma(\xi)} = \int_{s_0}^{\infty} \left(\frac{-1}{\xi}\right)^{\Delta}\Delta\xi < \infty.$$

We have

$$\begin{aligned} &\limsup_{s\to\infty}\left(\int_s^{\infty} \frac{\Delta\xi}{p(\xi)}\right)\left(\int_T^s q(\xi)\Delta\xi\right) \\ = \;& \alpha \limsup_{s\to\infty}\left((s-T)\int_s^{\infty} \frac{\Delta\xi}{\xi\sigma(\xi)}\right) \\ = \;& \alpha \limsup_{s\to\infty}\left((s-T)\int_s^{\infty} \left(\frac{-1}{\xi}\right)^{\Delta}\Delta\xi\right) = \alpha. \end{aligned}$$

According to Theorem 1, this implies that (28) oscillates if $\alpha > 1$.

3. Discussion and Conclusions

The results obtained in this paper apply to all time scales without restrictive conditions, such as $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, etc. (see [10]). And these results, in contrast to previous results in the literature, do not assume the fulfillment of condition (3) (canonical case) and therefore solve an open problem mentioned in many papers (see [19]). Furthermore, it would be interesting to find such criteria for half-linear dynamic equations of the form

$$\left(p\left|x^{\Delta}\right|^{\alpha-1}x^{\Delta}\right)^{\Delta}(s) + q(s)|x(\tau(s))|^{\alpha-1}x(\tau(s)) = 0,$$

where $\alpha > 0$ is a constant.

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