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NUMERICAL RECONSTRUCTION OF A SPACE-DEPENDENT SOURCE TERM FOR MULTIDIMENSIONAL SPACE-TIME FRACTIONAL DIFFUSION EQUATIONS

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Abstract. In this paper, we consider the problem of identifying the unknown source function in the time-space fractional diffusion equation from the final observation data. An implicit difference technique is proposed in conjunction with the matrix transfer scheme for approximating the solution of the direct problem. The challenge pertains to an inverse scenario encompassing a nonlocal ill-posed operator. The problem under investigation is formulated as a regularized optimization problem with a least-squares cost function minimization objective. An approximation for the source function is obtained using an iterative non-stationary Tikhonov regularization approach. Three numerical examples are reported to verify the efficiency of the proposed schemes.

Key words: Time-space fractional equation, Inverse problem, Fractional Laplacian, Iterative non-stationary Tikhonov regularization.

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1. INTRODUCTION

In this paper, we consider the problem of identifying the unknown source function $f(x)$ in the following time-space fractional diffusion equation from the terminal observation data $y(x,T)=\mathcal{O}(x)$:

$$\begin{cases} \partial_t^{\alpha} y + (-\Delta)^{\frac{s}{2}} y &= fg \quad \text{in} \quad \Theta\times(0,T], \\ y &= 0 \quad \text{in} \quad \mathbb{R}^d\setminus\Theta\times(0,T], \\ y(\cdot,0) &= y_0 \quad \text{in} \quad \Theta, \end{cases} \tag{1}$$

where

- The bounded open set $\Theta\subset\mathbb{R}^d (d=1,2,3)$ and the source term $f\in L^2(\Theta)$,

$g \in L^{\infty}(0,T)$, the initial value $y_0 \in L^2(\Theta)$, and the terminal time $T > 0$.

- The time-Caputo fractional derivative $\partial_t^{\alpha} y$ of order $\alpha \in (0,1)$ is defined as:

$$\partial_t^{\alpha} y = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} y'(w)\, dw, \tag{2}$$

where $\Gamma(z) = \displaystyle\int_0^{\infty} s^{z-1} \exp(-s)\, ds$ is the Γ function.

- The fractional operator $(-\Delta)^{\frac{s}{2}}$ is defined in the normal way, which can be represented by Laplace operator in terms of spectral decomposition for $s \in (1,2)$.

Let $\{\psi_k, \lambda_k\}$ be the corresponding eigenvectors and eigenvalues of the Laplacian operator $-\Delta$:

$$\begin{cases} -\Delta\psi_k & = \lambda_k \psi_k \text{ in } \Theta, \\ \psi_k & = 0 \text{ in } \partial\Theta. \end{cases}$$

Consider

$$\mathcal{G}_{\theta} = \left\{ f = \textstyle\sum_{i=1}^{\infty} b_i \psi_i, b_n = (f, \psi_i) \Big| \sum_{i=1}^{\infty} \Big| b_n \Big|^2 |\lambda_i|^{\theta} < \infty, \theta = \max(0,s) \right\}.$$

Hence, we can write

$$(-\Delta)^{\frac{s}{2}} = \sum_{i=1}^{\infty} b_i \lambda_i^{\frac{s}{2}} \psi_i.$$

The eigenvalues λ_i and eigenfunctions ψ_i depend on the boundaries and the geometry of the domain Θ. For example, in a rectangular domain, the eigenvalues and eigenfunctions can be found analytically using the separation of variables. In more complex geometries, numerical methods such as finite element analysis or spectral methods may be used to compute the eigenfunctions and the eigenvalues. The spectral decomposition of the Laplacian operator is a fundamental concept in the study of partial differential equations and plays an essential function in several disciplines of physics, engineering, and applied mathematics. It provides a powerful tool for solving boundary and eigenvalue problems and understanding the behavior of solutions to differential equations in different domains.

Fractional calculus and fractional differential equations have become vital tools for modeling a wide range of phenomena in many fields, see for example [1–7]. It is critical to find approximate and analytical solutions to these equations [8–11]. Moreover, the inverse problem for fractional differential equations has significant applications in many fields of applied sciences. In recent decades, numerous analytical and numerical techniques for solving various types of partial differential problems have been published [12–16]. The ability to identify unknown source terms from measurements or observations is crucial for understanding and modeling complex processes with memory effects and long-range interactions. Recently, several aspects of the inverse source problem have been investigated on a large scale. For example, Tatar *et al.* [17] studied space-dependent source determination on a space-time fractional inverse problem. Tuan and Long [18] discussed the use of the Fourier truncation technique for a time-space fractional inverse source problem for diffusion equation. However, to the best of our knowledge, there is no numerical study for this model. In [19] Dou and Hon developed a numerical strategy based on a kernel-based approximation technique for addressing a backward time-space fractional diffusion equation. Wei *et al.* [20] investigated the identification of a time-dependent source component using initial and boundary data as well as extra measurement data at an inner location in a space-time fractional diffusion problem.

In this work, we provide a finite difference approach in conjugation with the matrix transfer methodology for approximating the solution of the space-time fractional-order direct diffusion equation in one- and two-dimensional cases. Then, the Tikhonov regularization approach is used to solve the inverse source equation.

The paper is organized as follows: In Sec. 1, the problem formulation is provided, giving an overview of the research topic. In Sec. 2, a finite difference approach is provided in conjugation with the matrix transfer methodology for approximating the solution of the space-time fractional-order direct diffusion equation in one- and two-dimensions. In Sec. 3 the identification approach is constructed based on an iterative non-stationary Tikhonov regularization technique for the inverse source equation with a detailed description of the approach. In Sec. 4, the numerical results are reported showing the practical implications of the scheme. Finally, Sec. 5 draws conclusions summarizing the key findings.

2. NUMERICAL SCHEME OF THE DIRECT PROBLEM

In this Section, we construct the difference scheme for problem (1). Let $x_i = ih(i = 0.1, \cdots, N)$, where $h = 1/N$ is a step over space, and likewise $t_j = j\delta t(j = 0, 1, \cdots, M)$, with $\delta t = T/M$ is the grid step over time.

We begin with the following standard equation [21, 22]:

$$\begin{cases} \partial_t y - \partial_{xx} y &= f(x)\, g(t) \quad \text{in} \quad \Theta \times (0,T], \\ y &= 0 \quad\quad\quad\;\; \text{in} \quad \mathbb{R}^d \setminus \Theta \times (0,T], \\ y(\cdot, 0) &= z \quad\quad\quad\;\; \text{in} \quad \Theta. \end{cases} \tag{3}$$

Using the finite difference discretization, we get

$$\frac{dy_i}{dt} + \frac{1}{h^2}(-y_{i-1} + 2y_i - y_{i+1}) = g(t)\, f_i,$$

$$y_0 = y_N = 0,$$

$$y_i = z_i.$$

We can express equation (3) as a differential equation in the following form:

$$\frac{dY}{dt} + \zeta AY = g(t)\,F,$$

with $\zeta = \frac{1}{h^2}$, $Y = (y_1, y_2, \cdots, y_{N-1})^T$, $F = (f_1, f_2, \cdots, f_{N-1})^T$, $Y^0 = (z_1, z_2, \cdots, z_{N-1})^T$,

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & 1 & & & \\ & -1 & 2 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)}.$$

Hence

$$A = P\Psi P^{-1},$$

where P denotes the orthogonal matrix and the eigenvalues $\Psi = diag(\lambda_1, \lambda_2, \cdots, \lambda_{N-1})$ are for the matrix A. Thus, the problem (3) can be represented as:

$$\partial_t^\alpha Y + \bar{\zeta} A^{\frac{s}{2}} Y = g(t)\,f,$$

where $\bar{\zeta} = \frac{1}{h^\alpha}$, $A^{\frac{s}{2}} = P\Psi^{\frac{s}{2}}P^{-1}$. Hence, the discretization of the fractional derivative in time is given by:

$$\partial_t^\alpha y(x,t_j) = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big(b_0^{(\alpha)} y(x,t_j) - \sum_{k=1}^{j-1}\Big(b_{j-k-1}^{(\alpha)} - b_{j-k}^{(\alpha)}\Big) y(x,t_k) - b_{j-1}^{(\alpha)} y(x,t_0)\Big), \tag{4}$$

where $b_l^{(\alpha)} = (1+l)^{1-\alpha} - l^{1-\alpha}, l \geq 0$. Let $w_k = b_{j-k-1}^{(\alpha)} - b_{j-k}^{(\alpha)}$, then we have

$$\begin{cases} \partial_t^\alpha y_1^j & = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big(b_0^{(\alpha)} y_1^j - \sum_{k=1}^{j-1} w_k^{(\alpha)} y_1^k - b_{j-1}^{(\alpha)} y_1^0\Big) \\ \partial_t^\alpha y_2^j & = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big(b_0^{(\alpha)} y_2^j - \sum_{k=1}^{j-1} w_k^{(\alpha)} y_2^k - b_{j-1}^{(\alpha)} y_2^0\Big) \\ \cdots & \cdots \\ \partial_t^\alpha y_{N-1}^j & = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big(b_0^{(\alpha)} y_{N-1}^j - \sum_{k=1}^{j-1} w_k^{(\alpha)} y_{N-1}^k - b_{j-1}^{(\alpha)} y_{N-1}^0\Big). \end{cases} \tag{5}$$

Hence, we can write

$$BY^j = a,$$

with $Y^j = (y_1^j, y_2^j, \cdots, y_{N-1}^j)$, $B = b_0^{(\alpha)} \frac{\delta t^{-\beta}}{\Gamma(2-\beta)} I_{(N-1)\times(N-1)} + \bar{\zeta} A^{\frac{s}{2}}$ and

$$a = \frac{\delta t^{-\beta}}{\Gamma(2-\beta)} \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^{j-1} \\ y_2^1 & y_2^2 & \cdots & y_2^{j-1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{N-1}^1 & y_{N-1}^2 & \cdots & y_{N-1}^{j-1} \end{pmatrix} \begin{pmatrix} w_1^{(\alpha)} \\ w_2^{(\alpha)} \\ \cdots \\ w_{N-1}^{(\alpha)} \end{pmatrix} + b_{j-1}^{(\alpha)} \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_{N-1} \end{pmatrix} + Fg(t_j).$$

3. IDENTIFICATION APPROACH

In this part, we propose a non-stationary iterative Tikhonov regularization technique to identify the source function $f(x)$ using further data. Let f^0 be the initial state and f^k be the iterative value of the k-th step, hence

$$f^{k+1} = \frac{1}{2}\arg\min J_\eta = \left|\left|y(x,T) - \mathcal{O}(x)\right|\right|_{L^2(\Theta)} + \frac{\eta_{k+1}}{2}\left|\left|f - f^k\right|\right|_{L^2(\Theta)}. \tag{6}$$

Let

$$\Phi = span\{\psi_1, \psi_2, \cdots, \psi_s\}.$$

Then we can write

$$f(x) = \sum_{k=1}^{s} f^k(x)\psi_k(x).$$

This is to identify an approximation

$$\hat{f}(x) \in \Phi_s,$$

with a vector

$$f = (f^1, f^2, \cdots, f^s) \in \mathbb{R}^S.$$

Now, we propose an algorithm to identify the source function which depends on the space Θ. For any given $f^k \in \mathbb{R}^S$. Pose

$$f^{k+1} = f^k + \delta f^k,$$

with δf^k is a small perturbation of f^k. Hence, it suffices to obtain an optimal perturbation δf^k to get f^{k+1} from the given f^k. By the linearity of $y[f+\delta f](x,T)$ at f, we can get

$$y[f+\delta f](x,T) = y[f](x,T) + \delta f \mathcal{F}_f^T, \tag{7}$$

with $\mathcal{F}_f = y[e_1](x,T), \cdots, y[e_s](x,T)$, and $e_i = (0,0,\cdots,1,\cdots,0)$ for $i = 0,\cdots,s$. Thus using (7) we can write (6) in the following form, Find

$$\min_{f\in\mathbb{R}^s}\left(\left|\left|y(x,T) - \mathcal{O}(x) + \delta f \mathcal{F}_f^T\right|\right|^2_{L^2(\Theta)} + \frac{\eta_{k+1}}{2}\left|\left|\delta f C \delta f^T\right|\right|^2_{L^2(\Theta)}, \tag{8}$$

where $C(\psi_i, \psi_j)_{s\times s}$. Minimizing (6) implies solving the following problem

$$(\eta_{k+1}C + D)\delta f^T = H, \tag{9}$$

wher $D = (\mathcal{F}_i, \mathcal{F}_j)_{s\times s}$ and $H = (\mathcal{O}(x) - y[f](x,T), \mathcal{F}_j)_{s\times 1}$. So the iteration steps are then

$$f^{k+1} = f^k + \delta f^k, \tag{10}$$

until arriving at a stopping principle.

4. RESULTS

In this Section, we provide three numerical examples in multidimensions as an application of the proposed scheme. In our implementation procedure, the domain Θ is considered as $(0,1)$ for the one-dimensional case and $(0,1)\times(0,1)$ for two-dimensional, and the final time $T=1$. We take the temporal grid size and the spacial grid with $\delta t = \frac{1}{100}$ and $h = \frac{1}{100}$, respectively.

The noise is created by introducing the following random perturbation:

$$\mathcal{O}^{\epsilon} = \mathcal{O} + \epsilon\mathcal{O}.(2.rand(size(\mathcal{O})-1),$$

where the equivalent level of noise is computed by $\epsilon = \|\mathcal{O}^{\epsilon} - \mathcal{O}\|_{L^2(\Theta)}$. The following error function will be used to demonstrate the accuracy of the proposed scheme

$$e_m = \|f - f_m\|_{L^2(\Theta)},$$

where f_m is the reconstructed source function at m-th iteration and f is the exact solution.

4.1. THE ONE-DIMENSIONAL CASE

Example 1. *For this case, we take the basis function space* $\Phi_S = span\{1, \sqrt{2}\cos(\pi x), \cdots, \sqrt{2}\cos((s-1)\pi x)\}$. *In the first example, our objective is to identify the following example,*

$$f(x) = x^4 + x\sin(\pi x).$$

Figure 1 depicts the reconstruction findings.

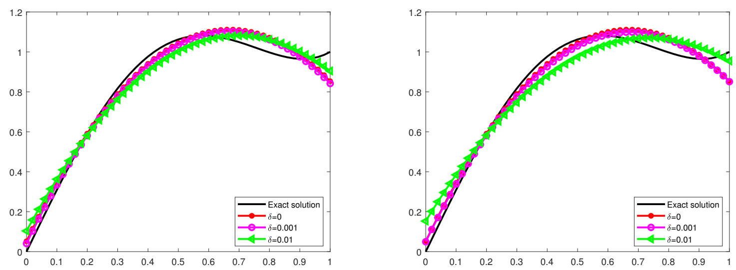

Figure 1 – Reconstruction outcomes for Example 1 with (a) $\alpha = 0.4, s = 1.2$ and (b) $\alpha = 0.7, s = 1.7$.

Example 2. *Now we apply our algorithm to identify the nonsmooth following function*

$$f(x) = \begin{cases} 1, & 0 \le x \le 0.1, \\ 1 + 2.5(x-0.1), & 0.1 \le x \le 0.5, \\ 2 + 2.5(0.5-x), & 0.5 \le x \le 0.9, \\ 1, & 0.9 \le x \le 1. \end{cases}$$

The numerical outcomes of the function, with various noise levels are depicted in Fig. 2.

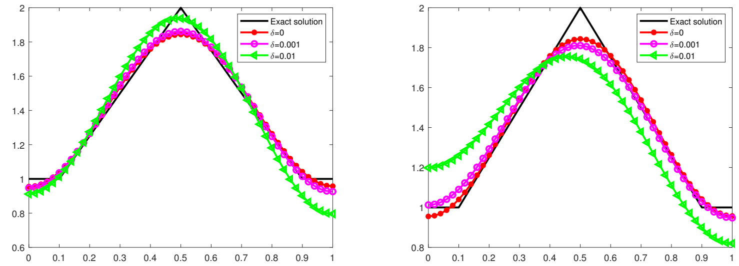

Figure 2 – Reconstruction results for Example 2 with (a) $\alpha = 0.4, s = 1.2$ and (b) $\alpha = 0.7, s = 1.7$.

4.2. THE TWO-DIMENSIONAL CASE

For this case we choose $x = (x_1, x_2)$ $\delta t = 1/100$, $\delta x_1 = 1/100$, $\delta x_2 = 1/100$, and $\Phi^{s_1 \times s_2} = \{\psi_i(x_1)\psi_j(x_2)\}$ with

$$\psi_i(x_1) = \begin{cases} 1 & \text{if} \quad i = 1, \\ \sqrt{2}\cos((i-1)\pi x), & i > 1, \end{cases}$$

and

$$\psi_i(x_2) = \begin{cases} 1 & \text{if} \quad j = 1, \\ \sqrt{2}\cos((j-1)\pi x), & j > 1, \end{cases}$$

$i = 1, 2, \cdots, S_1$, $j = 1, 2, \cdots, S_2$.

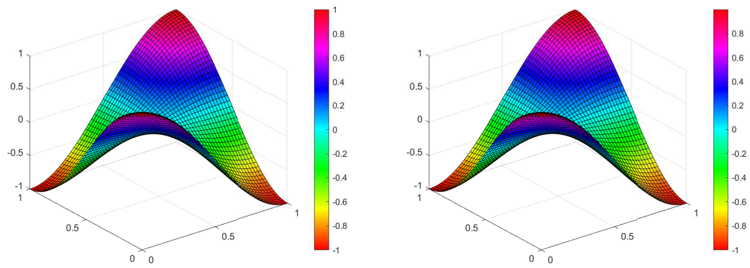

Figure 3 – Reconstruction outcomes, for Example 3 (a) the true solution and (b) the approximate solution.

Example 3. *We consider the following function*

$$f(x_1, x_2) = \cos(\pi x_1)\cos(\pi x_2). \tag{11}$$

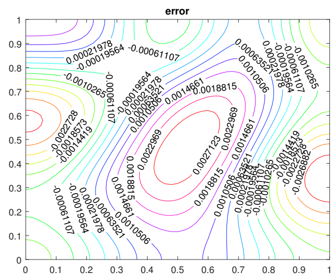

Figure 4 – The absolute true errors for Example 3.

Figures 3 and 4 display the reconstructed source functions, the true source function, and related absolute true errors for $\alpha = 0.5$ *and* $s = 1.2$ *when the level of the noise is* $epsilon = 0.01$.

The exact source solutions, reconstructed source terms, and the absolute errors between the exact source functions and the numerical solutions for $\alpha = 0.5$ *and* $s = 1.2$ *are shown in Figs. 3 and 4 by taking withe noise level* $\epsilon = 0.01$.

5. CONCLUSION

The problem of identifying the unknown source function in the time-space fractional diffusion equation from the final observation data has been considered. An iterative non-stationary Tikhonov regularization scheme coupled with a Tikhonov regularization is used.

The proposed method is evaluated in the simulation phase by numerically reconstructing three separate examples in the one- and two-dimensional cases. These simulations demonstrate the accuracy and efficiency of the proposed method.

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