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Research Article

Oscillatory Behavior of Noncanonical Quasilinear Second-Order Dynamic Equations on Time Scales

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The objective of this article is to examine the oscillatory behavior of a class of quasilinear second-order dynamic equations on time scales. Our focus will be on the noncanonical case, which has received relatively less attention compared to the more commonly studied canonical dynamic equations. Our approach involves transforming the noncanonical equation into a corresponding canonical equation. By utilizing this transformation and a range of techniques, we develop new, more efficient, and precise oscillation criteria. Finally, we demonstrate the significance and usefulness of our results by applying them to specific cases within the equation being studied.

1. Introduction

The origins of studying dynamic equations on time scales can be traced back to its founder Hilger[1], and it has since evolved into a significant area of mathematics. The purpose of this theory is to bring together the study of differential and difference equations. In recent times, there have been discussions about various theoretical aspects of this theory. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In order to have a comprehensive understanding, it is necessary to review some of the basic concepts of time-scale theory. The forward and backward jump operators σ, ρ: $\mathbb{T} \longrightarrow \mathbb{T}$ are defined by

$$\begin{aligned}\sigma(\zeta) &= \inf\{s \in \mathbb{T} | s > \zeta\} \text{ and}\\ \rho(\zeta) &= \sup\{s \in \mathbb{T} | s < \zeta\},\end{aligned} \tag{1}$$

supplemented by $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. A point $\zeta \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, and left-dense, if $\sigma(\zeta) > \zeta, \sigma(\zeta) = \zeta, \rho(\zeta) < \zeta, \rho(\zeta) = \zeta$ holds, respectively. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise, it is $\mathbb{T}$ without this left-scattered maximum. The graininess function μ: $\mathbb{T} \longrightarrow [0, \infty)$ is defined by $\mu(\zeta) = \sigma(\zeta) - \zeta$. Hence, the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant ζ for $\mathbb{T} = \mathbb{Z}$. However, a time scale $\mathbb{T}$ could have nonconstant graininess. A function h: $\mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous and is written $h \in C_{rd}(\mathbb{T}, \mathbb{R})$, provided that h is continuous at right-dense points, and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. We say that h: $\mathbb{T} \longrightarrow \mathbb{R}$ is differentiable at $\zeta \in \mathbb{T}$ whenever

$$h^\Delta \coloneqq \lim_{s \longrightarrow \zeta} \frac{h(\zeta) - h(s)}{\zeta - s}, \tag{2}$$

exists when $\sigma(\zeta) = \zeta$ (hereby, $s \longrightarrow \zeta$ it is understood that s approaches ζ in the time scale), and when h is continuous at ζ and $\sigma(\zeta) > \zeta$, it is

$$h^{\Delta} := \lim_{s \longrightarrow \zeta} \frac{h(\sigma(\zeta)) - h(\zeta)}{\mu(\zeta)}. \tag{3}$$

The product and quotient rules ([4], Theorem 1.20) for the derivative of the product hk and the quotient h/k of two differentiable functions h and k are as follows:

$$\begin{aligned} (hk)^{\Delta}(\zeta) &= h^{\Delta}(\zeta)k(\zeta) + h(\sigma(\zeta))k^{\Delta} = h(\zeta)k^{\Delta}(\zeta) + h^{\Delta}(\zeta)k(\sigma(\zeta)), \\ \left(\frac{h}{k}\right)^{\Delta}(\zeta) &= \frac{h^{\Delta}(\zeta)k(\zeta) - h(\zeta)k^{\Delta}(\zeta)}{k(\zeta)k(\sigma(\zeta))}. \end{aligned} \tag{4}$$

The chain rule ([4], Theorem 1.90) for the derivative of the composite function $h \circ k$ of a continuously differentiable function $h: \mathbb{R} \longrightarrow \mathbb{R}$ and a delta differentiable function $k: \mathbb{T} \longrightarrow \mathbb{R}$ results in

$$(h \circ k)^{\Delta} = \left\{ \int_0^1 h'\left(k + s\mu k^{\Delta}\right) ds \right\} g^{\Delta}. \tag{5}$$

For a great introduction to the fundamentals of time scales, see [2–4].

In this work, we investigate the oscillatory properties of solutions to the noncanonical second-order dynamic equations of the following form:

$$\left[r(\zeta)\left(x^{\Delta}(\zeta)\right)^{\alpha}\right]^{\Delta} + q(\zeta)x^{\beta}(\tau(\zeta)) = 0, \zeta \in [\zeta_0, \infty)_{\mathbb{T}}. \tag{6}$$

The following assumptions will be needed throughout the paper:

(H1) $\alpha \geq 1$ and β are ratios of odd positive integers such that $\alpha < \beta + 1$;

(H2) $r \in C_{rd}([\zeta_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that

$$\int_{\zeta_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} < \infty. \tag{7}$$

(H3) $q \in C_{rd}([\zeta_0, \infty)_{\mathbb{T}}, [0, \infty))$ such that $q \equiv 0$;

(H4) $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ is nondecreasing such that $\tau(\zeta) \leq \zeta$ and $\lim_{\zeta \longrightarrow \infty} \tau(\zeta) = \infty$.

A solution $x(\zeta)$ of (6) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Equation (6) is said to be oscillatory if all its solutions oscillate. Following Trench [5], we shall say that equation (6) is in canonical form if

$$\eta(\zeta) = \int_{\zeta_0}^{\zeta} \frac{\Delta s}{r^{1/\alpha}(s)}; \ \eta(\zeta) \longrightarrow \infty \text{ as } \zeta \longrightarrow \infty. \tag{8}$$

Conversely, we say that (6) is in noncanonical form if

$$\xi(\zeta_0) := \int_{\zeta_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} < \infty. \tag{9}$$

Oscillation phenomena arise in a variety of models based on real-world applications. Understanding and predicting oscillatory behavior is important for designing effective control strategies, optimizing performance, and improving predictions. As a result, the study of oscillation phenomena is a key area of research in many fields, and it continues to play a central role in the study of dynamic equations on time scales. For instance, when $\mathbb{T} = \mathbb{Z}$ and $\alpha = \beta$, Chatzarakis et al. [6] obtained new oscillation criteria for the half-linear retarded difference equation

$$\Delta(r(\zeta)\Delta(x(\zeta)))^{\alpha} + q(\zeta)x^{\alpha}(\tau(\zeta)) = 0, \tag{10}$$

where $\tau(\zeta) \leq \zeta$ in the noncanonical form via canonical transformation. In [7], authors studied the oscillatory behavior of the second-order quasilinear retarded difference equation

$$\Delta(r(\zeta)\Delta(x(\zeta)))^{\alpha} + q(\zeta)x^{\beta}(\tau(\zeta)) = 0, \tag{11}$$

where $\alpha \geq 1$ and β are ratios of odd positive integers such that $\beta > \alpha - 1$ under the condition $\sum_{\zeta=\zeta_0}^{\infty} r^{-1/\alpha}(\zeta) < \infty$. Newly, Grace [8] studied some new criteria for the oscillation of the nonlinear second-order delay difference equations of the following form:

$$\Delta\left(r(\zeta)(\Delta x(\zeta))^{\alpha}\right) + q(\zeta)x^{\beta}(\zeta - m + 1) = 0, \tag{12}$$

where α and β are ratios of positive odd integers and $m \geq 1$ is a positive integer under the condition $\sum_{s=\zeta_0}^{\zeta-1} r^{-1/\alpha}(s) \longrightarrow \infty$ as $t \longrightarrow \infty$. Grace provided some new oscillation criteria for (12) via comparison with a second-order linear difference equation or a first-order linear delay difference equation whose oscillatory behavior is discussed intensively in the literature.

As a special case of (6) for $\mathbb{T} = \mathbb{R}$ and $\alpha = \beta = 1$, Baculikova [9] obtained the oscillation of the delay and advanced differential equation

$$\left(r(\zeta)(x(\zeta))'\right)' + q(\zeta)x(\tau(\zeta)) = 0, \tag{13}$$

in the noncanonical form. With fewer restrictions than recently used, Džurina and Jadlovská [10] examined the oscillatory behavior of solutions of the delay differential equation

$$\left(r(\zeta)\left((x(\zeta))'\right)^{\alpha}\right)' + q(\zeta)x^{\alpha}(\tau(\zeta)) = 0, \tag{14}$$

where $\alpha > 0$ is a quotient of odd positive integers and $\int^{\infty} r^{-1/\alpha}(s)ds < \infty$.

Recently, Grace et al. [11] obtained some new unified oscillation criteria for solutions of (6) when $\alpha = \beta$ and $\tau(\zeta) = \zeta$ in the canonical and noncanonical forms. Very recently, Graef et al. [12] established some new oscillation criteria for the solutions of (6) when $\alpha = 1$, $\beta \in (0, 1]$ is a ratio of odd positive integers in the noncanonical form with the restriction

$$\int_{\zeta_0}^{\infty} \xi(s)q(s)\Delta s = \infty. \tag{15}$$

In recent times, numerous researchers have shown great interest in examining the oscillations of specific instances of equation (6) (see [11–14]). As a special case for $\mathbb{T} = \mathbb{R}$, see [15–21], and for $\mathbb{T} = \mathbb{Z}$, see [22–24]. Using techniques such as integral averaging, generalized Riccati transformations, and Kneser-type oscillation, the authors were able to achieve the oscillation criteria in both canonical and noncanonical cases. Nonoscillatory solutions (which become positive at some point) for equation (6) in canonical form have a structure where they maintain the same sign throughout and their derivative $x^{\Delta}(\zeta)$ becomes positive eventually. On the other hand, for the noncanonical equation, the derivative $x^{\Delta}(\zeta)$ may become eventually positive or negative. A commonly used approach in the literature is to analyze such equations to apply previous findings to canonical equations. However, this technique has some disadvantages, such as imposing additional conditions or not ensuring that all solutions are oscillatory (for details, see [25]).

Given this context, our objective is to introduce novel adequate criteria that guarantee the oscillatory nature of all solutions to equation (6). By identifying an appropriate transformation that can convert the equation (6) from the noncanonical form to a canonical form, we can proceed with our analysis. The outcomes derived in this manuscript enhance and supplement the preexisting literature's findings, even for the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

2. Preliminary Results

In what follows, we need only to consider the eventually positive solutions of equation (6), since if x satisfies equation (6), then $-x$ is also its solution. Without loss of generality, we only give proofs for the positive solutions. For brevity and clarity, let

$$\begin{aligned} a(\zeta) &= r^{1/\alpha}(\zeta)\xi(\sigma(\zeta))\xi(\zeta) \\ g(\zeta) &= \frac{1}{\alpha}q(\zeta)\xi^{\alpha-1}(\zeta)\xi(\sigma(\zeta))\xi^{1+\beta-\alpha}(\tau(\zeta)) \\ A(\zeta) &= \int_{\zeta_0}^{\zeta}\frac{\Delta s}{a(s)}. \end{aligned} \tag{16}$$

The following results will be crucial in proving our main results.

Lemma 1 (see [12]). *Let $x(\zeta)$ be an eventually positive solution of (6). Then, $x(t)$ satisfies one of the following two cases for all sufficiently large t:*

(i) $x(\zeta) > 0$, $r(\zeta)(x^{\Delta}(\zeta))^{\alpha} > 0$, and $(r(\zeta)(x^{\Delta}(\zeta))^{\alpha})^{\Delta} \le 0$;
(ii) $x(\zeta) > 0$, $r(\zeta)(x^{\Delta}(\zeta))^{\alpha} < 0$, and $(r(\zeta)(x^{\Delta}(\zeta))^{\alpha})^{\Delta} \le 0$.

Lemma 2 ([4], Theorem 1.14) (mean value theorem). *Let h be a continuous function on $[a, b]$ that is differentiable on $[a, b)$. Then, there exist $\lambda, \varsigma \in [a, b)$ such that*

$$h^{\Delta}(\lambda) \le \frac{h(b) - h(a)}{b - a} \le h^{\Delta}(\varsigma). \tag{17}$$

Theorem 3 (see [26]). *Let X be a Banach space and S be a bounded, convex, and closed subset of X. Consider two maps T_1 and T_2 of S into X such that*

(i) $T_1x + T_2y \in S$ for every pair $x, y \in S$;
(ii) T_1 is contraction mapping;
(iii) T_2 is completely continuous.

Then, the equation $T_1x + T_2x = x$ has a solution in S.

Lemma 4 (see [26]). *Suppose that $X = \{y\colon y \in C_{rd}([\zeta_0, \infty)_{\mathbb{T}}, \mathbb{R}), \sup_{\zeta\in[\zeta_0,\infty)_{\mathbb{T}}}|y(\zeta)| < \infty\}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[\zeta_0, \zeta_1]_{\mathbb{T}}$ for any $\zeta_1 \in [\zeta_0, \infty)_{\mathbb{T}}$. Then, X is relatively compact.*

Lemma 5. *Assume that $x(\zeta)$ is an eventually positive solution of (6). Then,*

$$\left(\frac{r^{1/\alpha}(\zeta)x^{\Delta}(\zeta)}{x(\tau(\zeta))}\right)^{\alpha-1} \le \xi^{1-\alpha}(\zeta). \tag{18}$$

Proof. Let $x(\zeta)$ be an eventually positive solution of (6). Then, using Lemma 1, we can see that $x^{\Delta}(\zeta) > 0$ or $x^{\Delta}(\zeta) < 0$, say for $\zeta \ge \zeta_1 \ge \zeta_0$. First, assume that $x^{\Delta}(\zeta) > 0$ for all $\zeta \ge \zeta_1$. Note that $\eta(\tau(\zeta)) + \xi(\tau(\zeta)) = \xi(\zeta_1) > 0$. This, with (9), leads to $\eta(\tau(\zeta)) \ge \xi(\zeta)$ for large $\zeta \ge \zeta_2$, for some $\zeta_2 \ge \zeta_1$. Taking into consideration the decreasing fact of $r(\zeta)(x^{\Delta}(\zeta))^{\alpha}$, we conclude that

$$\begin{aligned} x(\tau(\zeta)) &\ge \int_{\zeta_0}^{\tau(\zeta)}\frac{1}{r^{1/\alpha}(s)}\left(r^{1/\alpha}(s)x^{\Delta}(s)\right)\Delta s \\ &\ge \left(r^{1/\alpha}(\zeta)x^{\Delta}(\zeta)\right)\int_{\zeta_0}^{\tau(\zeta)}\frac{1}{r^{1/\alpha}(s)}\Delta s \\ &\ge \eta(\tau(\zeta))\left(r^{1/\alpha}(\zeta)x^{\Delta}(\zeta)\right) \\ &\ge \xi(\zeta)\left(r^{1/\alpha}(\zeta)x^{\Delta}(\zeta)\right), \end{aligned} \tag{19}$$

which is obviously equivalent to (18). Next, assume that $x^{\Delta}(\zeta) < 0$ for all $\zeta \ge \zeta_1$. Since $r^{1/\alpha}(\zeta)x^{\Delta}(\zeta)$ is decreasing, then

$$x(\tau(\zeta)) \geq x(\zeta) \geq \int_{\zeta}^{\infty} \frac{1}{r^{1/\alpha}(s)} \left(-r^{1/\alpha}(s) x^{\Delta}(s)\right) \Delta s \geq -\xi(\zeta)\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right) \geq 0, \tag{20}$$

using the fact that $\alpha \geq 1$ is a quotient of odd positive integers. Hence, it follows (18). This proves the lemma. □

3. Oscillation Results

In the following, we present oscillation results for (6) via comparison with canonical second-order linear dynamic equation.

Theorem 6. *Assume that the dynamic equation*

$$\left(a(\zeta) y^{\Delta}(\zeta)\right)^{\Delta} + g(\zeta) A^{\beta-\gamma}(\tau(\zeta)) y(\tau(\zeta)) = 0, \tag{21}$$

where $\gamma := \max\{\alpha, \beta\}$ *is oscillatory, then (6) is also oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(\zeta)$ of (6) such that $x(\zeta) > 0$ and $x(\tau(\zeta)) > 0$ for all $\zeta \geq \zeta_1 \geq \zeta_0$. By the chain rule ([4], Theorem 1.90), it is clear that

$$\left(r(\zeta)\left(x^{\Delta}(\zeta)\right)^{\alpha}\right)^{\Delta} = \left(\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\alpha}\right)^{\Delta} \geq \alpha\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\alpha-1}\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\Delta}. \tag{22}$$

Combining (6) with (22), we obtain

$$\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\Delta} + \frac{1}{\alpha}\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{1-\alpha} q(\zeta) x^{\beta}(\tau(\zeta)) \leq 0. \tag{23}$$

Substituting (18) into (23), we get

$$\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\Delta} + \frac{1}{\alpha} \xi^{\alpha-1}(\zeta) q(\zeta) x^{1+\beta-\alpha}(\tau(\zeta)) \leq 0. \tag{24}$$

Using a straightforward calculation (see [27]), we conclude that

$$\left(r^{1/\alpha}(\zeta) x^{\Delta}(\zeta)\right)^{\Delta} = \frac{1}{\xi(\sigma(\zeta))}\left(r^{1/\alpha}(\zeta) \xi(\sigma(\zeta)) \xi(\zeta)\left(\frac{x(\zeta)}{\xi(\zeta)}\right)^{\Delta}\right)^{\Delta}. \tag{25}$$

Consequently,

$$\frac{1}{\xi(\sigma(\zeta))}\left(r^{1/\alpha}(\zeta) \xi(\sigma(\zeta)) \xi(\zeta)\left(\frac{x(\zeta)}{\xi(\zeta)}\right)^{\Delta}\right)^{\Delta} + \frac{1}{\alpha} \xi^{\alpha-1}(\zeta) q(\zeta) x^{1+\beta-\alpha}(\tau(\zeta)) \leq 0. \tag{26}$$

Using the transformation $x(\zeta) = \xi(\zeta) y(\zeta)$, we get

$$\left(a(\zeta) y^{\Delta}(\zeta)\right)^{\Delta} + g(\zeta) y^{1+\beta-\alpha}(\tau(\zeta)) \leq 0. \tag{27}$$

It is worth noting that the transformation $x(\zeta) = \xi(\zeta) y(\zeta)$ preserves oscillation and $\int_{\zeta_0}^{\infty} \Delta s / a(s) = \infty$. This leads to $[a(\zeta) y^{\Delta}(\zeta)]^{\Delta} < 0$ and $y^{\Delta}(\zeta) > 0$ on $[\zeta_1, \infty)_{\mathbb{T}}$. Therefore,

$$y(\zeta) \geq \int_{\zeta_1}^{\zeta} \frac{a(s) y^{\Delta}(s)}{a(s) \Delta s} \geq \left(a(\zeta) y^{\Delta}(\zeta)\right) \int_{\zeta_1}^{\zeta} \frac{\Delta s}{a(s)} = A(\zeta)\left(a(\zeta) y^{\Delta}(\zeta)\right), \tag{28}$$

which implies

$$\left(\frac{y(\zeta)}{A(\zeta)}\right)^{\Delta} < 0. \tag{29}$$

If $\beta \leq \alpha$, by the fact that (29), we get for $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$,

$$y(\zeta) \leq \frac{y(\zeta_1)}{A(\zeta_1)} A(\zeta), \tag{30}$$

whereas if $\beta \geq \alpha$, by the fact that $y^{\Delta}(\zeta) > 0$ on $[\zeta_1, \infty)_{\mathbb{T}}$, we obtain for $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$,

$$y(\zeta) \geq y(\zeta_1). \tag{31}$$

Let $0 < k < 1$ be arbitrary. Combining (30) and (32), there exists a $\zeta_2 \in [\zeta_1, \infty)_{\mathbb{T}}$ such that

$$y^{\beta-\alpha}(\zeta) \geq k A^{\beta-\gamma}(\zeta) \text{ for } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}, \tag{32}$$

where $\gamma := \max\{\alpha, \beta\}$. Hence, for $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$,

$$\left[a(\zeta) y^{\Delta}(\zeta)\right]^{\Delta} + g(\zeta) A^{\beta-\gamma}(\tau(\zeta)) y(\tau(\zeta)) \leq 0, \tag{33}$$

because $k < 1$ is arbitrary. Using the fact that $a(\zeta) y^{\Delta}(\zeta)$ is decreasing and $y^{\Delta}(\zeta) > 0$, we get

$$\begin{aligned} a(\zeta)y^{\Delta}(\zeta) &\geq \int_{\zeta}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))y(\tau(s))\Delta s,\\ y(\zeta) &\geq \int_{\zeta_1}^{\zeta}\left(\frac{1}{a(v)}\int_{v}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))y(\tau(s))\Delta s\right)\Delta v. \end{aligned} \tag{34}$$

Let

$$X=\{z\colon z\in C_{\mathrm{rd}}([\zeta_0,\infty)_{\mathbb{T}},\mathbb{R})\}, \sup_{\zeta\in[\zeta_0,\infty)_{\mathbb{T}}}|z(\zeta)|<\infty. \tag{35}$$

Clearly, X is a Banach space with the norm $\|z\|\sup_{\zeta\in[\zeta_0,\infty)_{\mathbb{T}}}|z(\zeta)|$. Consider $S\subset X$ such that

$$S=\{z\in X\colon 0\leq z(\zeta)\leq y(\zeta),\quad \zeta\in[\zeta_0,\infty)_{\mathbb{T}}\}. \tag{36}$$

It is clear that S is a closed, bounded, and convex subset of $C_{\mathrm{rd}}([\zeta_0,\infty)_{\mathbb{T}},\mathbb{R})$. Define an operator $T\colon S\longrightarrow X$ by

$$(Tz)(\zeta)=\begin{cases}\displaystyle\int_{\zeta_1}^{\zeta}\left(\frac{1}{a(v)}\int_{v}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))y(\tau(s))\Delta s\right)\Delta v, & \zeta\in[\zeta_1,\infty)_{\mathbb{T}},\\ y(\zeta_1), & \zeta\in[\zeta_0,\zeta_1]_{\mathbb{T}}.\end{cases} \tag{37}$$

Now, T is continuous, and if $z\in S$, then

$$(Tz)(\zeta)\geq y(\zeta_1)\geq 0, \tag{38}$$

and

$$(Tz)(\zeta)\leq \int_{\zeta_1}^{\zeta}\left(\frac{1}{a(v)}\int_{v}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))y(\tau(s))\Delta s\right)\Delta v\leq y(\zeta). \tag{39}$$

Thus, $TS\subset S$. Therefore, by the Krasnoselskii's fixed point Theorem 3, T has a fixed point $z\in S$. Moreover, $z(\zeta)=(Tz)(\zeta)$ satisfies that

$$z(\zeta)=y(\zeta_1)+\int_{\zeta_1}^{\zeta}\left(\frac{1}{a(v)}\int_{v}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))z(\tau(s))\Delta s\right)\Delta v,\quad \zeta\geq\zeta_1. \tag{40}$$

It follows that $z(t)$ is a positive solution of the following dynamic equation:

$$\left(a(\zeta)z^{\Delta}(\zeta)\right)^{\Delta}+g(\zeta)A^{\beta-\gamma}(\tau(\zeta))z(\tau(\zeta))=0. \tag{41}$$

This completes the proof. □

Theorem 7. *If*

$$\limsup_{\zeta\longrightarrow\infty}\left(\begin{array}{c}\displaystyle\frac{1}{A(\tau(\zeta))}\int_{\zeta_0}^{\tau(\zeta)} A(\sigma(s))g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s\\ \displaystyle +\int_{\tau(\zeta)}^{\zeta} g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s+A(\tau(\zeta))\int_{\zeta}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))\Delta s\end{array}\right)>1, \tag{42}$$

then (6) is oscillatory.

Proof. Assume that (6) is not oscillatory. Then, by Theorem 6, we see that (21) is also not oscillatory. Let $y(\zeta)$ be an eventually positive solution of (21), then there exists $\zeta_1 \geq \zeta_0$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$, for all $\zeta \geq \zeta_1$. Integrating (21) and using the increasing fact of $y(\zeta)$ lead to

$$y^{\Delta}(\zeta) \geq \frac{1}{a(\zeta)} \int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s. \tag{43}$$

Integrating again yields

$$\begin{aligned} y(\zeta) &\geq \int_{\zeta_1}^{\zeta} \left(\frac{1}{a(v)} \int_{v}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \right) \Delta v \\ &= \int_{\zeta_1}^{\zeta} \left(\frac{1}{a(v)} \int_{v}^{\zeta} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \right) \Delta v + \int_{\zeta_1}^{\zeta} \left(\frac{1}{a(v)} \int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \right) \Delta v, \end{aligned} \tag{44}$$

for $\zeta_1 \geq \zeta_0$. Using integration by parts, we obtain

$$y(\zeta) \geq \int_{\zeta_1}^{\zeta} A(\sigma(s)) g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s + A(\zeta) \int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s. \tag{45}$$

Hence,

$$\begin{aligned} y(\tau(\zeta)) &\geq \int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s)) g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s + A(\tau(\zeta)) \int_{\tau(\zeta)}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \\ &= \int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s)) g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s + A(\tau(\zeta)) \int_{\tau(\zeta)}^{\zeta} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \\ &\quad + A(\tau(\zeta)) \int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s. \end{aligned} \tag{46}$$

Using the fact that $y(\zeta)/A(\zeta)$ is decreasing and $y(\zeta)$ is increasing, we see that if $s \geq \zeta$, we have $\tau(s) \geq \tau(\zeta)$, so

$$y(\tau(s)) \geq y(\tau(\zeta)) \text{ for } \zeta \geq \zeta_2, \tag{47}$$

for some $\zeta_2 \geq \zeta_1$. Now,

$$\int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \geq y(\tau(\zeta)) \int_{\zeta}^{\infty} g(s) A^{\beta-\gamma}(\tau(s)) \Delta s. \tag{48}$$

Also, for $\tau(s) \leq \tau(\zeta)$, we have

$$\frac{y(\tau(s))}{A(\tau(s))} \geq \frac{y(\tau(\zeta))}{A(\tau(\zeta))}. \tag{49}$$

Thus,

$$\int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s)) g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \geq \frac{y(\tau(\zeta))}{A(\tau(\zeta))} \int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s)) g(s) A^{\beta-\gamma+1}(\tau(s)) \Delta s, \tag{50}$$

$$\int_{\tau(\zeta)}^{\zeta} g(s) A^{\beta-\gamma}(\tau(s)) y(\tau(s)) \Delta s \geq \frac{y(\tau(\zeta))}{A(\tau(\zeta))} \int_{\tau(\zeta)}^{\zeta} g(s) A^{\beta-\gamma+1}(\tau(s)) \Delta s. \tag{51}$$

Using (48), (50), and (51) in (46) gives

$$
\begin{aligned}
y(\tau(\zeta)) \geq & \frac{y(\tau(\zeta))}{A(\tau(\zeta))}\int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s))g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s \\
& + y(\tau(\zeta))\int_{\tau(\zeta)}^{\zeta} g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s + y(\tau(\zeta))A(\tau(\zeta))\int_{\tau}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))\Delta s.
\end{aligned} \tag{52}
$$

It follows that

$$
\begin{aligned}
1 \geq & \frac{1}{A(\tau(\zeta))}\int_{\zeta_1}^{\tau(\zeta)} A(\sigma(s))g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s \\
& + \int_{\tau(\zeta)}^{\zeta} g(s)A^{\beta-\gamma+1}(\tau(s))\Delta s + A(\tau(\zeta))\int_{\zeta}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))\Delta s.
\end{aligned} \tag{53}
$$

This is a contradiction and completes the proof of the theorem. □

Theorem 8. *Assume that there exists a function $\varphi \in C_{rd}^{1}(\mathbb{T},\mathbb{R}^{+})$ such that*

$$
\limsup_{\zeta\longrightarrow\infty}\left(\varphi(\zeta)\int_{\zeta}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))\Delta s + \int_{\zeta_0}^{\zeta}\left(\varphi(s)g(s)A^{\beta-\gamma}(\tau(s)) - \frac{a^{*}(s)\left(\varphi_{+}^{\Delta}(s)\right)^{2}}{4\varphi(s)\tau^{\Delta}(s)}\right)\Delta s\right) = \infty, \tag{54}
$$

where $\varphi_{+}^{\Delta}(l) = \max\{\varphi^{\Delta}(\zeta), 0\}$ and $a^{*}(\zeta) = \max\{a(\zeta)|\tau(\zeta) \leq \varsigma \leq \tau(\sigma(\zeta))\}$, then (6) is oscillatory.

Proof. Assume that (6) is nonoscillatory, then, according to Theorem 6, it follows that (21) is also nonoscillatory. Let $y(\zeta)$ be an eventually positive solution of (21), then there exists an integer $\zeta_1 \geq \zeta_0$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for all $\zeta \geq \zeta_1$. Proceeding the proof in the same manner as in ([28], Theorem 6), we arrive at the contradiction. This completes the proof. □

Theorem 9. *Suppose that conditions (H1)–(H4) hold. Let $D_0 = \{(\zeta, s)\colon t > s \geq \zeta_0, \zeta, s \in \mathbb{T}\}$ and $D = \{(\zeta, s)\colon \zeta \geq s \geq \zeta_0, \zeta, s \in \mathbb{T}\}$. Moreover, suppose that there exist functions $H \in C(D,\mathbb{R}), h \in C(D_0,\mathbb{R})$ and $\varphi(t) \in C_{rd}^{1}(\mathbb{T},\mathbb{R}^{+})$ such that the following three conditions hold:*

- *(i) $H(\zeta,\zeta) = 0$ for all $t \geq \zeta_0, H(\zeta, s) > 0$ and for all $(\zeta, s) \in D_0$;*
- *(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;*
- *(iii) $[H(\zeta, s)\varphi(s)]^{\Delta_s} = h(\zeta, s)\sqrt{H(\zeta, s)\varphi(s)}$, for all $(\zeta, s) \in D_0$.*

If

$$
\limsup_{\zeta\longrightarrow\infty}\frac{1}{H(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}\left[H(\zeta, s)\varphi(s)g(s)A^{\beta-\gamma}(\tau(s)) - \frac{a^{*}(s)h^{2}(\zeta, s)}{4\tau^{\Delta}(s)}\right]\Delta s = \infty, \tag{55}
$$

then (6) is oscillatory.

Proof. Assume that (6) is nonoscillatory, then, according to Theorem 6, it follows that (21) is also nonoscillatory. Let $y(\zeta)$ be an eventually positive solution of (21), then there exists an integer $\zeta_1 \geq \zeta_0$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for all $\zeta \geq \zeta_1$. Proceeding the proof in the same manner as in ([28], Theorem 9), we arrive at the contradiction. This completes the proof. □

Theorem 10. *If the first-order dynamic equation*

$$
\chi^{\Delta}(\zeta) + g(\zeta)A^{\beta-\gamma+1}(\tau(\zeta))\chi(\tau(\zeta)) = 0, \tag{56}
$$

for all sufficiently large $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$, is oscillatory, then (6) is also oscillatory

Proof. Assume that (6) is not oscillatory. Then, by Theorem 6, we see that (21) is also not oscillatory. Let $y(\zeta)$ be an eventually positive solution of (21), then there exists $\zeta_1 \geq \zeta_0$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for all $\zeta \geq \zeta_1$. Let $\chi(\zeta) = a(\zeta)y^{\Delta}(\zeta)$ form (28), then we have

$$y(\tau(\zeta)) \geq a(\zeta)y^{\Delta}(\zeta)A(\tau(\zeta)) = A(\tau(\zeta))\chi(\tau(\zeta)). \quad (57)$$

Using (57) in (21), we see that $\chi(\zeta)$ is a positive solution of the dynamic inequality

$$\chi^{\Delta}(\zeta) + g(\zeta)A^{\beta-\gamma+1}(\tau(\zeta))\chi(\tau(\zeta)) \leq 0. \quad (58)$$

By ([6], Theorem 6), (56) also presents a nonoscillatory solution. This contradiction proves that (21) is oscillatory. □

Corollary 11. *If there exists $\delta \in [0,1]_{\mathbb{R}}$ such that*

$$\liminf_{\zeta\longrightarrow\infty} \int_{\tau(\zeta)}^{\zeta} g(s)\psi(\tau(s))A(\tau(s))\Delta s > \delta \text{ and } \limsup_{\zeta\longrightarrow\infty} \int_{\tau(\zeta)}^{\sigma(\zeta)} g(s)\psi(\tau(s))A(\tau(s))\Delta s > 1 - \left(1-\sqrt{1-\delta}\right)^2, \quad (59)$$

then every solution of (6) is oscillatory.

Proof. Assume that (6) is not oscillatory. Then, by Theorem 10, we see that (56) is also not oscillatory. Let $\chi(\zeta)$ be an eventually positive solution of (56), then there exists an integer $\zeta_1 \geq \zeta_0$ such that $\chi(\zeta)$ and $\chi(\tau(\zeta))$ for all $\zeta \geq \zeta_1$. Proceeding the proof in the same manner as in (24, Corollary 2), we arrive at the contradiction. This completes the proof. □

Example 1. Consider the second-order differential equation

$$\left(\zeta^{\nu+1}\left(y'(\zeta)\right)^{\nu}\right)' + q_0 y^{\nu}(\lambda\zeta) = 0, \zeta \geq 1, 0 < \lambda \leq 1, q_0 > 0, \nu \geq 1. \quad (60)$$

Here, $r(\zeta) = \zeta^{\nu+1}, q(t) = q_0, \tau(\zeta) = \lambda\zeta$. It is clear that

$$\xi(\zeta) = \int_{\zeta}^{\infty} s^{-(\nu+1)/\nu} ds = \nu\zeta^{-1/\nu} < \infty. \quad (61)$$

It follows that $a(\zeta) = \nu^2\zeta^{(\nu-1)/\nu}$ and $g(\zeta) = q_0\nu^{\nu}/\lambda^{1/\nu}\zeta^{-(\nu+1)/\nu}$, hence the transformed equation in the canonical form is as follows:

$$\left(\nu^2\zeta^{(\nu+1)/\nu}y'(\zeta)\right)' + \frac{q_0\nu^{\nu}}{\lambda^{1/\nu}}\zeta^{-(\nu+1)/\nu}y(\lambda\zeta) = 0. \quad (62)$$

Since

$$A(\zeta) = \frac{1}{\nu^2}\int_{\zeta_0}^{\zeta} s^{-(\nu-1)/\nu} ds = \frac{\zeta^{1/\nu}}{\nu}, \quad (63)$$

choose $\varphi(\zeta) = \zeta$, then condition (54) takes the following form:

$$\begin{aligned}
&\limsup_{\zeta\longrightarrow\infty}\left(\varphi(\zeta)\int_{\zeta}^{\infty} g(s)A^{\beta-\gamma}(\tau(s))\Delta s + \int_{\zeta_0}^{\zeta}\left(\varphi(s)g(s)A^{\beta-\gamma}(\tau(s)) - \frac{a^*(s)\left(\varphi_+^{\Delta}(s)\right)^2}{4\varphi(s)\tau^{\Delta}(s)}\right)\Delta s\right)\\
&= \limsup_{\zeta\longrightarrow\infty}\left(\frac{q_0\nu^{\nu+1}}{\lambda^{1/\nu}}\zeta^{(\nu-1)/\nu} + \int_{\zeta_0}^{\zeta}\left(s\frac{q_0\nu^{\nu}}{\lambda^{1/\nu}}s^{-(\nu+1)/\nu} - \frac{\nu^2 s^{(\nu-1)/\nu}}{4s\lambda}\right)ds\right)\\
&= \infty \text{ for } \left(q_0 > \frac{\nu}{4(\nu+1)}\lambda^{(1-\nu)/\nu}\right).
\end{aligned} \quad (64)$$

Therefore, when applying Theorem 7, we see that (60) is oscillatory, if $q_0 > \nu^{1-\nu}/2\nu - \ln(\lambda)$. Also, (60) is oscillatory provided that $q_0 > 0.75\nu^{1-\nu}/\ln(1/\lambda)$.

Let $\nu = 1$ and $\lambda = 0.5$; (60) is oscillatory by Theorem 8 if $q_0 > 0.25$ and by using Theorem 8, we see that (60) is oscillatory for $q_0 > 0.37$. By applying ([14], Theorem 4), we conclude that (60) is oscillatory for $q_0 > 0.53073$ and by using ([1], Theorem 1), $q_0 > 0.37$. Note that ([18], Theorems 3 and 2.2) cannot be applied to (60) for $\nu > 1$.

Example 2. Consider the following equation:

$$\left(\zeta^2 y'(\zeta)\right)' + q_0\zeta^2 y^3(\zeta) = 0, \zeta \geq 0. \quad (65)$$

Here, $\alpha = 1, \beta = 3$ and $(\zeta) = q_0\zeta^2$. It is clear that

$$\xi(\zeta) = \int_{\zeta}^{\infty}\frac{ds}{s^2} = \frac{1}{\zeta} < \infty. \quad (66)$$

It follows that $a(\zeta) = 1$ and $g(\zeta) = q_0/\zeta^2$, hence the transformed equation in the canonical form is

$$y''(\zeta) + \frac{q_0}{\zeta^2} y(\zeta) = 0. \tag{67}$$

Since $(\zeta) = \zeta$, by choosing $\varphi(\zeta) = \zeta$, the condition (54) guarantees that (65) is oscillatory for $q_0 > \gamma/4$, where $\gamma = c^{\beta-\alpha}, > 0$.

Example 3. Consider the second-order difference equation

$$\Delta\left(\zeta^{\alpha}(\zeta+1)^{\alpha}\left(\Delta y(\zeta)^{\alpha}\right)+q_0\zeta^{\beta-1}y^{\beta}(\zeta-1)=0,\quad \zeta\geq\zeta_0>1, \tag{68}$$

where $\alpha \geq 1, \beta$ are ratios of odd positive integers, q_0 is a positive real number, and $\sigma(\zeta) = \zeta + 1$. It is clear that

$$\xi(\zeta) = \sum_{s=\zeta}^{\infty} \frac{1}{s(s+1)} = \frac{1}{\zeta} < \infty. \tag{69}$$

It follows that $a(\zeta) = 1$ and $g(\zeta) = q_0/\alpha\zeta^{\beta-\alpha}/(\zeta+1)(\zeta-1)^{1+\beta-\alpha}$, hence the transformed equation in the canonical form is

$$\Delta^2(y(\zeta)) + \frac{q_0}{\alpha}\frac{\zeta^{\beta-\alpha}}{(\zeta+1)(\zeta-1)^{1+\beta-\alpha}} y(\zeta) = 0. \tag{70}$$

Choose $\varphi(\zeta) = \zeta$, then condition (54) takes the form

$$\begin{aligned}&\limsup_{\zeta\longrightarrow\infty}\left(\varphi(\zeta)\int_{\zeta}^{\infty}g(s)A^{\beta-\gamma}(\tau(s))\Delta s+\int_{\zeta_0}^{\zeta}\left(\varphi(s)g(s)A^{\beta-\gamma}(\tau(s))-\frac{a^{*}(s)\left(\varphi_{+}^{\Delta}(s)\right)^{2}}{4\varphi(s)\tau^{\Delta}(s)}\right)\Delta s\right)\\&=\limsup_{\zeta\longrightarrow\infty}\sum_{s=\zeta_0}^{\zeta}\left(\frac{q_0}{\alpha}\frac{s^{\beta-\alpha+2}}{(s+1)(s-1)^{2+\beta-\alpha}}-\frac{1}{4s}\right).\end{aligned} \tag{71}$$

It follows that (68) is oscillatory for $q_0 > \alpha/4\gamma$. We see that (68) is oscillatory for $q_0 > 1/4$ when $\alpha = \beta = 1$ which is consistent with the results in [6] and guarantee that every solution of (68) is oscillatory unlike [29].

4. Conclusion and Discussion

The outcomes of this paper are presented in a fundamentally novel and highly general form. These findings not only enrich but also complement the existing literature's discoveries, even in the particular instances when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Based on these outcomes, we can draw the following conclusions:

(i) The oscillatory behavior of the solution to (6) is examined for the case $\alpha = \beta$ in previous works. However, in this paper, we focus on studying the oscillation of (6) for the case when $\alpha \neq \beta$, see [6, 9, 10].

(ii) When $\mathbb{T} = \mathbb{Z}$, (6) becomes the difference (11) which is discussed in [8, 30, 31] with $\beta \in [0, 1)$; however, in our results $\beta \in \mathbb{Q}^{+}_{odd}$, where $\mathbb{Q}^{+}_{odd} := \{a/b\colon a, b \in \mathbb{Z}^{+}$ are odd$\}$.

(iii) In this paper, we improved the obtained results in [12], where we removed condition (15).

(iv) In contrast to [25], our results ensure that all solutions are oscillatory [28, 32, 33].

Data Availability

The data are not available as no new data were created or analyzed.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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