Space-dependent variable-order time-fractional wave equation: Existence and uniqueness of its weak solution

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Abstract

The investigation of an initial-boundary value problem for a fractional wave equation with space-dependent variable-order wherein the coefficients have a dependency on the spatial and time variables is the concern of this work. This type of variable-order fractional differential operator originates in the modelling of viscoelastic materials. The global in time existence of a unique weak solution to the model problem has been proved under appropriate conditions on the data. Rothe's time discretization method is applied to achieve that purpose.

Keywords: Time fractional wave equation, Variable coefficients, Uniqueness and existence, Non-autonomous, Variable-order, Rothe's time discretization

1. Introduction

The time-fractional diffusion or wave equation is the fractional-order generalization of the classical diffusion or wave equation, respectively. Some real-world applications for different kinds of fractional order wave equations can be found in [1, 2, 3]. For available applications of variable-order fractional diffusion and wave operators in the general area of scientific and engineering modelling, mathematical foundations and numerical methods, please see [4, 5, 6]. This work is concerned with the following variable-order fractional wave equation with space-dependent variable order. Let us fix these notations to identify the problem under study easily. We denote by $\Lambda \subset \mathbb{R}^d,\ d \in \mathbb{N}$, a bounded domain with boundary $\partial\Lambda$, which is Lipschitz continuous. The end of time interval is represented by T, $Q_T := (0,T] \times \Lambda$ and $\Sigma_T := (0,T] \times \partial\Lambda$. A linear differential operator of second-order in its general style is given by

$$\mathcal{L}(\mathbf{z},t)v(\mathbf{z},t) = -\nabla \cdot (\mathcal{R}(\mathbf{z},t)\nabla v(\mathbf{z},t)) + \boldsymbol{b}(\mathbf{z},t) \cdot \nabla v(\mathbf{z},t) + c(\mathbf{z},t)v(\mathbf{z},t), \tag{1}$$

where

$$\mathcal{R}(\mathbf{z},t) = \left(r_{i,j}(\mathbf{z},t)\right)_{i,j=1,\ldots,d}, \quad \mathcal{R}^\top = \mathcal{R},$$
$$\boldsymbol{b}(\mathbf{z},t) = (b_1(\mathbf{z},t), b_2(\mathbf{z},t), \ldots, b_d(\mathbf{z},t)).$$

We aim to prove the existence and uniqueness of a weak solution v for given f, $\tilde{u}_0$ and $\tilde{v}_0$ such that

$$\begin{cases} \partial_t\left(k * (\partial_t v - \tilde{v}_0)\right) + \mathcal{L}v &= f \quad \text{in } Q_T, \\ v &= 0 \quad \text{on } \Sigma_T, \\ v(\mathbf{z},0) &= \tilde{u}_0(\mathbf{z}) \quad \mathbf{z} \in \Lambda, \\ \partial_t v(\mathbf{z},0) &= \tilde{v}_0(\mathbf{z}) \quad \mathbf{z} \in \Lambda. \end{cases} \tag{2}$$

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The convolution product is represented by a symbol '$*$', i.e

$$(k * q)(\mathbf{z}, t) := \int_0^t k(\mathbf{z}, t-s) q(\mathbf{z}, s) \, \mathrm{d}s,$$

with the kernel k defined by

$$k(\mathbf{z}, t) = \frac{t^{1-\gamma(\mathbf{z})}}{\Gamma(2-\gamma(\mathbf{z}))}, \qquad t > 0, \quad \mathbf{z} \in \Lambda, \tag{3}$$

where Γ is the Gamma function and $\gamma \in \mathrm{C}(\Lambda)$ satisfies

$$1 < \gamma(\mathbf{z}) \leq \gamma_2 < 2.$$

The term $\partial_t (k * (\partial_t v - \tilde{v}_0))$ represents the Caputo fractional derivative of space-dependent variable-order $\gamma(\mathbf{z})$. We interpret this term as in [7, Definition 3.2], and specify throughout the paper in which sense it should be properly understood. This operator is used to describe phenomena in viscoelasticity [8, 4, 9, 10, 11, 12]. Note that a recent overview of the applications in mechanics, viscoelasticity and wave propagation is given in the work [6]. We want to point out that in these references (and also in the following), different definitions of variable-order differential operators are used (with different purposes). The order of the fractional integrals can be defined as functions of space and/or time [13] and it can be seen as a successful generalization of the constant-order fractional calculus.

This contribution studies the well-posedness of the fractional wave equation of space-dependent variable-order with homogeneous Dirichlet boundary conditions. In the recent work [14], the author investigated the existence and uniqueness of a weak solution to a non-autonomous time-fractional diffusion equation with space-dependent variable-order. The author employed the positive definiteness property of the corresponding kernel to obtain the well-posedness result. Although the kernel (3) has similar properties, this property seems not valuable for the analysis of problem (2). Instead, it is another result of that paper that turns out to be helpful when studying the existence and uniqueness of a solution to problem (2).

Next, we discuss the well-posedness results available in literature related to this topic. The solution to autonomous (time-independent elliptic part) constant-order fractional wave equations is proved to be existed and unique in [15, 16, 17, 18]. A priori estimates for solutions of boundary value problems for constant-order fractional wave equations are derived in [19]. A fundamental solution to the fractional wave equation of constant-order is determined in [3]. The existence and uniqueness of the solution to generalized fractional differential equations with variable-order operators have been discussed in [20, 21, 22, 23]. Analytical solutions to specific fractional time-dependent variable-order differential equations are obtained in [24]. Interesting studies about the numerical analysis of the variable and distributed order fractional wave equation can be found in [25, 26, 27, 28, 29]. However, to the best of our knowledge, no paper deals with the well-posedness of problem (2), which is the main goal of this paper.

For completeness, we mention that Rothe's discretization method has been recently used for other problems with fractional integral operators. We mention the following contributions for the sake of clarity. First, a class of multi-term time-fractional integral diffusion equations has been proposed in [30]. The existence, uniqueness and regularity of a strong solution are provided utilizing Rothe's method. Second, a fractional-order partial integrodifferential equation with initial/boundary conditions has been studied in [31]. Rothe's method was used to prove the existence and uniqueness of a strong solution to the integrodifferential equation, which ensures that the initial boundary value problem is well-posed. Third, a class of elliptic hemivariational inequalities involving the time-fractional order integral operator is investigated in [32]. A result of the existence of a solution to the problem is established using the Rothe method and the surjectivity of multivalued pseudomonotone operators. Accordingly, the weak solvability of a fractional viscoelastic contact problem is studied.

Finally, we mention that the authors in [33, Section 4.1] showed that if $\gamma(\mathbf{z}) = \gamma \in (1, 2)$, $\tilde{v}_0 = f = 0$, and the operator $\mathcal{L}$ is time-independent with $\boldsymbol{b} = \mathbf{0}$, the solution of problem (2) satisfies for $0 < t \leq T$ the bounds

$$\left| \frac{\partial^\ell v}{\partial t^\ell}(t) \right| \leq C(1 + t^{\gamma - \ell}) \quad \text{for } \ell = 0, 1, 2, \tag{4}$$

i.e. the second order derivative of u with respect to the time t is unbounded as $t \searrow 0$. It holds that $v \in \mathrm{C}^1([0, T]) \cap \mathrm{C}^2((0, T])$ with $\partial_{tt} v \in \mathrm{L}^1(0, T)$. We may not contradict this result in the analysis of problem (2) as it is a special case of problem (2).

We complete this section by giving the outline of the paper. In Section 2, the weak formulation is derived, and the uniqueness of a solution is shown. Note that the coefficients in the operator $\mathcal{L}$ are time-dependent (i.e. non-autonomous system), which implies that the Mittag-Leffler analysis is not appropriate and one has to use other tools to succeed. For this reason, in Section 3 and 4, we apply Rothe's method to show the existence of a solution to the variational problem. First, a priori estimates are derived in Section 3, then followed by a limit transition in Section 4. In Section 5, we discuss the situation that γ is constant. We conclude this section by introducing the notations used in this paper.

Remark 1.1. *We denote by $(\cdot,\cdot)$ the standard inner product in $\mathrm{L}^2(\Lambda)$ with induced norm $\|\cdot\|$. Let Z represent an abstract Banach space with norm $\|\cdot\|_Z$. The $\mathrm{L}^p\left((0,T),Z\right)$ such that $p \geq 1$ is a space of measurable functions $v : (0,T) \to Z$ which achieves*

$$\|v\|_{\mathrm{L}^p((0,T),Z)} = \left(\int_0^T \|v(t)\|_Z^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

The $\mathrm{C}\left([0,T],Z\right)$ is a space of all continuous functions $v : [0,T] \to Z$ satisfying

$$\|v\|_{\mathrm{C}([0,T],Z)} = \max_{t\in[0,T]} \|v(t)\|_Z < \infty.$$

The $\mathrm{L}^\infty\left((0,T),Z\right)$ is a space of all measurable functions $v : (0,T) \to Z$ that are essentially bounded. The space $\mathrm{H}^1((0,T),Z)$ consists of functions $v : (0,T) \to Z$ such that the weak derivative v' exists and

$$\|v\|_{\mathrm{H}^1((0,T),Z)} = \left(\int_0^T \|v(t)\|_Z^2 + \left\|v'(t)\right\|_Z^2 \mathrm{d}t\right)^{\frac{1}{2}} < \infty.$$

The following notations will be fixed through the paper. This means that C, ε and C_ε are generic and positive constants (has no dependence of the discretization parameter), where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C\left(1+\varepsilon+\frac{1}{\varepsilon}\right)$. The same notation for different constants is used, but the meaning should be clear from the context.

2. Uniqueness of a solution

The assumptions on the data functions are stated firstly in the operator (1). They will be invoked throughout the rest of this contribution:

- *AS-1*: The matrix $\boldsymbol{\mathcal{R}} = \left(r_{ij}(\mathbf{z},t)\right)$ is a symmetric $d\times d$ matrix-valued function such that $\boldsymbol{\mathcal{R}} \in \left(\mathrm{C}\left([0,T],\mathrm{L}^\infty(\Lambda)\right)\right)^{d\times d}$ is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\alpha \left|\boldsymbol{\eta}\right|^2 \leq \sum_{i,j=1}^{d} r_{ij}(\mathbf{z},t)\eta_i\eta_j,$$

  for a.a. $(\mathbf{z},t) \in Q_T$ and $\boldsymbol{\eta} \in \mathbb{R}^d$;

- *AS-2*: We assume that $\boldsymbol{b} \in \mathrm{C}\left([0,T],\mathbf{L}^\infty(\Lambda)\right)$ and $c \in \mathrm{C}\left([0,T],\mathrm{L}^\infty(\Lambda)\right)$ such that

$$-\frac{\|\boldsymbol{b}\|^2_{\mathbf{L}^\infty\left(\overline{Q_T}\right)}}{2\alpha} + c(\mathbf{z},t) \geq 0, \quad \text{for a.a. } (\mathbf{z},t) \in Q_T,$$

- *AS-3*: $\partial_t\boldsymbol{\mathcal{R}} \in \left(\mathrm{L}^\infty(Q_T)\right)^{d\times d}$ and $\partial_t\boldsymbol{b} \in \mathbf{L}^\infty(Q_T)$;

- *AS-4*: $\tilde{u}_0 \in \mathrm{H}^1_0(\Lambda)$, $\tilde{v}_0 \in \mathrm{L}^2(\Lambda)$ and $f \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Lambda)\right)$.

A bilinear form $\mathbf{L}$ is associated with the differential operator $\mathcal{L}$ defined in (1) as follows

$$\mathbf{L}(t)\,(v(t),\phi) := (\mathcal{L}v,\phi) = (\mathcal{R}(t)\nabla v(t),\nabla\phi) + (\boldsymbol{b}(t)\cdot\nabla v(t),\phi) + (c(t)v(t),\phi)\,,$$

with $v(t),\phi \in \mathrm{H}_0^1(\Lambda)$. Using the properties above, we obtain for all $v,\phi \in \mathrm{H}_0^1(\Lambda)$ that

$$\mathbf{L}(t)\,(v,\phi) \le C\,\|v\|_{\mathrm{H}^1(\Lambda)}\,\|\phi\|_{\mathrm{H}^1(\Lambda)}\,,$$

and

$$\frac{\alpha}{2}\,\|\nabla\phi\|^2 \le \mathbf{L}(t)\,(\phi,\phi)\,,\quad \forall\phi\in\mathrm{H}_0^1(\Lambda). \tag{5}$$

The convolution kernel k has the following properties

- $k(\cdot,t)\in\mathrm{L}^\infty(\Lambda)$ for all $t>0$ since

$$0 \le \frac{\min\{1,t^{1-\gamma_2}\}}{\Gamma(2-\gamma_2)} \le k(\mathbf{z},t) \le \max\left\{1,t^{1-\gamma_2}\right\},\quad \mathbf{z}\in\Lambda, \tag{6}$$

  since $\Gamma(z)\ge\Gamma(1)=1$ for all $z\in(0,1)$;

- $k(\mathbf{z},\cdot)\in\mathrm{L}^1(0,T)$ for all $\mathbf{z}\in\Lambda$ with

$$\int_0^T |k(\mathbf{z},t)|\,\mathrm{d}t \le 2\max\{1,T\}, \tag{7}$$

  since $\Gamma(z)\ge\frac{1}{2}$ for $z\in(1,2)$;

- $\partial_t k(\mathbf{z},\cdot)\in\mathrm{L}^1_{\mathrm{loc}}(0,T)$ for all $\mathbf{z}\in\Lambda$;

- $\partial_t k(\mathbf{z},t)\le 0$ for all $t>0$ and for all $\mathbf{z}\in\Lambda$ and

$$k(\mathbf{z},t) \ge \frac{\min\{1,T^{1-\gamma_2}\}}{\Gamma(2-\gamma_2)} =: \tilde{k} > 0, \tag{8}$$

  for $t\in(0,T]$ and $\mathbf{z}\in\Lambda$.

As $k(\mathbf{z},t)=\frac{t^{1-\gamma(\mathbf{z})}}{\Gamma(2-\gamma(\mathbf{z}))}=\frac{t^{\alpha(\mathbf{z})}}{\Gamma(1-\alpha(\mathbf{z}))}$ with $\alpha(\mathbf{z})=\gamma(\mathbf{z})-1\in(0,\gamma_2-1)$, we can apply [14, Corollary 3.1] to see that the following lemma is satisfied.

Lemma 2.1. *For any $v:[0,T]\to\mathrm{L}^2(\Lambda)$ satisfying*

$$v\in\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right),\, k*v\in\mathrm{H}^1\left((0,T),\mathrm{L}^2(\Lambda)\right),$$

it holds for all $\rho\in[0,T]$ that

$$\int_0^\rho (\partial_t(k*v)(t),v(t))\,\mathrm{d}t \ge \frac{\tilde{k}}{2}\int_0^\rho \|v(t)\|^2\,\mathrm{d}t.$$

The weak formulation of problem (2) is given by:

search $v\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)\right)$ with $\partial_t(k*(\partial_t v-\tilde{v}_0))\in\mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Lambda)^*\right)$ such that for a.a. $t\in(0,T)$ and for all $\phi\in\mathrm{H}_0^1(\Lambda)$ it holds that

$$\langle\partial_t(k*(\partial_t v-\tilde{v}_0))(t),\phi\rangle_{\mathrm{H}_0^1(\Lambda)^*\times\mathrm{H}_0^1(\Lambda)} + \mathbf{L}(t)\,(v(t),\phi) = (f(t),\phi)\,. \tag{9}$$

In the following theorem, the uniqueness of a solution is firstly illustrated. The existence of a solution will be discussed in the next section.

Theorem 2.1 (Uniqueness)**.** *There exists at most one solution $v\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)\right)\cap\mathrm{C}\left([0,T],\mathrm{L}^2(\Lambda)\right)$ with $\partial_t v\in\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)$ and $\partial_t(k*(\partial_t v-\tilde{v}_0))\in\mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Lambda)^*\right)$ satisfying the weak formulation* (9).

Proof. In order to show the uniqueness of a solution, we need to prove that from $f = 0$ in Q_T, and $\tilde{u}_0 = \tilde{v}_0 = 0$ in Λ, it follows that $v = 0$ a.e. in Q_T. We first integrate (9) in time over $t \in (0,\rho) \subset (0,T)$ and afterwards we take $\phi = v(\rho) \in \mathrm{H}^1_0(\Lambda)$ in the result, and we integrate again in time over $\rho \in (0,\eta) \subset (0,T)$. Considering that $(k * \partial_t v)(\rho) = \partial_t(k * v)(\rho)$ (using e.g. [7, Lemma 3.5] for a.a. $\mathbf{x} \in \Lambda$), we get that

$$\int_0^\eta \left(\partial_t \left(k * v\right)(\rho), v(\rho)\right) \mathrm{d}\rho + \int_0^\eta \int_0^\rho \mathbf{L}(t)(v(t), v(\rho))\, \mathrm{d}t\, \mathrm{d}\rho = 0. \tag{10}$$

From Young's inequality for convolutions

$$\|f_1 * f_2\|_{\mathrm{L}^r(0,T)} \le \|f_1\|_{\mathrm{L}^p(0,T)}\, \|f_2\|_{\mathrm{L}^q(0,T)} \tag{$\star$}$$

for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ with $1 \le p, q \le r \le \infty$, we get that

$$\int_\Lambda \|(k * \partial_t v)(\mathbf{z})\|^2_{\mathrm{L}^2(0,T)}\, \mathrm{d}\mathbf{z} \overset{(7)}{\le} C\, \|\partial_t v\|^2_{\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)},$$

and thus $\partial_t(k * v) \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Lambda)\right)$. Therefore, we can apply Lemma 2.1 to obtain that

$$\int_0^\eta \left(\partial_t \left(k * v\right)(\rho), v(\rho)\right) \mathrm{d}\rho \ge \frac{\tilde{k}}{2} \int_0^\eta \|v(t)\|^2\, \mathrm{d}t.$$

From the symmetry of $\mathcal{R}$, we get that

$$\begin{aligned}
\int_0^\eta \left(\int_0^\rho \mathcal{R}(t)\nabla v(t)\, \mathrm{d}t, \nabla v(\rho)\right) \mathrm{d}\rho = {} & \frac{1}{2}\left(\mathcal{R}(\eta)\left[\int_0^\eta \nabla v(s)\ \mathrm{d}s\right], \int_0^\eta \nabla v(t)\, \mathrm{d}t\right) \\
& + \frac{1}{2}\int_0^\eta \left(\int_0^\rho \nabla v(t)\, \mathrm{d}t, \partial_t\mathcal{R}(\rho)\left[\int_0^\rho \nabla v(s)\ \mathrm{d}s\right]\right) \mathrm{d}\rho \\
& - \left(\int_0^\eta \partial_t\mathcal{R}(t)\left[\int_0^t \nabla v(s)\ \mathrm{d}s\right] \mathrm{d}t, \int_0^\eta \nabla v(t)\, \mathrm{d}t\right).
\end{aligned}$$

Employing the ε-Young inequality and $\partial_t\mathcal{R} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$, we have that

$$\int_0^\eta \left(\int_0^\rho \mathcal{R}(t)\nabla v(t)\, \mathrm{d}t, \nabla v(\rho)\right) \mathrm{d}\rho \ge \left(\frac{\alpha}{2} - \varepsilon_1\right)\left\|\int_0^\eta \nabla v(t)\, \mathrm{d}t\right\|^2 - C_{\varepsilon_1}\int_0^\eta \left\|\int_0^\rho \nabla v(t)\, \mathrm{d}t\right\|^2 \mathrm{d}\rho.$$

Using $\partial_t \boldsymbol{b} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^d$, we have that

$$\begin{aligned}
\left|\int_0^\eta \left(\int_0^\rho \boldsymbol{b}(t)\cdot\nabla v(t)\, \mathrm{d}t, v(\rho)\right) \mathrm{d}\rho\right| &= \left|\int_0^\eta \left(-\int_0^\rho \left[\partial_t\boldsymbol{b}(t)\cdot\left(\int_0^t \nabla v(s)\ \mathrm{d}s\right)\right] \mathrm{d}t\ + \boldsymbol{b}(\rho)\cdot\left(\int_0^\rho \nabla v(s)\ \mathrm{d}s\right), v(\rho)\right) \mathrm{d}\rho\right| \\
&\le C_{\varepsilon_2}\int_0^\eta \left\|\int_0^\rho \nabla v(s)\ \mathrm{d}s\right\|^2 \mathrm{d}\rho + \varepsilon_2\int_0^\eta \|v(t)\|^2\, \mathrm{d}t,
\end{aligned}$$

and

$$\left|\int_0^\eta \left(\int_0^\rho c(t)v(t)\, \mathrm{d}t, v(\rho)\right) \mathrm{d}\rho\right| \le C_{\varepsilon_3}\int_0^\eta \left(\int_0^\rho \|v(t)\|^2\, \mathrm{d}t\right) \mathrm{d}\rho + \varepsilon_3\int_0^\eta \|v(\rho)\|^2\, \mathrm{d}\rho.$$

All the above calculations can be calculated to obtain from (10) that

$$\begin{aligned}
&\left(\frac{\tilde{k}}{2} - \varepsilon_2 - \varepsilon_3\right)\int_0^\eta \|v(\rho)\|^2\, \mathrm{d}\rho + \left(\frac{\alpha}{2} - \varepsilon_1\right)\left\|\int_0^\eta \nabla v(\rho)\, \mathrm{d}\rho\right\|^2 \\
&\le C_{\varepsilon_3}\int_0^\eta \left(\int_0^\rho \|v(t)\|^2\, \mathrm{d}t\right) \mathrm{d}\rho + (C_{\varepsilon_1} + C_{\varepsilon_2})\int_0^\eta \left\|\int_0^\rho \nabla v(s)\ \mathrm{d}s\right\|^2 \mathrm{d}\rho.
\end{aligned}$$

First, we fix $\varepsilon_i > 0, i = 1,2,3$ sufficiently small such that $\varepsilon_2 + \varepsilon_3 < \frac{\tilde{k}}{2}$ and $\varepsilon_1 < \frac{\alpha}{2}$. Then, we apply the Grönwall argument, and consequently we obtain that $v = 0$ a.e. in Q_T. □

3. A priori estimates

Here, Rothe's method [34] is invoked to address the existence of a weak solution to problem (9). Firstly, a discretization of the time interval $(0, T]$ into $n \in \mathbb{N}$ equidistant subintervals $\overline{\tau}_i := (t_{i-1}, t_i]$ with length $\tau = \frac{T}{n} < 1$. Secondly, the approximation of v at time $t = t_i$ $(0 \le i \le n)$ is denoted by v_i, the backward Euler finite-difference formulas $(1 \le i \le n)$ are recalled for approximating the first and second order time derivative at time $t = t_i$:

$$\partial_t v(t_i) \approx \delta v_i = \frac{v_i - v_{i-1}}{\tau},$$
$$\partial_{tt} v(t_i) \approx \delta^2 v_i = \frac{v_i - v_{i-1}}{\tau^2} - \frac{\delta v_{i-1}}{\tau}.$$

Please note that $v_0 := \tilde{u}_0$ and $\delta v_0 := \tilde{v}_0$. These notations are also used for any function $u \neq v$. Finally, for $u : \Lambda \times [0, T] \to \mathbb{R}$ satisfying

$$(k * u)(\cdot, 0) = \lim_{t \searrow 0} (k * u)(\cdot, t) = 0, \tag{11}$$

and

$$\int_\Lambda (k * u^2)(\mathbf{z}, 0)\,\mathrm{d}\mathbf{z} = \lim_{t \searrow 0} \int_\Lambda (k * u^2)(\mathbf{z}, t)\,\mathrm{d}\mathbf{z} = 0, \tag{12}$$

the time discrete convolution is defined as follows (a.a. $\mathbf{z} \in \Lambda$)

$$(k * u)(\mathbf{z}, t_i) \approx (k * u)_i(\mathbf{z}) := \sum_{l=1}^{i} k_{i+1-l}(\mathbf{z}) u_l(\mathbf{z}) \tau, \tag{13}$$

with

$$(k * u)_0(\mathbf{z}) := 0 \quad \text{and} \quad \int_\Lambda (k * u^2)_0(\mathbf{z})\,\mathrm{d}\mathbf{z} := 0.$$

In the discrete convolution (13), a possible singularity of k at $t = 0$ is avoided. The following lemma can be found in [14, Lemma 3.3].

Lemma 3.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $u : [0, T] \to \mathrm{L}^2(\Lambda)$ be satisfying* (11) *and* (12)*. Then, for all $j = 1, \ldots, n$ (with n the number of time discretization intervals) it holds that*

$$\sum_{i=1}^{j} (\delta(k * u)_i, u_i)\,\tau \ge \frac{1}{2} \int_\Lambda (k * u^2)_j(z)\,\mathrm{d}z + \frac{1}{2} \sum_{i=1}^{j} \left\| \sqrt{k_i} u_i \right\|^2 \tau.$$

Remark 3.1. *The conditions* (11) *and* (12) *are satisfied for $u = v$ and $u = \partial_t v$ satisfying* (4).

We propose the following time-discrete variational problems:

$$\text{Find } v_i \in \mathrm{H}_0^1(\Lambda),\ i = 1, 2, \ldots, n, \text{ such that}$$
$$\left((k * \delta^2 v)_i, \phi\right) + \mathbf{L}_i(v_i, \phi) = (f_i, \phi), \quad \forall \phi \in \mathrm{H}_0^1(\Lambda). \tag{14}$$

The discrete problem can reformulated by using the time discrete convolution (13), i.e

$$a_i(v_i, \phi) = \langle F_i, \phi \rangle, \quad \forall \phi \in \mathrm{H}_0^1(\Lambda), \tag{15}$$

with

$$a_i(v, \phi) := \frac{1}{\tau} (k(\tau) v, \phi) + \mathbf{L}_i(v, \phi)$$

and

$$\langle F_i, \phi \rangle := (f_i, \phi) + \frac{1}{\tau} (k(\tau) v_{i-1}, \phi) + (k(\tau) \delta v_{i-1}, \phi) - \sum_{l=1}^{i-1} \left(k_{i+1-l} \delta^2 v_l, \phi\right) \tau.$$

Under appropriate assumptions on the data, the well-posedness of this problem is proved in the next lemma.

Lemma 3.2. *Let the assumptions AS-(1–4) be fulfilled. For any $i = 1, 2, \dots, n$, there exists a unique $v_i \in \mathrm{H}_0^1(\Lambda)$ solving* (14).

Proof. Invoking the properties of $\mathcal{L}$ and w, it follows that the bilinear form a_i is $\mathrm{H}_0^1(\Lambda)$-elliptic and continuous for $i = 1, \dots, n$. For instance, the $\mathrm{H}_0^1(\Lambda)$-ellipticity of a_i follows from

$$a_i(v, v) \overset{(5),(8)}{\geq} \frac{\tilde{k}}{\tau} \|v\|^2 + \frac{\alpha}{2} \|\nabla v\|^2 \geq \min\left\{\frac{\tilde{k}}{\tau}, \frac{\alpha}{2}\right\} \|v\|^2_{\mathrm{H}^1(\Lambda)}, \quad \forall v \in \mathrm{H}_0^1(\Lambda).$$

If $\tilde{u}_0, v_1, \dots, v_{i-1}, \tilde{v}_0 \in \mathrm{L}^2(\Lambda)$, then the linear functional F_i is bounded since

$$\begin{aligned}|\langle F_i, \phi \rangle| &\overset{(6)}{\leq} \|f_i\| \|\phi\| + \tau^{-\gamma_2} \|v_{i-1}\| \|\phi\| + \tau^{-\gamma_2} \|v_i - v_{i-1}\| \|\phi\| + \tau^{1-\gamma_2} \|\phi\| \sum_{l=1}^{i-1} \|\delta v_l - \delta v_{l-1}\| \\ &\leq C(\tau^{-\gamma_2}) \|\phi\|_{\mathrm{H}^1(\Lambda)}.\end{aligned}$$

The existence and uniqueness of $v_i \in \mathrm{H}_0^1(\Lambda)$ to problem (15) for any $i = 1, \dots, n$ follows now inductively from the Lax-Milgram lemma. □

In what follows, a priori estimates are addressed, which will be helpful in the next section.

Lemma 3.3. *Assume that AS-(1–4) are fulfilled. Then, there exists positive constants C and τ_0 such that for any $\tau < \tau_0$ and for all $j = 1, 2, \dots, n$, it holds that*

$$\int_\Lambda (k * (\delta v)^2)_j(z)\, \mathrm{d}z + \sum_{i=1}^{j} \left\| \sqrt{k_i} \delta v_i \right\|^2 \tau + \sum_{i=1}^{j} \|\delta v_i\|^2 \tau + \left\| \nabla v_j \right\|^2 + \sum_{i=1}^{j} \|\nabla v_i - \nabla v_{i-1}\|^2 \leq C.$$

Proof. Choose $\phi = \delta v_i \tau$ in (14) and sum up for $i = 1, \dots, j$ with $1 \leq j \leq n$. By the aid of the next relation for any sequence $\{z_i\}_{i \in \mathbb{N}} \subset \mathrm{L}^2(\Lambda)$, cfr. [14, Lemma 3.3]:

$$\delta(k * z)_i(\cdot) = k_i(\cdot) z_0(\cdot) + (k * \delta z)_i(\cdot), \tag{16}$$

we deduce that

$$\sum_{i=1}^{j} (\delta(k * \delta v)_i, \delta v_i)\, \tau + \sum_{i=1}^{j} \mathbf{L}_i(v_i, \delta v_i) \tau = \sum_{i=1}^{j} (f_i, \delta v_i)\, \tau + \sum_{i=1}^{j} (k_i \tilde{v}_0, \delta v_i)\, \tau. \tag{17}$$

For the first term on the LHS, we apply Lemma 3.1 to get that

$$\sum_{i=1}^{j} (\delta(k * \delta v)_i, \delta v_i)\, \tau \overset{(8)}{\geq} \frac{1}{2} \int_\Lambda (k * (\delta v)^2)_j(\mathbf{z})\, \mathrm{d}\mathbf{z} + \frac{1}{4} \sum_{i=1}^{j} \left\| \sqrt{k_i} \delta v_i \right\|^2 \tau + \frac{\tilde{k}}{4} \sum_{i=1}^{j} \|\delta v_i\|^2 \tau. \tag{18}$$

To handle the second term on the LHS of (17), we invoke the following per parts formula for a symmetric bilinear form [35, Eq. 3.16]:

$$\sum_{i=1}^{j} r(t_i; z_i, z_i - z_{i-1}) = \frac{1}{2} r(t_j; z_j, z_j) - \frac{1}{2} r(0; z_0, z_0) + \frac{1}{2} \sum_{i=1}^{j} \left(r(t_i; \delta z_i, \delta z_i) \tau^2 - \delta r(t_i; z_{i-1}, z_{i-1}) \tau \right). \tag{19}$$

Hence, due to the symmetry of $\mathcal{R}$, we have that

$$\begin{aligned}\sum_{i=1}^{j} (\mathcal{R}_i \nabla v_i, \nabla \delta v_i)\, \tau = \frac{1}{2} \left(\mathcal{R}_j \nabla v_j, \nabla v_j \right) - \frac{1}{2} (\mathcal{R}_0 \nabla \tilde{v}_0, \nabla \tilde{v}_0) - \frac{1}{2} \sum_{i=1}^{j} ((\delta \mathcal{R}_i) \nabla v_{i-1}, \nabla v_{i-1})\, \tau \\ + \frac{1}{2} \sum_{i=1}^{j} (\mathcal{R}_i (\nabla v_i - \nabla v_{i-1}), \nabla v_i - \nabla v_{i-1}),\end{aligned}$$

and thus we get that

$$\sum_{i=1}^{j} (\mathcal{R}_i \nabla v_i, \nabla \delta v_i)\,\tau \geq \frac{\alpha}{2} \left\| \nabla v_j \right\|^2 - C - C \sum_{i=1}^{j-1} \|\nabla v_i\|^2 \,\tau + \frac{\alpha}{2} \sum_{i=1}^{j} \|\nabla v_i - \nabla v_{i-1}\|^2 \,. \tag{20}$$

By recalling ε-Young inequality and $v_j = \tilde{u}_0 + \sum_{l=1}^{j} \delta v_l \tau$, we obtain that

$$\left| \sum_{i=1}^{j} (c_i v_i, \delta v_i)\,\tau \right| \leq C_{\varepsilon_1} \sum_{i=1}^{j} \|v_i\|^2 \,\tau + \varepsilon_1 \sum_{i=1}^{j} \|\delta v_i\|^2 \,\tau \leq C_{\varepsilon_1} + C_{\varepsilon_1} \sum_{i=1}^{j} \left(\sum_{l=1}^{i} \|\delta v_l\|^2 \,\tau \right) \tau + \varepsilon_1 \sum_{i=1}^{j} \|\delta v_i\|^2 \,\tau,$$

and

$$\left| \sum_{i=1}^{j} (\boldsymbol{b}_i \cdot \nabla v_i, \delta v_i)\,\tau \right| \leq C_{\varepsilon_2} \sum_{i=1}^{j} \|\nabla v_i\|^2 \,\tau + \varepsilon_2 \sum_{i=1}^{j} \|\delta v_i\|^2 \,\tau.$$

Moreover, we estimate the first and last term in the right-hand side (RHS) of (17) as follows:

$$\left| \sum_{i=1}^{j} (f_i, \delta v_i)\,\tau \right| \leq C_{\varepsilon_3} + \varepsilon_3 \sum_{i=1}^{j} \|\delta v_i\|^2 \,\tau$$

and

$$\begin{aligned} \left| \sum_{i=1}^{j} (k_i \tilde{v}_0, \delta v_i)\,\tau \right| &\leq C_{\varepsilon_4} \int_\Lambda \tilde{v}_0^2(\mathbf{z}) \left(\sum_{i=1}^{j} k_i(\mathbf{z}) \tau \right) \mathrm{d}\mathbf{z} + \varepsilon_4 \sum_{i=1}^{j} \tau \int_\Lambda k_i(\mathbf{z})\, (\delta v_i)^2\,(\mathbf{z})\,\mathrm{d}\mathbf{z} \\ &\leq C_{\varepsilon_4} \|\tilde{v}_0\|^2 + \varepsilon_4 \sum_{i=1}^{j} \left\| \sqrt{k_i} \delta v_i \right\|^2 \tau. \end{aligned}$$

Finally, the previous estimates are combined to get that

$$\begin{aligned} &\frac{1}{2} \int_\Lambda (k * (\delta v)^2)_j(\mathbf{z})\,\mathrm{d}\mathbf{z} + \left(\frac{1}{4} - \varepsilon_4 \right) \sum_{i=1}^{j} \left\| \sqrt{k_i} \delta v_i \right\|^2 \tau + \left(\frac{\tilde{k}}{4} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 \right) \sum_{i=1}^{j} \|\delta v_i\|^2 \,\tau + \frac{\alpha}{2} \left\| \nabla v_j \right\|^2 \\ &+ \frac{\alpha}{2} \sum_{i=1}^{j} \|\nabla v_i - \nabla v_{i-1}\|^2 \leq C_{\varepsilon_{1,3,4}} + C_{\varepsilon_2} \sum_{i=1}^{j} \|\nabla v_i\|^2 \,\tau + C_{\varepsilon_1} \sum_{i=1}^{j} \left(\sum_{l=1}^{i} \|\delta v_l\|^2 \,\tau \right) \tau. \end{aligned}$$

Fixing $\varepsilon_i > 0, i = 1, \ldots, 4$, sufficiently small and applying Grönwall's lemma conclude the proof. □

Corollary 3.1. *Assume that AS-(1–4) are fulfilled. Then, there exists positive constants C and τ_0 such that for any $\tau < \tau_0$ and for all $j = 1, 2, \ldots, n$, it holds that*

$$\left\| (k * \delta^2 v)_j \right\|^2_{\mathrm{H}_0^1(\Lambda)^*} \leq C.$$

Proof. The estimate follows from

$$\begin{aligned} \left\| (k * \delta^2 v)_i \right\|_{\mathrm{H}_0^1(\Lambda)^*} &= \sup_{\|\phi\|_{\mathrm{H}_0^1(\Lambda)} = 1} \left| \langle (k * \delta^2 v)_i, \phi \rangle_{\mathrm{H}_0^1(\Lambda)^* \times \mathrm{H}_0^1(\Lambda)} \right| \\ &= \sup_{\|\phi\|_{\mathrm{H}_0^1(\Lambda)} = 1} |(f_i, \phi) - \mathbf{L}_i(v_i, \phi)| \\ &\leq C\,[1 + \|v_i\| + \|\nabla v_i\|]\,, \end{aligned}$$

and by invoking the result of Lemma 3.3. □

4. Existence of a solution

We firstly exhibit the following piecewise linear function with respect to time $\mathcal{V}_n : [0,T] \to \mathrm{L}^2(\Lambda)$

$$\mathcal{V}_n : [0,T] \to \mathrm{L}^2(\Lambda) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0 \\ v_{i-1} + (t - t_{i-1})\delta v_i & t \in \overline{\tau}_i \end{cases},$$

and the piecewise constant in time function $\overline{v}_n : [0,T] \to \mathrm{L}^2(\Lambda)$

$$\overline{v}_n : [0,T] \to \mathrm{L}^2(\Lambda) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0 \\ v_i & t \in \overline{\tau}_i \end{cases}.$$

Similarly, we define $\overline{k}_n$, $\overline{\mathcal{R}}_n$, $\overline{\boldsymbol{b}}_n$, $\overline{c}_n$, $\overline{\mathbf{L}}_n$ and $\overline{f}_n$. Moreover, we define

$$C_n : [0,T] \to \mathrm{L}^2(\Lambda) : t \mapsto \begin{cases} 0 & t = 0 \\ (k * \delta v)_{i-1} + (t - t_{i-1})\delta(k * \delta v)_i & t \in \overline{\tau}_i \end{cases},$$

$$\overline{C}_n : [0,T] \to \mathrm{L}^2(\Lambda) : t \mapsto \begin{cases} 0 & t = 0 \\ (k * \delta v)_i & t \in \overline{\tau}_i \end{cases}.$$

By the aid of Rothe's functions and Eq. (16), then, on the whole time frame, Eq. (14) can be formulated as

$$\left(\partial_t C_n(t) - \overline{k}_n(t)\tilde{v}_0, \phi\right) + \overline{\mathbf{L}}_n(t)(\overline{v}_n(t), \phi) = \left(\overline{f}_n(t), \phi\right), \quad \forall \phi \in \mathrm{H}_0^1(\Lambda), \tag{21}$$

where

$$\overline{\mathbf{L}}_n(t)\left(\overline{v}_n(t), \phi\right) = \left(\overline{\mathcal{R}}_n(t)\nabla\overline{v}_n(t), \nabla\phi\right) + \left(\overline{\boldsymbol{b}}_n(t) \cdot \nabla\overline{v}_n(t), \phi\right) + \left(\overline{c}_n(t)\overline{v}_n(t), \phi\right).$$

The existence of a unique weak solution is addressed in the next theorem. In this setting, we can make use of the classical lemma [34, Lemma 1.3.13] for the convergence of the Rothe sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$, whilst for the diffusion equation with space-dependent variable order the Riesz-Frechét-Kolmogorov theorem was needed (see [14, Theorem 6.1]).

Theorem 4.1 (Existence)**.** *Assume that AS-(1–4) are fulfilled. Then, a unique weak solution v exists to the problem* (9) *with* $v \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Lambda)\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Lambda)\right)$ *with* $\partial_t v \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Lambda)\right)$ *and* $\partial_t(k*(\partial_t v - \tilde{v}_0)) \in \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Lambda)^*\right)$.

Proof. Some convergence results that can be deduced from Lemma 3.3 will be firstly stated. The Rellich-Kondrachov theorem [36, Theorem 6.6-3] implies the compact embedding

$$\mathrm{H}_0^1(\Lambda) \hookrightarrow\hookrightarrow \mathrm{L}^2(\Lambda).$$

From Lemma 3.3 and Friedrichs' inequality, it follows that we have for all $n \in \mathbb{N}$ that

$$\max_{t\in[0,T]} \|\overline{v}_n(t)\|^2_{\mathrm{H}_0^1(\Lambda)} \le C$$

and

$$\int_0^T \|\partial_t \mathcal{V}_n(s)\|^2 \,\mathrm{d}s \le C.$$

Hence, the conditions of [34, Lemma 1.3.13] are satisfied. Therefore, there exists a function $v \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Lambda)\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Lambda)\right)$ and a subsequence $\{\mathcal{V}_{n_k}\}_{k\in\mathbb{N}}$ of $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ (denoted by the same symbol again) such that

$$\begin{cases} \mathcal{V}_n \to v & \text{in } \mathrm{C}\left([0,T], \mathrm{L}^2(\Lambda)\right), \\ \mathcal{V}_n(t) \rightharpoonup v(t) & \text{in } \mathrm{H}_0^1(\Lambda), \text{ for all } t \in [0,T], \\ \overline{v}_n(t) \rightharpoonup v(t) & \text{in } \mathrm{H}_0^1(\Lambda), \text{ for all } t \in [0,T], \\ \partial_t \mathcal{V}_n \rightharpoonup \partial_t v & \text{in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Lambda)\right). \end{cases}$$

Invoking Lemma 3.3 also implies that

$$\int_0^T \|\mathcal{V}_n(t) - \overline{v}_n(t)\|^2 \, \mathrm{d}t \to 0 \text{ as } n \to \infty,$$

i.e. $\{\mathcal{V}_n\}$ and $\{\overline{v}_n\}$ have the same limit in $\mathrm{L}^2\left((0,T), \mathrm{L}^2(\Lambda)\right)$. Then, we integrate eq. (21) with respect to time over $(0,\rho) \subset (0,T)$ for the resulting subsequence to deduce that

$$(C_n(\rho), \phi) - \int_0^\rho \left(\overline{k}_n(t)\tilde{v}_0, \phi\right) \mathrm{d}t + \int_0^\rho \overline{\mathbf{L}}_n(t)(\overline{v}_n(t), \phi) \, \mathrm{d}t = \int_0^\rho \left(\overline{f}_n(t), \phi\right) \mathrm{d}t, \quad \forall \phi \in \mathrm{H}_0^1(\Lambda). \tag{22}$$

We have that $\left\|\overline{\mathcal{R}}_n(t) - \mathcal{R}(t)\right\|_{(\mathrm{L}^\infty(\Lambda))^{d\times d}} \to 0$, $\left\|\overline{\boldsymbol{b}}_n(t) - \boldsymbol{b}(t)\right\|_{\mathrm{L}^\infty(\Lambda)} \to 0$ and $\|\overline{c}_n(t) - c(t)\|_{\mathrm{L}^\infty(\Lambda)} \to 0$ for all $t \in (0,T)$ as $n \to \infty$. Hence, employing the results above, for $\rho \in (0,T)$, we can see that

$$\left|\int_0^\rho \overline{\mathbf{L}}_n(t)(\overline{v}_n(t), \phi) \, \mathrm{d}t - \int_0^\rho \mathbf{L}(t)(v(t), \phi) \, \mathrm{d}t\right| \stackrel{n\to\infty}{\longrightarrow} 0.$$

Similarly,

$$\left|\int_0^\rho \left(\overline{f}_n(t), \phi\right) \mathrm{d}t - \int_0^\rho (f(t), \phi) \, \mathrm{d}t\right| \to 0$$

as $n \to \infty$. In order to be able to make the limit transition in the remaining terms of (22), we integrate the equation (22) once more in time over $\rho \in (0,\eta) \subset (0,T)$. We obtain that

$$\begin{aligned} \int_0^\eta (C_n(\rho), \phi) \, \mathrm{d}\rho - \int_0^\eta \left(\tilde{v}_0 \int_0^\rho \overline{k}_n(t) \, \mathrm{d}t, \phi\right) \mathrm{d}\rho + \int_0^\eta \int_0^\rho \overline{\mathbf{L}}_n(t)(\overline{v}_n(t), \phi) \, \mathrm{d}t \, \mathrm{d}\rho \\ = \int_0^\eta \int_0^\rho \left(\overline{f}_n(t), \phi\right) \mathrm{d}t \, \mathrm{d}\rho, \quad \forall \phi \in \mathrm{H}_0^1(\Lambda). \end{aligned} \tag{23}$$

The following steps should be considered

$$\begin{aligned} &\text{(i)} \quad \left|\int_0^T (C_n(\rho), \phi) \, \mathrm{d}\rho - \int_0^T \left(\overline{C}_n(\rho), \phi\right) \mathrm{d}\rho\right| \to 0; \\ &\text{(ii)} \quad \left|\int_0^T \left(\overline{C}_n(\rho), \phi\right) \mathrm{d}\rho - \int_0^T ((k * \partial_t \mathcal{V}_n)(\rho), \phi) \, \mathrm{d}\rho\right| \to 0; \\ &\text{(iii)} \quad \left|\int_0^T ((k * \partial_t \mathcal{V}_n)(\rho), \phi) \, \mathrm{d}\rho - \int_0^T ((k * \partial_t v)(\rho), \phi) \, \mathrm{d}\rho\right| \to 0. \end{aligned}$$

Invoking Corollary 3.1, we get that

$$\begin{aligned} \left|\int_0^T (C_n(\rho), \phi) \, \mathrm{d}\rho - \int_0^T \left(\overline{C}_n(\rho), \phi\right) \mathrm{d}\rho\right| &= \left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} ((t - t_i)\delta(k * \delta v)_i, \phi) \, \mathrm{d}t\right| \\ &\stackrel{(16)}{=} \left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t - t_i) \left((k * \delta^2 v)_i - k_i \tilde{v}_0, \phi\right) \mathrm{d}t\right| \\ &\le \sum_{i=1}^n \tau^2 \left|\langle (k * \delta^2 v)_i, \phi \rangle_{\mathrm{H}_0^1(\Lambda)^* \times \mathrm{H}_0^1(\Lambda)}\right| \\ &\quad + \sum_{i=1}^n \tau^2 \int_\Lambda |k_i(\mathbf{z}, t)| \, |\tilde{v}_0(\mathbf{z})| \, |\phi(\mathbf{z})| \, \mathrm{d}\mathbf{z} \\ &\stackrel{(6)}{\le} C\tau + C\tau^{2-\gamma_2} \to 0 \text{ as } n \to \infty. \end{aligned}$$

It holds that $\overline{C}_n(t) = (\overline{k}_n * \partial_t \mathcal{V}_n)(\lceil t \rceil_\tau)$ for any $t \in (0,T)$, where $\lceil t \rceil_\tau = t_i$ when $t \in (t_{i-1}, t_i]$. We have that

$$\left((\overline{k}_n * \partial_t \mathcal{V}_n)(\lceil t \rceil_\tau), \phi\right) - ((k * \partial_t \mathcal{V}_n)(t), \phi)$$
$$= \left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\lceil t \rceil_\tau - s\right) \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right) + \left(\int_0^t \left[\overline{k}_n\left(\lceil t \rceil_\tau - s\right) - k\left(t-s\right)\right] \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right).$$

Next, we will show that

$$\left|\int_0^T \left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(-s + \lceil t \rceil_\tau\right) \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right) \mathrm{d}t\right| \to 0 \tag{24}$$

and

$$\left|\int_0^T \left(\int_0^t \left[\overline{k}_n\left(-s + \lceil t \rceil_\tau\right) - k\left(t-s\right)\right] \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\, \mathrm{d}t\right)\right| \to 0 \tag{25}$$

as $n \to \infty$. Then, the limit transition (ii) is satisfied. First, we have for $t \in (t_{i-1}, t_i]$ that

$$\left|\left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\lceil t \rceil_\tau - s\right) \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right)\right|$$
$$\leq \|\phi\| \left[\int_\Lambda \left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)\ \mathrm{d}s\right)\left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)\left(\partial_t \mathcal{V}_n\right)^2\left(\mathbf{z}, s\right)\ \mathrm{d}s\right) \mathrm{d}\mathbf{z}\right]^{\frac{1}{2}}.$$

For $t \in (t_{i-1}, t_i]$ and a.a. $\mathbf{z} \in \Lambda$, it holds that

$$\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)\ \mathrm{d}s \leq \tau^{2-\gamma_2}.$$

Thus, using Lemma 3.3, we obtain for $t \in (t_{i-1}, t_i]$ that

$$\begin{aligned}
\left|\left(\int_t^{\lceil t \rceil_\tau} \overline{k}_n\left(\lceil t \rceil_\tau - s\right) \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right)\right|^2 &\leq \tau^{2-\gamma_2} \|\phi\|^2 \int_\Lambda \int_0^{t_i} \overline{k}_n\left(\mathbf{z}, t_i - s\right)\left(\partial_t \mathcal{V}_n\right)^2(\mathbf{z}, s)\ \mathrm{d}s\, \mathrm{d}\mathbf{z} \\
&\leq \tau^{2-\gamma_2} \|\phi\|^2 \int_\Lambda \left(k * (\delta v)^2\right)_i (\mathbf{z})\, \mathrm{d}\mathbf{z} \\
&\leq C\tau^{2-\gamma_2},
\end{aligned}$$

so (24) is valid. Secondly, we see that

$$\left|\int_0^T \left(\int_0^t \left[\overline{k}_n\left(\lceil t \rceil_\tau - s\right) - k\left(t-s\right)\right] \partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right) \mathrm{d}t\right|$$
$$\leq C\tau^{\frac{1}{2}(2-\gamma_2)} \sqrt{T} \int_\Lambda |\phi(\mathbf{z})| \left[\int_0^T \left(\int_0^t \left|\overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right) - k\left(\mathbf{z}, t-s\right)\right| (\partial_t \mathcal{V}_n)^2(\mathbf{z}, s)\ \mathrm{d}s\right) \mathrm{d}t\right]^{\frac{1}{2}} \mathrm{d}\mathbf{z}.$$

As

$$\int_0^t \left|\overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right) - k\left(\mathbf{z}, t-s\right)\right|\ \mathrm{d}s \leq C\tau^{2-\gamma_2},$$

we conclude that

$$\begin{aligned}
&\int_0^T \left(\int_0^t \left|\overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right) - k\left(\mathbf{z}, t-s\right)\right| (\partial_t \mathcal{V}_n)^2(\mathbf{z}, s)\ \mathrm{d}s\right) \mathrm{d}t \\
&= \int_0^T \left(\left(k * (\partial_t \mathcal{V}_n)^2\right)(\mathbf{z}, t) - \int_0^t \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)(\partial_t \mathcal{V}_n)^2(\mathbf{z}, s)\ \mathrm{d}s\right) \mathrm{d}t \\
&\overset{(\star)}{\leq} \left\|\overline{k}_n(\mathbf{z})\right\|_{\mathrm{L}^1(0,T)} \left\|(\partial_t \mathcal{V}_n)^2(\mathbf{z})\right\|_{\mathrm{L}^1(0,T)} + \int_0^T \left(\int_0^t \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)(\partial_t \mathcal{V}_n)^2(\mathbf{z}, s)\ \mathrm{d}s\right) \mathrm{d}t \\
&\overset{(7)}{\leq} C \left\|(\partial_t \mathcal{V}_n)^2(\mathbf{z})\right\|_{\mathrm{L}^1(0,T)} + \int_0^T \left(\int_0^t \overline{k}_n\left(\mathbf{z}, \lceil t \rceil_\tau - s\right)(\partial_t \mathcal{V}_n)^2(\mathbf{z}, s)\ \mathrm{d}s\right) \mathrm{d}t.
\end{aligned}$$

Therefore, using Lemma 3.3, we finally obtain that

$$
\begin{aligned}
&\left|\int_0^T \left(\int_0^t \left[\overline{k}_n\left(\lceil t\rceil_\tau - s\right) - k\left(t-s\right)\right]\partial_t \mathcal{V}_n(s)\ \mathrm{d}s, \phi\right)\mathrm{d}t\right| \\
&\le C\tau^{\frac{1}{2}(2-\gamma_2)}\,\|\phi\|\left[\int_\Lambda \left\|(\partial_t\mathcal{V}_n)^2(\mathbf{z})\right\|_{\mathrm{L}^1(0,T)}\mathrm{d}\mathbf{z} + \int_\Lambda\left(\int_0^T\left(\int_0^t \overline{k}_n\left(\mathbf{z},\lceil t\rceil_\tau - s\right)(\partial_t\mathcal{V}_n)^2(\mathbf{z},s)\ \mathrm{d}s\right)\mathrm{d}t\right)\mathrm{d}\mathbf{z}\right]^{\frac{1}{2}} \\
&\le C\tau^{\frac{1}{2}(2-\gamma_2)}\left[\sum_{i=1}^n \|\delta v_i\|^2\,\tau + \sum_{i=1}^n \tau\int_\Lambda\left(k*(\delta v)^2\right)_i(\mathbf{z})\,\mathrm{d}\mathbf{z}\right]^{\frac{1}{2}} \\
&\le C\tau^{\frac{1}{2}(2-\gamma_2)}.
\end{aligned}
$$

Hence, Eq. (24) is satisfied. The limit transmission (iii) follows from $\partial_t\mathcal{V}_n \rightharpoonup \partial_t v$ in $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)$ since

$$
\begin{aligned}
&\left|\int_0^\rho ((k*\partial_t\mathcal{V}_n)(t),\phi)\,\mathrm{d}t\right| \\
&\le \sqrt{T}\int_\Lambda |\phi(\mathbf{z})|\left(\int_0^\rho (k*|\partial_t\mathcal{V}_n|)^2(\mathbf{z},t)\,\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}\mathbf{z} \\
&\overset{(\star)}{\le} \sqrt{T}\int_\Lambda |\phi(\mathbf{z})|\,\|k(\mathbf{z})\|_{\mathrm{L}^1(0,\rho)}\,\|\partial_t\mathcal{V}_n(\mathbf{z})\|_{\mathrm{L}^2(0,\rho)}\,\mathrm{d}\mathbf{z} \\
&\overset{(7)}{\le} C\,\|\phi\|\,\|\partial_t\mathcal{V}_n\|_{\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)},
\end{aligned}
$$

because the estimated integral represents a linear bounded functional on the space $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)$. Finally, the limit transition $n\to\infty$ in (23) can be achieved, and therefore we get that

$$
\begin{aligned}
\int_0^\eta ((k*\partial_t v)(\rho),\phi)\,\mathrm{d}\rho - \int_0^\eta\left(\tilde{v}_0\int_0^\rho k(t)\,\mathrm{d}t,\phi\right)\mathrm{d}\rho + \int_0^\eta\int_0^\rho \mathbf{L}(t)(v(t),\phi)\,\mathrm{d}t\,\mathrm{d}\rho \\
= \int_0^\eta\int_0^\rho (f(t),\phi)\,\mathrm{d}t\,\mathrm{d}\rho.
\end{aligned}
$$

Differentiating the previous relation with respect to η gives

$$
((k*\partial_t v)(\eta),\phi) - \left(\tilde{v}_0\int_0^\eta k(t)\,\mathrm{d}t,\phi\right) + \int_0^\eta \mathbf{L}(t)(v(t),\phi)\,\mathrm{d}t = \int_0^\eta (f(t),\phi)\,\mathrm{d}t. \tag{26}
$$

Since $v\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)\right)$ and $f\in\mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Lambda)\right)$, we obtain from (26), that $(k*(\partial_t v-\tilde{v}_0))(\eta)$ is absolutely continuous in time with values in $\mathrm{H}_0^1(\Lambda)^*$ and with $(k*(\partial_t v-\tilde{v}_0))(0)=0$ as

$$
(k*(\partial_t v-\tilde{v}_0))(\eta) = \int_0^\eta f(t)\,\mathrm{d}t - \int_0^\eta \mathcal{L}(t)v(t)\,\mathrm{d}t \quad \text{in } \mathrm{H}_0^1(\Lambda)^*,
$$

where

$$
\langle f(t)-\mathcal{L}(t)v(t),\phi\rangle = (f(t),\phi) - \mathbf{L}(t)(v(t),\phi).
$$

This relation is differentiated with respect to η to obtain for a.a. $t\in(0,T)$ that (replace η by t)

$$
\partial_t(k*(\partial_t v-\tilde{v}_0))(t) = f(t)-\mathcal{L}(t)v(t) \quad \text{in } \mathrm{H}_0^1(\Lambda)^*, \tag{27}
$$

i.e. v satisfies the weak formulation (9). Moreover, again using that $v\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)\right)$ and $f\in\mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Lambda)\right)$, we get that

$$
\partial_t(k*(\partial_t v-\tilde{v}_0))\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)^*\right).
$$

The uniqueness of a solution follows from Theorem 2.1. The proof is finally concluded. □

Remark 4.1. *The results in this paper are also valid when an additional non-linear source $F(v)$ is considered in the governing PDE in RHS if F is Lipschitz continuous. Moreover, the results stay also valid (with the space $\mathrm{H}_0^1(\Lambda)$ replaced by $\mathrm{H}^1(\Lambda)$) when a Neumann condition is considered on the whole boundary of the domain ($\mathcal{R}(t)\nabla v(t) \cdot \boldsymbol{\nu} = 0$ on $\partial\Lambda$ for $t > 0$) if we assume that*

$$c(z,t) - \frac{\|\boldsymbol{b}\|^2_{\mathbf{L}^\infty(\overline{Q_T})}}{2\alpha} \geq \tilde{c}_0 > 0, \quad \text{for a.a. } (z,t) \in Q_T.$$

Then, we obtain (similar to Eq. (5)) the $\mathrm{H}^1(\Lambda)$-ellipticity of the bilinear form:

$$\mathbf{L}(t)(\phi,\phi) \geq \min\left\{\frac{\alpha}{2}, \tilde{c}_0\right\} \|\phi\|^2_{\mathrm{H}^1(\Lambda)}, \quad \forall \phi \in \mathrm{H}^1(\Lambda).$$

Remark 4.2. *It should be mentioned that showing that $\partial_t v(0) = \tilde{v}_0$ for the Caputo fractional derivative with space-dependent variable order cannot be achieved following our approach. However, we will show in the next section that this result can be obtained in the case that $\gamma(z)$ is constant in Λ.*

5. Existence of a weak solution for the wave equation with fractional derivative of constant order

We consider the constant order fractional case in this section, i.e. $\gamma(\mathbf{z}) = \gamma$ with $\gamma \in (1,2)$. Thus, the kernel k only depends on the time-variable. We have the existence of the function $l(t) = \frac{t^{\gamma-2}}{\Gamma(\gamma-1)}$ such that $(l * k)(t) = 1$ for all $t \in [0,T]$. Next, the convolution operation with kernel l can be applied on (27), i.e.

$$(l * [\partial_t(k * (\partial_t v - \tilde{v}_0))])(t) = (l * [f - \mathcal{L}v])(t) \quad \text{in } \mathrm{H}_0^1(\Lambda)^*.$$

From the absolute continuity of $(k * (\partial_t v - \tilde{v}_0))(t)$, $(k * (\partial_t v - \tilde{v}_0))(0) = 0$ and $(l * k)(t) = 1$, we have that

$$(l * [\partial_t(k * (\partial_t v - \tilde{v}_0))])(t) = \partial_t (l * k * (\partial_t v - \tilde{v}_0))(t) = \partial_t v(t) - \tilde{v}_0.$$

Hence,

$$\partial_t v(t) - \tilde{v}_0 = (l * [f - \mathcal{L}v])(t) \quad \text{in } \mathrm{H}_0^1(\Lambda)^*.$$

Since $v \in \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Lambda)\right)$ and $f \in \mathrm{L}^\infty\left((0,T), \mathrm{L}^2(\Lambda)\right)$, we obtain for all $\phi \in \mathrm{H}_0^1(\Lambda)$ that

$$\begin{aligned}\lim_{t \searrow 0} \left| \langle \partial_t v(t) - \tilde{v}_0, \phi \rangle_{\mathrm{H}_0^1(\Lambda)^* \times \mathrm{H}_0^1(\Lambda)} \right| &= \lim_{t \searrow 0} \left| \int_0^t l(t-s) \left[(f(s), \phi) - \mathbf{L}(s)(v(s), \phi) \right] \, \mathrm{d}s \right| \\ &\leq C \lim_{t \searrow 0} \int_0^t l(t-s) \, \mathrm{d}s = 0.\end{aligned}$$

Thus $\partial_t v(0) = \tilde{v}_0$ in $\mathrm{H}_0^1(\Lambda)^*$ and

$$\partial_t v \in \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Lambda)^*\right)$$

as e.g. it similarly holds that

$$\begin{aligned}&\lim_{t \searrow s} \left| \langle \partial_t v(t) - \partial_t v(s), \phi \rangle_{\mathrm{H}_0^1(\Lambda)^* \times \mathrm{H}_0^1(\Lambda)} \right| \\ &= \lim_{t \searrow s} \left| \int_s^t l(t-r) \left[(f(r), \phi) - \mathbf{L}(r)(v(r), \phi) \right] \, \mathrm{d}r + \int_0^s (l(t-r) - l(s-r)) \left[(f(r), \phi) - \mathbf{L}(r)(v(r), \phi) \right] \, \mathrm{d}r \right| \\ &\leq C \lim_{t \searrow s} \left[\int_s^t l(t-r) \, \mathrm{d}r + \int_0^s (l(s-r) - l(t-r)) \, \mathrm{d}r \right] = 0.\end{aligned}$$

This observation leads to an improvement of the results obtained in [18, Theorem 2], which we state in the following theorem.

Theorem 5.1 (Fractional superdiffusion equation)**.** *Assume that AS-(1–4) are fulfilled. Then, a unique weak solution v to the problem* (9) *is existed with* $v \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Lambda)\right)\cap\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)\right)$, $\partial_t v \in \mathrm{C}\left([0,T],\mathrm{H}_0^1(\Lambda)^*\right)\cap\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Lambda)\right)$ *and* $\partial_t(k * (\partial_t v - \tilde{v}_0)) \in \mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Lambda)^*\right)$.

Remark 5.1. *Theorem 5.1 is also valid when the kernel k is replaced by*

$$k(t) = \int_1^2 \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\mu(\gamma)\,\mathrm{d}\gamma,$$

where $\mu \in \mathrm{L}^1(1,2)$ *satisfies* $\mu \geq 0$ *and* $\mu \not\equiv 0$, *i.e. considering a time fractional derivative of distributed order. For more details, we refer the reader to [18].*

6. Conclusion

We have investigated an initial-boundary value problem for an autonomous fractional wave equation with space-dependent variable order wherein the coefficients have a dependency on the spatial and time variables. We have shown the uniqueness of a weak solution. A priori estimates which are required to prove the existence of a solution by Rothe's method have been deduced. Accordingly, we have established the existence of a unique weak solution to the problem under consideration. Future work can be concerned with investigating whether the scheme proposed in this paper is not only valuable from a theoretical viewpoint but also from a numerical viewpoint.

Acknowledgement

K. Van Bockstal is supported by a postdoctoral fellowship of the Research Foundation - Flanders (106016/12P2919N). A. S. Hendy wishes to acknowledge the support of the RSF grant, project 22-21-00075.

Conflict of interest

The authors declare that they have no conflict of interest.

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