

Article

Exact Solutions of the Oberbeck–Boussinesq Equations for the Description of Shear Thermal Diffusion of Newtonian Fluid Flows

Sergey Ershkov ^{1,*} , Natalya Burmasheva ^{2,3} , Dmytro D. Leshchenko ⁴  and Evgeniy Yu. Prosviryakov ^{2,3} 

¹ Department of Scientific Researches, Plekhanov Russian University of Economics, Scopus Number 60030998, 36 Streymyanny Lane, 117997 Moscow, Russia

² Sector of Nonlinear Vortex Hydrodynamics, Institute of Engineering Science of Ural Branch of the Russian Academy of Sciences, 34 Komsomolskaya St., 620049 Ekaterinburg, Russia; nat_burm@mail.ru (N.B.); evgen_pros@mail.ru (E.Y.P.)

³ Academic Department of Information Technologies and Control Systems, Ural Federal University, 19 Mira St., 620049 Ekaterinburg, Russia

⁴ Department of Theoretical Mechanics, Odessa State Academy of Civil Engineering and Architecture, 4 Didrikhson St., 65029 Odessa, Ukraine; leshchenkodmytro@gmail.com

* Correspondence: sergej-ershkov@yandex.ru

Abstract: We present a new exact solution of the thermal diffusion equations for steady-state shear flows of a binary fluid. Shear fluid flows are used in modeling and simulating large-scale currents of the world ocean, motions in thin layers of fluid, fluid flows in processes, and apparatuses of chemical technology. To describe the steady shear flows of an incompressible fluid, the system of Navier–Stokes equations in the Boussinesq approximation is redefined, so the construction of exact and numerical solutions to the equations of hydrodynamics is a very difficult and urgent task. A non-trivial exact solution is constructed in the Lin–Sidorov–Aristov class. For this class of exact solutions, the hydrodynamic fields (velocity field, pressure field, temperature field, and solute concentration field) were considered as linear forms in the x and y coordinates. The coefficients of linear forms depend on the third coordinate z . Thus, when considering a shear flow, the two-dimensional velocity field depends on three coordinates. It is worth noting that the solvability condition given in the article imposes a condition (relation) only between the velocity gradients. A theorem on the uniqueness of the exact solution in the Lin–Sidorov–Aristov class is formulated. The remaining coefficients of linear forms for hydrodynamic fields have functional arbitrariness. To illustrate the exact solution of the overdetermined system of Oberbeck–Boussinesq equations, a boundary value problem was solved to describe the complex convection of a vertical swirling fluid without its preliminary rotation. It was shown that the velocity field is highly stratified. Complex countercurrents are recorded in the fluid.

Keywords: Soret effect; Dufour effect; convection; diffusion; thermal diffusion; exact solution; overdetermined system; counterflow



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1. Introduction

World science knows the difficulty of integrating the Navier–Stokes equations for incompressible fluids (with constant density) [1–4]. The complexity of the analytical and numerical integration of the equations of motion of dissipative continuous media is a generally recognized Sixth Millennium Problem [5]. There is no doubt about the relevance of finding new classes of exact solutions of the three-dimensional Navier–Stokes equations for approaching the formulation of the existence and uniqueness theorem. It is clear that when describing real fluids and flows in technological, technical, natural, and biological processes, one must take into account that density is a thermodynamic medium (it depends on one or more physical parameters).

The functional dependence of density on temperature determines the basis for describing the convective motion of a fluid [6–8]. The first equations to describe the convective motion of a viscous fluid were obtained experimentally and theoretically from the Oberbeck–Boussinesq equations. In their derivation, a linear dependence of density on temperature was used [7–14]. It was further established that an impurity (solute) in a fluid similarly causes convection [15–17]. It should be noted that the thermal and concentration mechanisms of convection generation are fundamentally different physically. Mathematically speaking, the Oberbeck–Boussinesq equations for thermal convection and concentration convection differ only in dissipative coefficients in the heat conduction equation and in the diffusion equation, respectively [14,18–24]. These coefficients differ from each other by several orders of magnitude; therefore, one should expect differences in the structure and evolution of thermal and thermal solution flows.

Binary fluids (mixtures) are characterized by the appearance of a new source for the Archimedes force (heterogeneity of the concentration of a dissolved substance) and the competition of dissipative effects, due to which, unlike a pure medium, even the state of mechanical equilibrium in a mixture can be unstable. The interaction between convection, thermal conductivity, and diffusion is complicated by the presence of reverse effects: the appearance of a concentration gradient due to a change in temperature (thermal diffusion or the Soret effect) and the appearance of a temperature gradient due to a change in the concentration field (diffusion thermal conductivity or the Dufour effect) [15–17]. This physical mechanism further complicates the study of the Oberbeck–Boussinesq equations for thermal diffusion and the construction of exact solutions. A typical example illustrating the reasoning given above is seawater.

To solve problems, one can often use the reduction in the dimension of the velocity vector for three-dimensional fields of pressure, temperature, and concentration. In this case, mathematical models based on unidirectional and shear flows are used [18–21,25–31]. Exact solutions for unidirectional thermal diffusion flows are built on the basis of the Ostroumov–Birich class of exact solutions for a horizontal fluid layer or the Gershuni–Batchelor family of exact solutions for a vertical fluid layer [7,32–38]. Reviews on exact solutions for problems of convective, concentration, and thermal diffusion flows are contained in bibliographic sources [39–45].

In the transition to the study of shear flows, the researcher encounters an overdetermined Oberbeck–Boussinesq system [14,20,26,46,47]. The overdetermination of the system of equations arises due to the existence of a zero component of the velocity vector. For isothermal flows, first, in the Lin–Sidorov–Aristov class, the first non-trivial exact solution for isobaric flows was constructed [48–50]. Next, exact solutions were constructed for gradient and convective flows. In articles [20,21], the exact solutions for the thermal diffusion equations were obtained taking into account one of the Soret and Dufour effects. It was shown that these solutions differ due to different dissipative coefficients.

In this article, a new exact solution is constructed for steady-state shear thermal diffusion flows of viscous fluids, taking into account both cross effects (the Soret–Dufour effect). The corresponding theorem on the existence of a solution as a solution of the obtained overdetermined system in the class of functions linearly dependent on a part of spatial coordinates is formulated and proved. It is shown that the results obtained generalize the conclusions made by the authors earlier when analyzing more particular cases (taking into account only one of the thermal diffusion effects or completely ignoring them). The relevance of finding a new exact solution for describing shear flows of a viscous incompressible fluid is due to the fact that pioneering exact solutions for describing unidirectional Ostroumov–Birich and Gershuni–Batchelor flows are used as a basic flow for studying the hydrodynamic stability of convective and thermal diffusion flows [7,32–38]. It is clear that it is important to have an exact solution for a two-dimensional velocity field with a dependence on three coordinates in order to describe new non-trivial physical effects for steady flows, followed by a study of the hydrodynamic stability of new exact solutions for classical and non-classical perturbations.

2. Problem Statement

We consider the steady-state thermal diffusion flow of a Newtonian incompressible fluid, taking into account the cross-dissipative Soret and Dufour effects. The motion of a binary fluid is described by a system of thermal diffusion equations [7]:

$$(\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P + \nu \Delta \mathbf{V} + g(\beta_1 T + \beta_2 C) \mathbf{k}, \quad (1)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$

$$(\mathbf{V} \cdot \nabla) T = \chi \Delta T + \chi \delta \Delta C, \quad (3)$$

$$(\mathbf{V} \cdot \nabla) C = d \Delta C + \alpha \Delta T. \quad (4)$$

To study the binary fluid flows, a three-dimensional rectangular Cartesian coordinate system was chosen. In the system of Equations (1)–(4), the following notations are introduced: $\mathbf{V}(t, x, y, z) = (V_x, V_y, V_z)$ is fluid velocity; $P(t, x, y, z)$ is the deviation of pressure from hydrostatic, normalized to the average constant density of the fluid ρ ; $T(t, x, y, z)$ is the temperature deviation from the equilibrium state; $C(t, x, y, z)$ is the deviation of the concentration of the light phase (impurity) of the binary fluid mixture from the equilibrium value; g is the gravitational acceleration, β_1 and β_2 are the coefficients of thermal and concentration expansion, respectively; $\mathbf{k}$ is the unit vector of the Oz axis (applicate); ν is kinematic (molecular) viscosity; χ is the thermal diffusivity, d is the diffusion coefficient; α is the thermal diffusion coefficient (the Soret parameter); δ is the Dufour parameter; $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is the Hamilton operator; $(\mathbf{V} \cdot \nabla) = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}$ is the convective derivative; and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

We note that in the Navier–Stokes Equation (1), the last term describes the influence of the Archimedes force on the Boussinesq [9] approximation. System (1)–(4) is closed, since the number of equations in it coincides with the number of unknown functions for describing the distribution of hydrodynamic fields.

Next, we consider the shear flow of a fluid. In this case, the velocity vector contains one zero coordinate: $V_z = 0$. The use of shear flows is justified in describing large-scale fluid flows, taking into account the geometric anisotropy of the fluid layer (thin fluid layer). For large-scale fluid flows (in the world ocean), free-fall acceleration is a weakly changing parameter (up to 1% of the average value). Despite the fact that the real movement of the fluid is three-dimensional, for small values of one of the velocity components, it can be set equal to zero in the initial studies. The study of fluid motions for boundary conditions such as a “solid” cover can be carried out exhaustively using the model of shear flows [18–21,25–31].

We now write the reduced Equations (1)–(4) in coordinate form, taking into account the equality $V_z = 0$ [21,51]:

$$\begin{aligned} V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} &= -\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V_x, \\ V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} &= -\frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V_y, \\ \frac{\partial P}{\partial z} &= g(\beta_1 T + \beta_2 C), \\ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} &= 0, \\ V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} &= \chi \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \chi \delta \Delta C, \\ V_x \frac{\partial C}{\partial x} + V_y \frac{\partial C}{\partial y} &= d \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right) + \alpha \Delta T. \end{aligned} \quad (5)$$

The nonlinear overdetermined system (5) describes the flows of a binary fluid with a two-dimensional velocity field. Note that all hydrodynamic fields are three-dimensional.

System (5) does not contain an isolated equation for the hydrodynamic field that could be used to find the compatibility condition (resolvability condition). Next, we pose the problem of obtaining an exact solution to the nonlinear overdetermined system (5) that differs from the trivial (zero) solution.

We find a nontrivial solution to the system (5) in the Lin–Sidorov–Aristov class [48–50]. Let hydrodynamic fields be represented by linear forms in terms of the coordinates:

$$\begin{aligned} V_x &= U(z) + u_1(z)x + u_2(z)y, \\ V_y &= V(z) + v_1(z)x + v_2(z)y, \\ T &= T_0(z) + T_1(z)x + T_2(z)y, \\ C &= C_0(z) + C_1(z)x + C_2(z)y, \\ P &= P_0(z) + P_1(z)x + P_2(z)y. \end{aligned} \quad (6)$$

The linear forms that define class (6) describe the nonlinear properties of a moving fluid due to the complex dependence of the coefficients of the exact solution (6) on the vertical coordinate z .

The structure of the family of exact solutions (6) using the method of indefinite coefficients makes it possible to make an equivalent transition from system (5) to a system of ordinary differential equations. To perform this, it suffices to substitute expressions (6) into each of the equations of system (5) and equate the coefficients in the resulting relations for the same degrees of horizontal coordinates x and y . As a result, we obtain a system of ordinary differential equations written in a way convenient for integrating the overdetermined system:

$$\begin{aligned} u_1^2 + u_2v_1 &= \nu u_1'', \\ u_1u_2 + u_2v_2 &= \nu u_2'', \\ u_1v_1 + v_1v_2 &= \nu v_1'', \\ u_2v_1 + v_2^2 &= \nu v_2'', \\ u_1 + v_2 &= 0; \end{aligned} \quad (7)$$

$$\begin{aligned} u_1T_1 + v_1T_2 &= \chi T_1'' + \chi\delta C_1'', \\ u_2T_1 + v_2T_2 &= \chi T_2'' + \chi\delta C_2''; \end{aligned} \quad (8)$$

$$\begin{aligned} u_1C_1 + v_1C_2 &= dC_1'' + \alpha dT_1'', \\ u_2C_1 + v_2C_2 &= dC_2'' + \alpha dT_2''; \end{aligned} \quad (9)$$

$$\begin{aligned} P_1' &= g(\beta_1T_1 + \beta_2C_1), \\ P_2' &= g(\beta_1T_2 + \beta_2C_2); \end{aligned} \quad (10)$$

$$\begin{aligned} Uu_1 + Vu_2 &= -P_1 + \nu U'', \\ Uv_1 + Vv_2 &= -P_2 + \nu V''; \end{aligned} \quad (11)$$

$$\begin{aligned} UT_1 + VT_2 &= \chi T_0'' + \chi\delta C_0'', \\ UC_1 + VC_2 &= dC_0'' + \alpha dT_0''; \end{aligned} \quad (12)$$

$$P_0' = g(\beta_1T_0 + \beta_2C_0). \quad (13)$$

The prime in system (7)–(13) denotes taking the total derivative with respect to the variable z . System (7)–(13) inherits from system (5) the properties of nonlinearity and overdetermination. Note that the balance between the number of unknowns and the number of equations is violated only in system (7). In other words, if it is possible to find a non-trivial joint solution of the overdetermined subsystem (7), then a solution to any of the systems (8)–(13) can be found by successive integration.

In [14], a theorem was formulated and proved that determines the conditions for the existence of a necessary joint solution to subsystem (7). According to this theorem, the overdetermined system (7)–(13) has a non-trivial exact solution in class (6), which is an

exact solution of system (5) if and only if the spatial accelerations u_1 , u_2 , v_1 , and v_2 satisfy the relations:

$$u_1 = u \cos \theta \sin \theta = -v_2, \quad u_2 = u \cos^2 \theta, \quad v_1 = -u \sin^2 \theta, \quad (14)$$

where $u = c_1 z + c_2$ is a function that satisfies the simplest second-order differential equation $u'' = 0$; and θ is a real number.

A similar problem of finding a non-trivial joint solution of an overdetermined non-linear system of constitutive relations arises not only when solving problems of thermal diffusion [20,51], but also when considering both convective flows [14] and isothermal flows [46] of a viscous incompressible fluid.

3. Construction of the Exact Solution

Furthermore, it will be established that the exact solution describing the steady flow of a binary fluid in class (6) is described by a polynomial in three variables. To perform this, we prove the following theorem.

Theorem 1. *For any values of the parameters θ , c_1 , c_2 , defining the set of functions (14), for $1 - \alpha\delta \neq 0$ the overdetermined system of differential Equations (8)–(13) has a unique solution.*

For the case $\alpha = 0$ considered in [20], and in the case $\delta = 0$ considered in [51], the condition $1 - \alpha\delta \neq 0$ is automatically satisfied. In other words, the problem posed in this article can be considered a generalization of the results obtained in [20,51]. Note that both dissipative coefficients (the Soret coefficient and the Dufour coefficient) are small in magnitude; therefore, the assumption of the theorem on the fulfillment of the condition $1 - \alpha\delta \neq 0$ is physically justified and interesting [52].

System (1)–(4) is used both for describing incompressible fluids and for modeling the flows of incompressible gases, which, in a certain range of physical conditions, is applicable to real (i.e., compressible) gases. It is for this reason that none of the coefficients are neglected in this case, and both dissipative coefficients are used in writing the basic equations of the model to create the most symmetrical form of writing, suitable for both fluid and gas flows.

We will prove the formulated theorem in a constructive way, i.e., we indicate an algorithm for integrating the equations of system (7)–(13). After the joint solution (14) of the overdetermined system (7) is obtained, the difficulty in obtaining a solution to system (8)–(13) lies only in constructing the order of integration of the remaining equations. For this reason, further proof of the formulated theorem with an indication of the order of integration of the equations and comments are given regarding the properties of the resulting solution.

Proof. Equations (8) and (9) will be considered jointly since they constitute an isolated subsystem, taking into account expressions (14):

$$\begin{aligned} \chi T_1'' + \chi \delta C_1'' &= u \cos \theta \sin \theta T_1 - u \sin^2 \theta T_2, \\ \chi T_2'' + \chi \delta C_2'' &= u \cos^2 \theta T_1 - u \cos \theta \sin \theta T_2, \\ dC_1'' + \alpha dT_1'' &= u \cos \theta \sin \theta C_1 - u \sin^2 \theta C_2, \\ dC_2'' + \alpha dT_2'' &= u \cos^2 \theta C_1 - u \cos \theta \sin \theta C_2; \end{aligned} \quad (15)$$

We transform Equation (15) as follows: we multiply the first equation of the system by and subtract from it the multiplied by the second equation of this system. After that, we integrate the resulting equation:

$$\cos \theta T_1 + \delta \cos \theta C_1 - \sin \theta T_2 - \delta \sin \theta C_2 = \gamma_1 z + \gamma_2. \quad (16)$$

From the third and fourth equations of system (15), by analogous transformations, we obtain the following expression:

$$\cos \vartheta C_1 + \alpha \cos \vartheta T_1 - \sin \vartheta C_2 - \alpha \sin \vartheta T_2 = \gamma_3 z + \gamma_4. \quad (17)$$

Let us solve relations (16), (17) with respect to horizontal gradients T_2 and C_2 :

$$\begin{pmatrix} -\sin \vartheta & -\delta \sin \vartheta \\ -\alpha \sin \vartheta & -\sin \vartheta \end{pmatrix} \begin{pmatrix} T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\cos \vartheta T_1 - \delta \cos \vartheta C_1 + \gamma_1 z + \gamma_2 \\ -\alpha \cos \vartheta T_1 - \cos \vartheta C_1 + \gamma_3 z + \gamma_4 \end{pmatrix}, \quad (18)$$

The main determinant of the resulting matrix Equation (18) is as follows:

$$\begin{vmatrix} -\sin \vartheta & -\delta \sin \vartheta \\ -\alpha \sin \vartheta & -\sin \vartheta \end{vmatrix} = \sin^2 \vartheta (1 - \alpha \delta).$$

The last factor is assumed to be different from zero due to the conditions of the theorem being proved. Thus, the solvability of Equation (18) (and hence other equations of the system (10)–(13)) depends on the value of the parameter ϑ .

When $\sin \vartheta = 0$, system (15) is greatly simplified and contains an isolated subsystem with respect to gradients T_1 and C_1 :

$$\begin{aligned} T_1'' + \delta C_1'' &= 0, \quad C_1'' + \alpha T_1'' = 0; \\ \chi T_2'' + \chi \delta C_2'' &= u T_1, \quad d C_2'' + \alpha d T_2'' = u C_1. \end{aligned} \quad (19)$$

The first two equations of system (19) are a system of homogeneous linear equations with respect to derivatives T_1'' and C_1'' , with a non-degenerate (due to the conditions of the theorem) matrix. Hence, linear functions are its solution:

$$T_1 = c_3 z + c_4, \quad C_1 = c_5 z + c_6. \quad (20)$$

The last two equations of system (19) are (taking into account expressions (20)) a system of already inhomogeneous linear equations with respect to derivatives T_2'' and C_2'' , which is uniquely integrated:

$$C_2'' = \frac{u}{\chi d (\alpha \delta - 1)} (\alpha d T_1 - \chi C_1), \quad (21)$$

$$T_2'' = \frac{u}{\chi d (\alpha \delta - 1)} (-d T_1 + \chi \delta C_1). \quad (22)$$

The solution of the Equations (21) and (22) is a set of fourth-order polynomial functions. Thus, the solution of Equations (8) and (9) is constructed. The exact solution of Equation (10) for horizontal pressure gradients is easily obtained after a single solving of Equations (8) and (9). They are described by a polynomial of the fifth degree.

Equation (11) for the case are written more compactly:

$$\nu U'' = V u_2 + P_1, \quad \nu V'' = P_2. \quad (23)$$

First, the isolated equation of system (23) for the velocity V is integrated. A seventh-order polynomial is obtained, further substitution of which into the first equation of system (24) (presented below) makes it possible to determine the form of the velocity U as a tenth-order polynomial.

After this, we represent Equation (12) as a non-homogeneous matrix algebraic equation with respect to the second derivatives of the background components T_0 and C_0 of linear forms (6) with known polynomial homogeneity and a nonsingular matrix:

$$\begin{pmatrix} \chi \delta & \chi \\ d & \alpha d \end{pmatrix} \begin{pmatrix} C_0'' \\ T_0'' \end{pmatrix} = \begin{pmatrix} U T_1 + V T_2 \\ U C_1 + V C_2 \end{pmatrix}$$

Having inverted the matrix of the system, we construct a solution after double integration and obtain a solution for the homogeneous components of the concentration and pressure fields. Then, based on them, we integrate Equation (13) and obtain a unique solution for the background pressure. The above-mentioned exact solutions for the components of class (6) are not given because of their cumbersomeness and elementary nature of calculations.

Let us return to the general case of Equation (19), setting $\sin \vartheta \neq 0$. The matrix of the system will be non-degenerate, which means that the equation is solvable in a unique way:

$$\begin{aligned} T_2 &= \cot \vartheta T_1 + \frac{(-\gamma_1 + \gamma_3 \delta)z + (-\gamma_2 + \gamma_4 \delta)}{(1 - \alpha \delta) \sin \vartheta}, \\ C_2 &= \cot \vartheta C_1 + \frac{(\alpha \gamma_1 - \gamma_3)z + (\alpha \gamma_2 - \gamma_4)}{(1 - \alpha \delta) \sin \vartheta}. \end{aligned} \quad (24)$$

Substituting expressions (24) into the first and third equations of system (15), we reach the following relations:

$$\begin{aligned} \chi T_1'' + \chi \delta C_1'' &= \frac{\sin \vartheta ((\gamma_1 - \gamma_3 \delta)z + (\gamma_2 - \gamma_4 \delta))u}{1 - \alpha \delta}, \\ dC_1'' + \alpha dT_1'' &= \frac{\sin \vartheta ((-\alpha \gamma_1 + \gamma_3)z + (-\alpha \gamma_2 + \gamma_4))u}{1 - \alpha \delta}. \end{aligned} \quad (25)$$

Resolving system (33) (its matrix is again non-singular) with respect to the second derivatives of the gradients T_1 and C_1 , we obtain:

$$\begin{aligned} T_1'' &= \frac{\sin \vartheta ((d(\gamma_1 - \gamma_3 \delta) + \delta \chi (\alpha \gamma_1 - \gamma_3))z + d(\gamma_2 - \gamma_4 \delta) + \delta \chi (\alpha \gamma_2 - \gamma_4))u}{d(1 - \alpha \delta)^2 \chi}, \\ C_1'' &= \frac{\sin \vartheta ((d\alpha(-\gamma_1 + \gamma_3 \delta) + \chi(-\alpha \gamma_1 + \gamma_3))z + d\alpha(-\gamma_2 + \gamma_4 \delta) + \chi(-\alpha \gamma_2 + \gamma_4))u}{d(1 - \alpha \delta)^2 \chi}. \end{aligned} \quad (26)$$

The right-hand sides of system (26) contain polynomials of the second degree, so it can be easily integrated. After the exact solutions for the gradients T_1 and C_1 are found, according to (26), expressions for the gradients T_2 and C_2 are easily written out. Thus, system (8) and (9) are completely solved.

Further proof of the theorem is reduced to sequential integration of systems (10)–(13), the algorithm of which (and the justification for the uniqueness of the resulting solution) is given above when considering the case. Thus, the formulated theorem is completely proved. $\square$

4. Boundary Value Problem

As an interpretation of the structure of the exact solution obtained by proving the theorem formulated above, we formulate a boundary value problem. We will consider a steady thermal diffusion shear flow of a viscous incompressible fluid in a horizontal infinitely extended layer with both non-deformable boundaries (Figure 1).

The solution of system (7)–(13) we will look for the exact solution of the Oberbeck–Boussinesq system (5) in the following form:

$$\begin{aligned} V_x &= U(z) + u_2(z)y, \quad V_y = V(z), \\ T &= T_0(z) + T_1(z)x + T_2(z)y, \\ C &= C_0(z) + C_1(z)x + C_2(z)y, \\ P &= P_0(z) + P_1(z)x + P_2(z)y. \end{aligned} \quad (27)$$

Class (27) is a special case of class (6), when the spatial gradients u_1 , v_1 , and v_2 are assumed to be equal to zero. In this case, the overdetermined system (7) is reduced to one isolated equation for determining the only non-zero spatial acceleration:

$$u_2'' = 0.$$

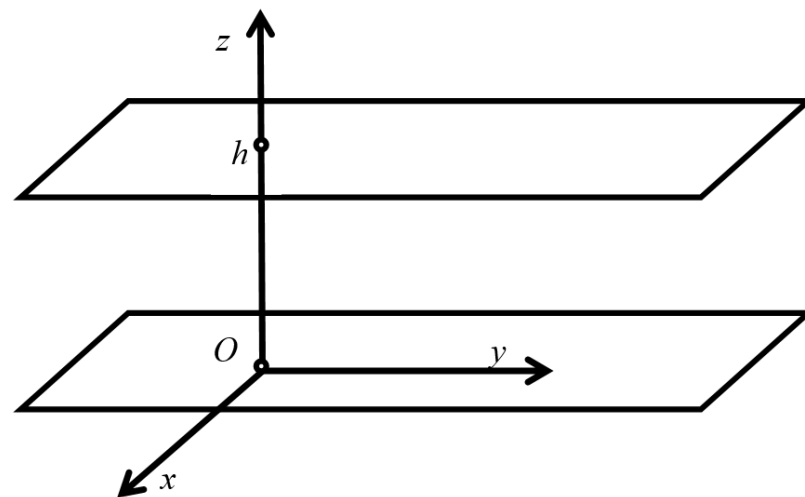


Figure 1. The flow scheme.

This solution satisfies relations (14) for $\sin \vartheta = 0$. Previously, the exact solution (27) was proposed to study a vertically swirling fluid in the absence of rotation for isothermal, gradient, and convective flows [14,18,25,26,51]. In articles [19–21,46,47,51], it was shown that the velocity field (27) can be used to model and simulate equatorial countercurrents. The formulas given below will take into account not only the effect of heat on the structure of the ocean, but also the distribution of salt in water.

To determine the constants of integration arising in the process of constructing a solution to system (27), we formulate the boundary conditions. We will assume that the no-slip condition is satisfied at the lower boundary (bottom):

$$V_x(0) = V_y(0) = 0.$$

In addition, the lower boundary will be considered impenetrable and thermally insulated:

$$C' = 0, T' = 0.$$

On the upper boundary, we consider that the inhomogeneous distribution of the given velocities corresponds to the movement of the upper boundary as a solid surface:

$$V_x(h) = W \cos \varphi + \Omega y, V_y(h) = W \sin \varphi.$$

Here, W is the value of the velocity at the upper boundary and φ is the angle that the velocity vector forms with the abscissa axis Ox .

We represent the boundary condition for pressure in the following form:

$$P(h) = S_0 + S_1x + S_2y,$$

where S_0 corresponds to the atmospheric pressure.

In addition, we assume that the temperature and concentration are distributed nonuniformly at the upper boundary of the layer:

$$T(h) = ax + by, C(h) = mx + ny.$$

According to the representation (27) for hydrodynamic fields, the formulated boundary conditions are reduced to the following conditions:

$$U = u_2 = V = 0, T'_0 = T'_1 = T'_2 = 0, C'_0 = C'_1 = C'_2 = 0 \quad (28)$$

on the lower border;

$$\begin{aligned} U &= W \cos \varphi, \quad u_2 = \Omega, \quad V = W \sin \varphi, \quad T_0 = 0, \quad T_1 = a, \quad T_2 = b, \\ P_0 &= S_0, \quad P_1 = S_1, \quad P_2 = S_2, \quad C_0 = 0, \quad C_1 = m, \quad C_2 = n \end{aligned} \quad (29)$$

at the upper boundary of the considered layer.

The solution of the boundary value problem (27)–(29) is a set of polynomial functions:

$$\begin{aligned} u_2 &= \Omega Z, \\ U &= \frac{Z}{51840dh^2(\alpha\delta-1)v^2\chi} \left[g\Omega^2h^7Z^8(\beta_2(ad\alpha - m\chi) + \beta_1(m\delta\chi - ad)) + \right. \\ &\quad + 48gh^5\Omega Z^5 \{ \beta_1(6bd(-1+\alpha\delta)\chi + h^2(ad - m\delta\chi)\Omega) + \\ &\quad + \beta_2(h^2m\chi\Omega + d(6n(-1+\alpha\delta)\chi - ah^2\alpha\Omega)) \} + \\ &\quad + 162h^4Z^4 \{ gh\Omega\beta_2(-h^2m\chi\Omega + d(-8n(-1+\alpha\delta)\chi + ah^2\alpha\Omega)) + \\ &\quad + 8dS_2(-1+\alpha\delta)\chi\Omega - gh\Omega\beta_1(8bd(-1+\alpha\delta)\chi + h^2(ad - m\delta\chi)\Omega) \} + \\ &\quad + h^2Z^3 \{ -12gh^3\Omega\beta_2(-13h^2m\chi\Omega + d(-120n(-1+\alpha\delta)\chi + 13ah^2\alpha\Omega)) + \\ &\quad + 12gh^3\Omega\beta_1((120bd(-1+\alpha\delta)\chi + 13h^2(ad - m\delta\chi)\Omega)) - \\ &\quad - 2160d(-1+\alpha\delta)\chi\Omega(h^2S_2 - 2Wv\sin\varphi) \} + \\ &\quad + 8640dgh^3Z^2(a\beta_1 + m\beta_2)(-1+\alpha\delta)v\chi - \\ &\quad - 25920dh^2Z(-S_1 + gh(a\beta_1 + m\beta_2))(-1+\alpha\delta)v\chi + \\ &\quad + \{ -864dh^3(-1+\alpha\delta)\chi(30S_1v - h^2S_2\Omega) + \\ &\quad + 51840dhW(-1+\alpha\delta)v^2\chi\cos\varphi - 4320dh^3W(-1+\alpha\delta)v\chi\Omega\sin\varphi + \\ &\quad + gh^4\beta_2(41h^4(-m\chi + ad\alpha)\Omega^2 + 432(-1+\alpha\delta)d\chi(40mv - nh^2\Omega) + \\ &\quad + gh^4\beta_1(h^2\chi\Omega(-432bd(-1+\alpha\delta) + 41h^2m\delta\Omega) + \\ &\quad + ad(17280(-1+\alpha\delta)v\chi - 41h^4\Omega^2)) \} \}; \\ V &= \frac{Z}{720d(-1+\alpha\delta)v\chi} \left[gh^5Z^5(ad(-\beta_1 + \alpha\beta_2) + m(-\beta_2 + \beta_1\delta)\chi)\Omega + \right. \\ &\quad + 20gh^3Z^2 \{ 6bd\beta_1(-1+\alpha\delta)\chi + h^2m(\beta_2 - \beta_1\delta)\chi\Omega + \\ &\quad + d(6n\beta_2(-1+\alpha\delta)\chi + ah^2(\beta_1 - \alpha\beta_2)\Omega) \} - \\ &\quad - 45h^2Z \{ gh^3m - (\beta_2 - \beta_1\delta)\chi\Omega - 8dS_2(-1+\alpha\delta)\chi + \\ &\quad + gh d(8(b\beta_1 + n\beta_2)(-1+\alpha\delta)\chi + ah^2(\beta_1 - \alpha\beta_2)\Omega) \} + \\ &\quad + \{ 360d(S_2d - 2Wv\sin\varphi)(1 - \alpha\delta)\chi + \\ &\quad + 2gh^3(120dn(1 - \alpha\delta)\chi - 13h^2(m\chi - ad\alpha)\Omega) + \\ &\quad + 2gh^3\beta_1(120bd(-1+\alpha\delta)\chi + 13h^2(ad - m\delta\chi)\Omega) + \\ &\quad + 2gh^3\beta_2(13h^2(m\chi - ad\alpha)\Omega + 120nd(-1+\alpha\delta)\chi) \} \]. \end{aligned} \quad (30)$$

In the representation of solution (30), a dimensionless vertical coordinate $Z = z/h$ is introduced, the values of which vary within the considered layer in the range $[0, 1]$. This makes it possible to analyze the behavior of flows in layers of different thicknesses from a unified standpoint. Note that the solution to the boundary value problem is given incompletely due to two circumstances:

- (1) the expressions for the fields of temperature, concentration, and pressure turn out to be too cumbersome;
- (2) the main interest is the structure (profile) of the velocity field and its mapping onto the phase space.

The homogeneous components of the velocity field (27) are described by high-order polynomials, which indicates the possibility of the existence of several stagnant points of the field. Passing through the latter, the fluid flow is able to change the direction of the flow. In other words, multiple stratification of the velocity field is possible due to the mutual influence of thermal and diffusion factors.

The velocity U is described by a polynomial of the ninth degree; however, the number of zero points of this speed does not exceed seven, and one of these zero points is located on the lower boundary of the layer. Similar conclusions are given for the velocity V : this polynomial of the sixth degree vanishes within the considered layer no more than three times. The corresponding profiles are shown in Figures 2 and 3, respectively.

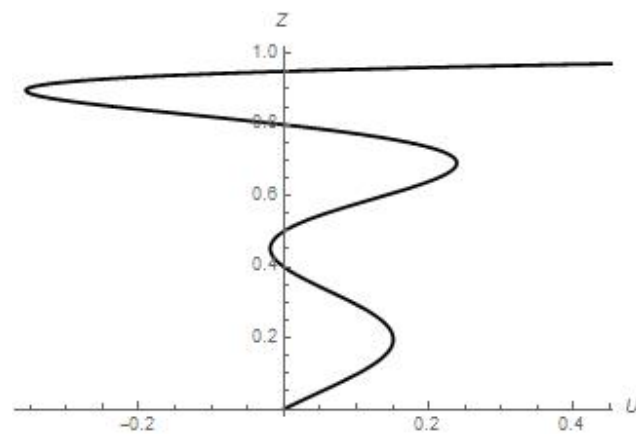


Figure 2. The velocity U profile.

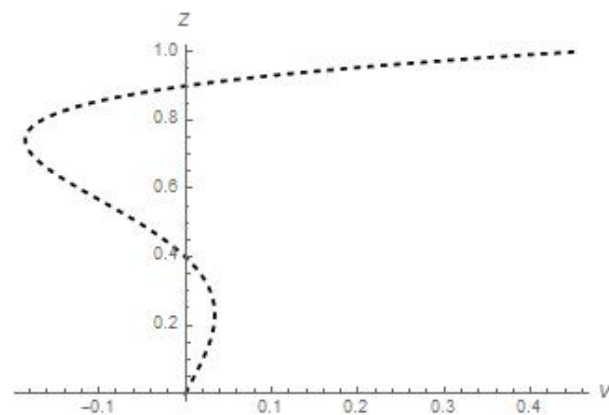


Figure 3. The velocity V profile.

The existence of a large number of zero points (stratification points) for these velocity field components leads to the fact that the shape of the velocity field hodograph turns out to be extremely complex (Figure 4).

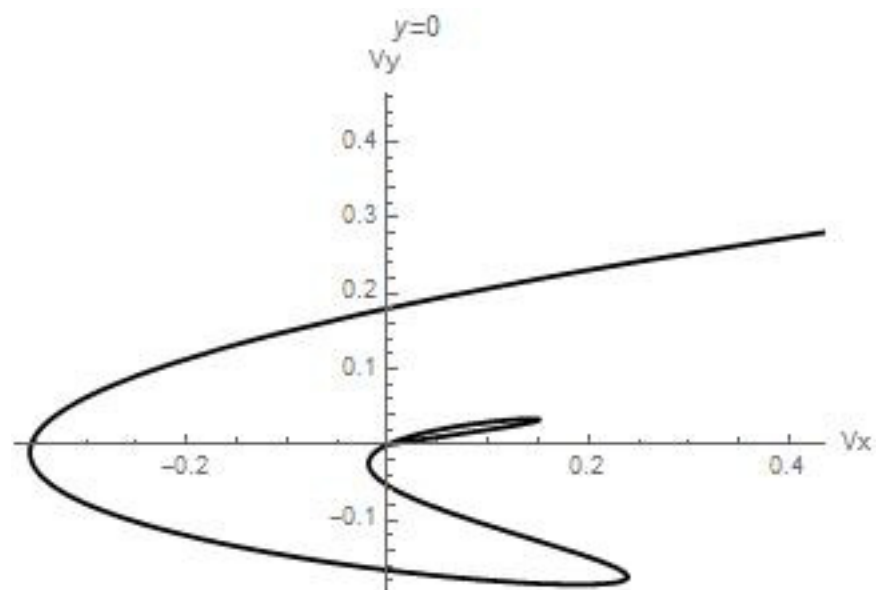


Figure 4. The hodograph of the velocity field corresponding to the profiles of the field components shown in Figures 2 and 3.

Note that the view of the hodograph $\{U, V\}$, shown in Figure 4 is valid only for the cross-section $y = 0$, since, strictly speaking, due to expressions (27), it is also necessary to take into account the inhomogeneous terms in the expression for the velocity V_x . The corresponding term u_2y , according to solution (30), is a linear dependence on the vertical coordinate Z , which leads to a “straightening” of the velocity profile U with distance from the control section $y = 0$ (Figures 5–10).

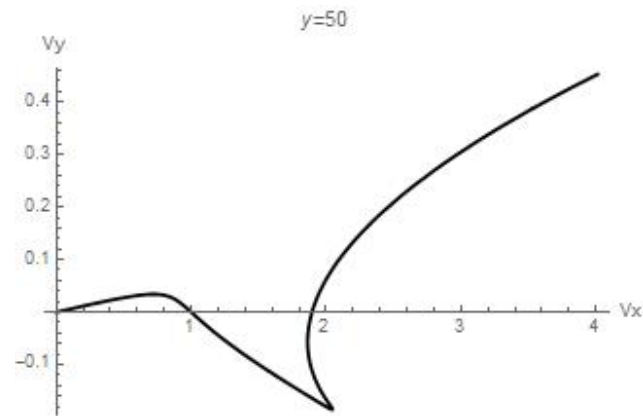


Figure 5. The hodograph of the velocity field.

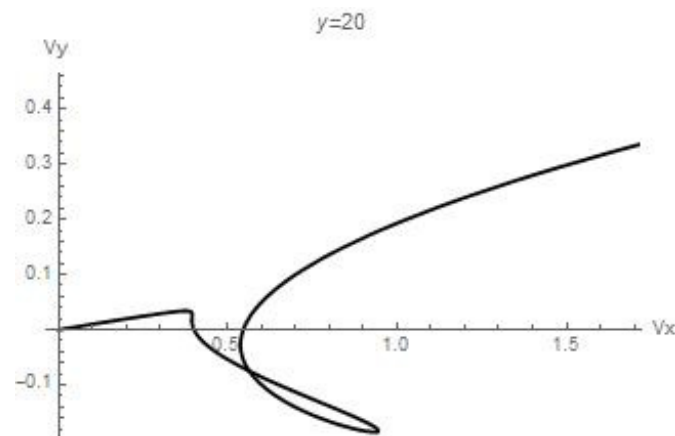


Figure 6. The hodograph of the velocity field.

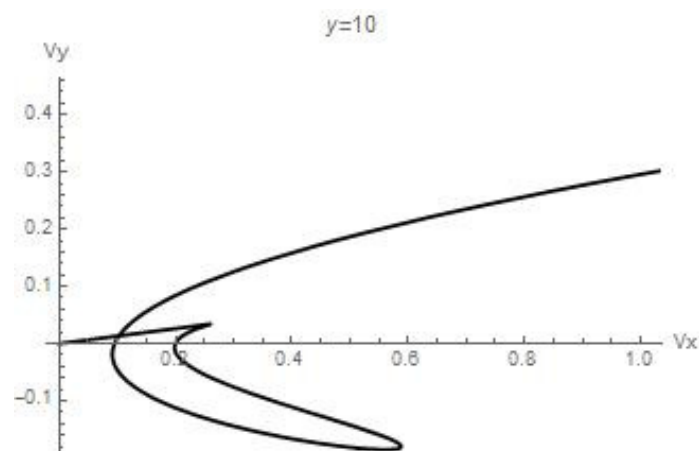


Figure 7. The hodograph of the velocity field.

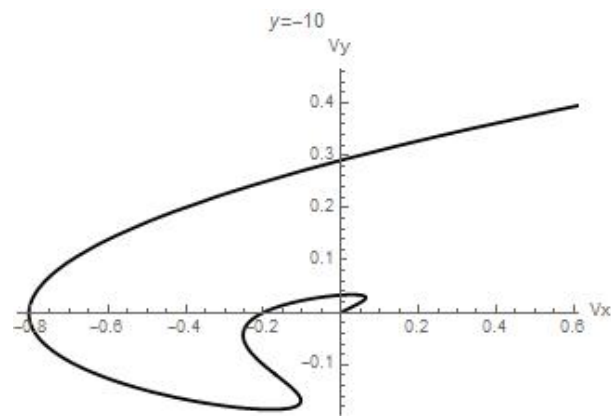


Figure 8. The hodograph of the velocity field.

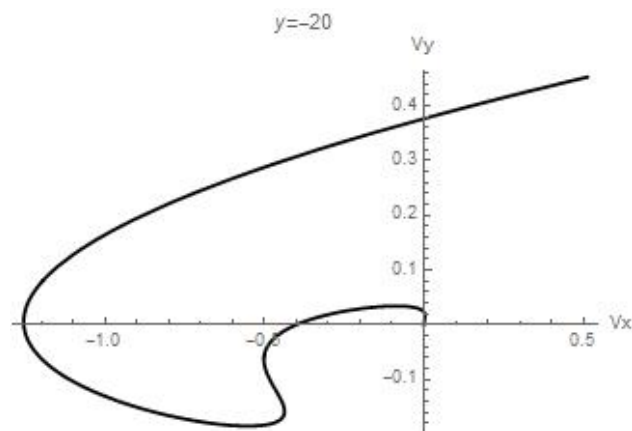


Figure 9. The hodograph of the velocity field.

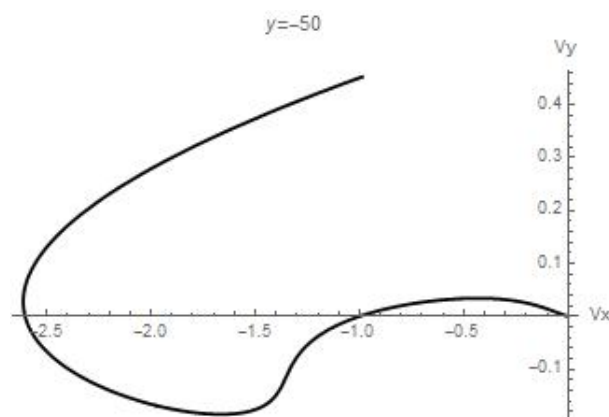


Figure 10. The hodograph of the velocity field.

An interesting effect of localization of vortex structures can be observed on the graph of the stream function corresponding to the constructed solution (Figure 11). It illustrates the complex flow structure. Counterflows with several zones of return flows are registered in the fluid. Such a complex flow structure is due to the inhomogeneous velocity distribution, thermal convection, concentration convection, and thermal diffusion.

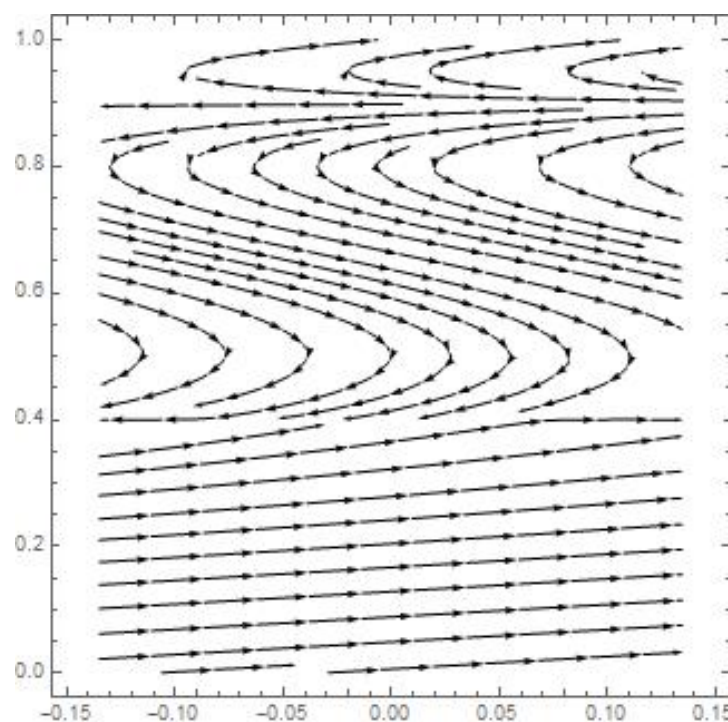


Figure 11. The stream function.

The stream function characterizes the value of the volumetric flow rate of the fluid through the cross-section of the channel built on streamlines. The curves in Figure 11 illustrate the non-uniformity of this flow rate. Consideration of both dissipative Soret and Dufour effects when describing thermal diffusion flows leads to the appearance of an additional stagnation point for the velocity field (29) and (30) if we consider the exact solutions only with the Soret effect [21] or only with the Dufour effect [51] taken into account. Thus, the competing interaction of the Soret and Dufour effects leads not only to quantitative changes for hydrodynamic fields, but also to a qualitative (topological) change in the stratification of the velocity field, pressure field, temperature field, and solute concentration field.

5. Conclusions

This article considers the problem of describing thermal diffusion flows, taking into account both cross effects (the Soret–Dufour effect) for shear flows. It is shown that consideration of shear flows simplifies the structure of the velocity field; however, it leads to overdetermination of the system of model equations. The solution of the overdetermined system is sought in the class of functions that can be represented as complete linear forms in two spatial coordinates. This class is capable of describing the nonlinear effects observed in a fluid in view of the nonlinear dependence of the coefficients of the mentioned linear forms on the third coordinate, despite the external simplicity of its structure. A theorem on the solvability of the resulting overdetermined nonlinear system is formulated and proved under the compatibility conditions presented in the earlier studies of the authors [14,20,26,51] and the fulfillment of an additional condition on thermal diffusion coefficients. The study (the formulated solvability theorem and its proof) generalizes the results obtained by the authors earlier [20,51] for thermal diffusion shear flows. In addition, while proving the central theorem of the article, we identified the conditions under which branching into the structures of the constructed exact solution of the overdetermined system appears. To illustrate the exact solution found, we consider a boundary value problem for describing an inhomogeneous shear flow of a vertically swirling fluid. The field of velocities described by polynomials for the ninth degree is investigated. It is shown that the velocity field is stratified depending on the boundary conditions and dissipative coefficients in different ways.

In this case, counterflows are recorded in the liquid, and the flow itself is spiral. In addition, an example of velocity profiles that differ from studies [21,51] is given. Accounting for both the Soret and Dufour effects leads to the appearance of new effects even for steady flows with subsequent ones. Thus, it will be important to use the exact solution obtained in the article as a basic flow for studying hydrodynamic instability. Let us also mention works among [53–85], where these refs. are within the framework of the analytical approach to the study of mathematical hydrodynamical models.

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