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Bernstein-Szegő inequality for the Riesz derivative of trigonometric polynomials in L_p-spaces, $0 \leqslant p \leqslant \infty$, with classical value of the sharp constant

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Abstract. The Bernstein-Szegő inequality for the Weyl derivative of real order $\alpha \geqslant 0$ of trigonometric polynomials of degree n is considered. The aim is to find values of the parameters for which the sharp constant in this inequality is equal to n^α (the classical value) in all L_p-spaces, $0 \leqslant p \leqslant \infty$. The set of all such α is described for some important particular cases of the Weyl-Szegő derivative, namely, for the Riesz derivative and for the conjugate Riesz derivative, for all $n \in \mathbb{N}$.

Bibliography: 22 titles.

Keywords: trigonometric polynomial, Riesz derivative, Bernstein-Szegő inequality, space L_0.

§ 1. Introduction

1.1. Notation. The statement of the problem. Let $\mathscr{T}_n = \mathscr{T}_n(\mathbb{C})$ be the class of all trigonometric polynomials

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) = \sum_{k=-n}^{n} c_k e^{ikt} \tag{1.1}$$

with complex coefficients. For a parameter p, $0 \leqslant p \leqslant \infty$, consider the following functionals on $\mathscr{T}_n$:

$$\|f_n\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f_n(t)|^p \, dt \right)^{1/p}, \qquad 0 < p < \infty,$$
$$\|f_n\|_\infty = \lim_{p \to +\infty} \|f_n\|_p = \|f_n\|_{C_{2\pi}} = \max\{|f_n(t)| \colon t \in \mathbb{R}\},$$
$$\|f_n\|_0 = \lim_{p \to +0} \|f_n\|_p = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln |f_n(t)| \, dt \right);$$

these functionals define a norm only for $1 \leqslant p \leqslant \infty$.

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In 1917 Weyl [1] introduced fractional derivatives of a periodic function. On the class $\mathscr{T}_n$, the fractional (or *Weyl*) derivative of real order $\alpha \geqslant 0$ is defined by

$$D^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha \left(a_k \cos\left(kt + \frac{\pi\alpha}{2}\right) + b_k \sin\left(kt + \frac{\pi\alpha}{2}\right)\right)$$
$$= \sum_{k=-n}^{n} c_k |k|^\alpha e^{(i\pi\alpha/2)\operatorname{sign} k} e^{ikt}.$$

For any positive integer α the Weyl derivative coincides with the classical derivative: $D^\alpha f_n = f_n^{(\alpha)}$. In the case when $\alpha = 0$ the operator D^0 removes the constant term of the polynomial: $D^0 f_n(t) = f_n(t) - c_0$. The Weyl derivatives have the following semigroup property: $D^\beta D^\alpha = D^{\alpha+\beta}$, $\alpha, \beta \geqslant 0$.

Along with the polynomial (1.1) we consider the conjugate polynomial

$$\widetilde{f}_n(t) = \sum_{k=1}^{n} (b_k \cos kt - a_k \sin kt) = i \sum_{k=-n}^{n} c_k (\operatorname{sign} k) e^{ikt};$$

note that this definition of the conjugate polynomial, in which we follow [2], differs in sign from the classical definition (see, for example, [3], Vol. 1, § 2.5).

Given a real θ, consider the *Weyl-Szegő operator*

$$D_\theta^\alpha f_n(t) = f_n^{(\alpha)}(t) \cos\theta + \widetilde{f}_n^{(\alpha)}(t) \sin\theta$$
$$= \sum_{k=1}^{n} k^\alpha \left(a_k \cos\left(kt + \frac{\pi\alpha}{2} + \theta\right) + b_k \sin\left(kt + \frac{\pi\alpha}{2} + \theta\right)\right)$$
$$= \sum_{k=-n}^{n} c_k |k|^\alpha e^{i(\pi\alpha/2+\theta)\operatorname{sign} k} e^{ikt}. \tag{1.2}$$

For $\theta = -\pi\alpha/2$ we have the operator

$$D_{-\pi\alpha/2}^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha (a_k \cos kt + b_k \sin kt) = \sum_{k=-n}^{n} c_k |k|^\alpha e^{ikt}, \tag{1.3}$$

and for $\theta = \pi(1-\alpha)/2$, the operator

$$D_{\pi(1-\alpha)/2}^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha (b_k \cos kt - a_k \sin kt) = i \sum_{k=-n}^{n} c_k |k|^\alpha (\operatorname{sign} k) e^{ikt}. \tag{1.4}$$

The operator (1.3) is known as the *Riesz derivative*; for some of its properties, see § 5.25.4 in [4]. The operator (1.4) will be called the *conjugate Riesz derivative*, since the polynomial $D_{\pi(1-\alpha)/2}^\alpha f_n$ is conjugate to $D_{-\pi\alpha/2}^\alpha f_n$ for any $f_n \in \mathscr{T}_n$. In what follows we write D_R^α and $\widetilde{D}_R^\alpha$ for $D_{-\pi\alpha/2}^\alpha$ and $D_{\pi(1-\alpha)/2}^\alpha$, respectively.

An important property of such operators is that the Riesz derivative of an even polynomial is an even polynomial again, and the conjugate Riesz derivative of an even polynomial is an odd one. Note that D_R^α for even $\alpha \in \mathbb{N}$ and $\widetilde{D}_R^\alpha$ for odd $\alpha \in \mathbb{N}$

are the classical derivatives of order α. On the other hand, D_R^α for odd $\alpha \in \mathbb{N}$ and $\widetilde{D}_R^\alpha$ for even $\alpha \in \mathbb{N}$ are the conjugate derivative operators of order α. Also note that $D_R^0 = D^0$, and $\widetilde{D}_R^0$ is the conjugation operator.

The Weyl-Szegő operator (1.2) can be written in a different way in terms of Riesz derivatives:

$$D_\theta^\alpha f_n(t) = D_R^\alpha f_n(t) \cos\tau + \widetilde{D}_R^\alpha f_n(t) \sin\tau, \qquad \tau = \frac{\pi\alpha}{2} + \theta. \tag{1.5}$$

We are interested in the norm of operator (1.2) on the set $\mathscr{T}_n$, that is, the least constant $B_n(\alpha, \theta)_p$ in the inequality

$$\|D_\theta^\alpha f_n\|_p \leqslant B_n(\alpha, \theta)_p \|f_n\|_p, \qquad f_n \in \mathscr{T}_n. \tag{1.6}$$

Inequalities of this kind are called *Bernstein-Szegő inequalities* (*Bernstein inequalities* for $\theta = 0$, and *Szegő inequalities* for $\theta = \pi/2$). It is easily checked that the constant $B_n(\alpha, \theta)_p$ is π-periodic in θ; so we assume in what follows that $\theta \in [0, \pi]$.

Since D_θ^α is the convolution operator, for the constant $B_n(\alpha, \theta)_p$ in (1.6) we have

$$B_n(\alpha, \theta)_p \leqslant B_n(\alpha, \theta)_\infty, \qquad 1 \leqslant p \leqslant \infty, \tag{1.7}$$

and

$$n^\alpha = B_n(\alpha, \theta)_2 \leqslant B_n(\alpha, \theta)_p \leqslant B_n(\alpha, \theta)_0, \qquad 0 \leqslant p \leqslant \infty. \tag{1.8}$$

Inequality (1.7) is known; the first inequality in (1.8) is quite clear — the corresponding lower estimate is given by the polynomial e^{int}. The last inequality in (1.8) was proved by Arestov [5]. This inequality means that $B_n(\alpha, \theta)_p$ assumes its largest value over $p \in [0, \infty]$ at $p = 0$; thus, the case $p = 0$ is of great value in this field of research.

In this paper, for all $n \in \mathbb{N}$ we characterize the α for which the constants $B_n(\alpha, R)_p$ and $\widetilde{B}_n(\alpha, R)_p$ in the inequalities

$$\|D_R^\alpha f_n\|_p \leqslant B_n(\alpha, R)_p \|f_n\|_p \tag{1.9}$$

and

$$\|\widetilde{D}_R^\alpha f_n\|_p \leqslant \widetilde{B}_n(\alpha, R)_p \|f_n\|_p \tag{1.10}$$

are equal to n^α for all p, $0 \leqslant p \leqslant \infty$. In view of (1.8), to do this it suffices to investigate the inequalities

$$\|D_R^\alpha f_n\|_0 \leqslant B_n(\alpha, R)_0 \|f_n\|_0 \tag{1.11}$$

and

$$\|\widetilde{D}_R^\alpha f_n\|_0 \leqslant \widetilde{B}_n(\alpha, R)_0 \|f_n\|_0, \tag{1.12}$$

or, more precisely, to characterize n and α for which the constants $B_n(\alpha, R)_0$ and $\widetilde{B}_n(\alpha, R)_0$ are n^α.

1.2. Historical remarks. Inequalities of the form (1.6) have been studied for over 90 years. For a historical account, see [2], [6]–[12] and the books [13], Ch. 3, [14], § 8.1, and [15], §§ 6.1.2 and 6.1.7. For $1 \leqslant p \leqslant \infty$ and $\alpha \geqslant 1$, for each $\theta \in [0, \pi]$ the sharp inequality

$$\big\| f_n^{(\alpha)} \cos\theta + \widetilde{f}_n^{(\alpha)} \sin\theta \big\|_p \leqslant n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{T}_n, \tag{1.13}$$

holds with the classical constant $B_n(\alpha, \theta)_p = n^\alpha$. For the first-order derivative, inequality (1.13) in the uniform norm is due to Bernstein, M. Riesz and Szegő. For $\alpha \in \mathbb{N}$ and $p \geqslant 1$ it was established by Zygmund. For real $\alpha \geqslant 1$ and $p \geqslant 1$ inequality (1.13) is due to Lizorkin for $\theta = 0$ and Kozko for all $\theta \in [0, \pi]$. Moreover, Kozko [16] examined conditions on the parameters α and θ under which (1.13) holds with the constant n^α for all L_p, $1 \leqslant p \leqslant \infty$.

For $0 < p < 1$ even Bernstein's inequality (for the first-order derivative) was quite a challenge. It was proved by Arestov, who created in [6] and [17] a new method for dealing with extremal problems for algebraic polynomials on the unit circle, and, as a consequence, with trigonometric polynomials on the period with respect to the norms generated by functions φ in the class Φ^+, which he introduced. In particular, his method also works in the L_p-spaces, $0 \leqslant p \leqslant \infty$. Using this approach, for all $0 \leqslant p \leqslant \infty$ he proved the sharp Bernstein inequality

$$\|f_n^{(r)}\|_p \leqslant n^r \|f_n\|_p, \qquad f_n \in \mathscr{T}_n, \tag{1.14}$$

on the class of all trigonometric polynomials $\mathscr{T}_n$ with positive integer r.

In 1994 Arestov [2] considered Szegő's inequality for the derivatives of nonnegative integer order r for conjugate trigonometric polynomials in L_0, that is,

$$\|\widetilde{f}_n^{(r)}\|_0 \leqslant B_n\biggl(r, \frac{\pi}{2}\biggr)_0 \|f\|_0, \qquad f_n \in \mathscr{T}_n. \tag{1.15}$$

He showed that, for a fixed nonnegative integer r, the constant in Szegő's inequality behaves as

$$B_n\biggl(r, \frac{\pi}{2}\biggr)_0 = 4^{n+o(n)} \quad \text{as } n \to \infty. \tag{1.16}$$

Thus, the behaviour of $B_n(r, \pi/2)_0$ differs substantially from that of the constant $B_n(r, 0)_0 = n^r$ in Bernstein's inequality (1.14) for $r \in \mathbb{N}$ in L_0.

In the same paper [2] Arestov raised the problem of characterizing r and n such that inequality (1.15) for the derivative of the conjugate polynomial holds with the classical constant $B_n(r, \pi/2)_0 = n^r$. He showed that this inequality holds with this constant if $r \geqslant n \ln 2n$. In 1994, on the basis of computer experiments, Arestov put forward the following conjecture regarding the constant $B_n(r, \pi/2)_0$ in (1.15).

Conjecture A. A necessary and sufficient condition for Szegő's inequality (1.15) to hold in L_0 for the derivative of order $r \in \mathbb{N}$ of the conjugate polynomial of order n with constant n^r is $r \geqslant 2n - 2$.

In 2014 Arestov and Glazyrina [10] investigated the Bernstein-Szegő inequality for real $\alpha \geqslant 0$ and arbitrary real θ. In this and more general settings, they examined conditions on n, α and θ under which the Bernstein-Szegő inequality in L_0 (and, as a consequence of (1.8), in all L_p-spaces, $0 < p \leqslant \infty$) holds with constant n^α. They showed that for all $\theta \in [0, \pi]$ a sufficient condition for this is $\alpha \geqslant n \ln 2n$.

Let $A_n(\theta)$ be the set of all $\alpha \geqslant 0$ such that $B_n(\alpha, \theta)_0 = n^\alpha$.

Arestov and Glazyrina made the following two conjectures.

Conjecture 1. If $\alpha \in \mathbb{R}$ and $\alpha \geqslant 2n-2$, then the Bernstein-Szegő inequality in L_0 for the derivative of order α of a polynomial of degree n holds with constant n^α for each θ.

Conjecture 2. For $\theta = 0$ Bernstein's inequality holds with constant n^α if and only if $\alpha \in \mathbb{N}$ or $\alpha \geqslant 2n-2$, that is,

$$A_n(0) = \{1, 2, 3, \dots, 2n-3\} \cup [2n-2, \infty).$$

Arestov and Glazyrina [10] proved these two conjectures for $n = 2$. For each θ, they described the set $A_2(\theta)$. Namely, they showed that $A_2(0) = \{1\} \cup [2, \infty)$ and $A_2(\theta) = [\alpha^*(\theta), \infty)$ for $\theta \in (0, \pi)$, where $\alpha^*(\theta) \in (1, 2)$ is a root of a certain equation.

Popov announced (in his talks at several conferences of 2017–2021; see [18] and the references given there) that Conjecture 2 holds for $n \leqslant 10$ and $\theta = 0$ (that is, in the case of Bernstein's inequality).

In 2022 this author [12] showed that, for any $\theta \in [0, \pi]$, the condition $\alpha \geqslant 2n-2$ ensures the Bernstein-Szegő inequality with the classical constant n^α, that is, Conjecture 1 was confirmed for all $n \in \mathbb{N}$.

1.3. The main results. In this paper we prove the following results on the Riesz and conjugate Riesz derivatives.

Theorem 1. *A necessary and sufficient condition that inequality* (1.9) *hold for the Riesz derivative with constant* $B_n(\alpha, R)_p = n^\alpha$ *for all* $0 \leqslant p \leqslant \infty$ *is that*

$$\alpha \in \{2, 4, 6, \dots, 2n-4\} \cup [2n-2, \infty). \tag{1.17}$$

Theorem 2. *A necessary and sufficient condition that inequality* (1.10) *hold for the conjugate Riesz derivative with constant* $\widetilde{B}_n(\alpha, R)_p = n^\alpha$ *for all* $0 \leqslant p \leqslant \infty$ *is that*

$$\alpha \in \{1, 3, 5, \dots, 2n-5\} \cup [2n-3, \infty). \tag{1.18}$$

For $n \in \mathbb{N}$ let α_n^* be the least nonnegative number such that $B_n(\alpha, \theta)_0 = n^\alpha$ for all $\alpha \geqslant \alpha_n^*$ and all $\theta \in [0, \pi]$. The result of this author confirming Conjecture 1 (see [12]) means that $\alpha_n^* \leqslant 2n-2$. Theorem 1 implies that $\alpha_n^* \geqslant 2n-2$. Therefore, $\alpha_n^* = 2n-2$.

Below, in § 5, for each p, $0 \leqslant p < \infty$, we describe the set of trigonometric polynomials for which inequalities (1.9) and (1.10) for α given by conditions (1.17) and (1.18), respectively, become equalities.

§ 2. The method

2.1. Arestov's method for extremal problems. Below we attack the Bernstein-Szegő inequality using Arestov's method (see [5], [6] and [17]), which is capable of dealing with extremal problems for algebraic polynomials on the unit circle of the complex plane, or, which is the same, for trigonometric polynomials on the period, in view of the formula

$$P_{2n}(e^{it}) = e^{int} f_n(t). \tag{2.1}$$

Let $\mathscr{P}_m = \mathscr{P}_m(\mathbb{C})$ be the set of algebraic polynomials of degree at most m with complex coefficients. For a polynomial of degree $s < m$ it is convenient to assume that it has a zero of multiplicity $m - s$ at the point at infinity $z = \infty$.

It is clear that $P_m(e^{it}) \in \mathscr{T}_m$. Given a polynomial $P_m \in \mathscr{P}_m$, for $0 \leqslant p \leqslant \infty$, we write for brevity $\|P_m\|_p = \|P_m(e^{it})\|_p$; in particular,

$$\|P_m\|_0 = \|P_m(e^{it})\|_0 = \exp\biggl(\frac{1}{2\pi}\int_0^{2\pi} \ln|P_m(e^{it})|\,dt\biggr). \tag{2.2}$$

For a polynomial P_m with nonzero leading coefficient $c_m \neq 0$ and zeros $\{z_j\}_{j=1}^m$, it follows from Jensen's formula (see, for example, [19], Vol. 1, Problem 175) that

$$\|P_m\|_0 = |c_m| \prod_{j=1}^m \max\{1, |z_j|\}. \tag{2.3}$$

Given two polynomials

$$P_m(z) = \sum_{k=0}^m C_m^k a_k z^k \in \mathscr{P}_m \quad \text{and} \quad \Lambda_m(z) = \sum_{k=0}^m C_m^k \lambda_k z^k \in \mathscr{P}_m,$$

the polynomial

$$\Lambda_m P_m(z) = \sum_{k=0}^m C_m^k \lambda_k a_k z^k \tag{2.4}$$

is known as their *Szegő composition*. For some properties of the Szegő composition, see [19], Vol. 2, Pt. V, Ch. 2, and [20], Ch. 4. For fixed Λ_m the Szegő composition (2.4) is a linear operator in $\mathscr{P}_m$. The polynomial

$$I_m(z) = (1+z)^m = \sum_{k=0}^m C_m^k z^k \tag{2.5}$$

has the following property: for each $\Lambda_m \in \mathscr{P}_m$,

$$\Lambda_m I_m(z) = I_m \Lambda_m(z) = \Lambda_m(z). \tag{2.6}$$

In view of this I_m can be regarded as the 'identity element' for Szegő composition.

The following result is due to Arestov [5].

Theorem A. *For all polynomials $\Lambda_m, P_m \in \mathscr{P}_m$ and $0 \leqslant p \leqslant \infty$,*

$$\|\Lambda_m P_m\|_p \leqslant \|\Lambda_m\|_0 \|P_m\|_p. \tag{2.7}$$

For $p = 0$ inequality (2.7) is sharp for any Λ_m and is attained at the polynomial (2.5).

2.2. Turning to the investigation of the (conjugate) Riesz derivative of the extremal polynomial. Let us verify that the Riesz derivative operator D_R^α and the conjugate Riesz derivative operator $\widetilde{D}_R^\alpha$, as defined by (1.3) and (1.4) on the set $\mathscr{T}_n$ via (2.1), can be represented as the Szegő composition operator (see (2.4)) on the set $\mathscr{P}_{2n}$ for some polynomials Λ_{2n}^α and $\widetilde{\Lambda}_{2n}^\alpha$. More precisely,

$$D_R^\alpha f_n(t) = e^{-int}(\Lambda_{2n}^\alpha P_{2n})(e^{it}), \qquad f_n(t) = e^{-int}P_{2n}(e^{it}), \tag{2.8}$$

and

$$\widetilde{D}_R^\alpha f_n(t) = e^{-int}(\widetilde{\Lambda}_{2n}^\alpha P_{2n})(e^{it}), \qquad f_n(t) = e^{-int}P_{2n}(e^{it}). \tag{2.9}$$

The following notation will be adhered to in what follows:

$$Q_n^\alpha(z) = \sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^k. \tag{2.10}$$

Proposition 1. *For $\alpha \geqslant 0$, to the Riesz derivative D_R^α on the class $\mathscr{T}_n$ there corresponds via (2.8) the operator of Szegő composition with the polynomial*

$$\Lambda_{2n}^\alpha(z) = \sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^{n-k} + \sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^{n+k} = z^n Q_n^\alpha\left(\frac{1}{z}\right) + z^n Q_n^\alpha(z) \tag{2.11}$$

on the set $\mathscr{P}_{2n}$, and to the conjugate Riesz derivative $\widetilde{D}_R^\alpha$ there corresponds via (2.9) the operator of Szegő composition with the polynomial

$$\widetilde{\Lambda}_{2n}^\alpha(z) = -\sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^{n-k} + \sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^{n+k} = -z^n Q_n^\alpha\left(\frac{1}{z}\right) + z^n Q_n^\alpha(z). \tag{2.12}$$

Proof. We write a trigonometric polynomial $f_n \in \mathscr{T}_n$ as

$$f_n(t) = \sum_{k=-n}^{n} C_{2n}^{n+k} c_k e^{ikt}.$$

Using (1.3) and (1.4),

$$D_R^\alpha f_n(t) = \sum_{k=-n}^{n} C_{2n}^{n+k} |k|^\alpha c_k e^{ikt} \quad \text{and} \quad \widetilde{D}_R^\alpha f_n(t) = \sum_{k=-n}^{n} C_{2n}^{n+k} |k|^\alpha (\operatorname{sign} k) c_k e^{ikt}.$$

The algebraic polynomial

$$P_{2n}(z) = \sum_{k=-n}^{n} C_{2n}^{n+k} c_k z^{n+k}$$

is defined from the polynomial f_n via (2.1), and, in a similar way, the polynomials

$$R_{2n}^\alpha(z) = \sum_{k=-n}^{n} C_{2n}^{n+k} |k|^\alpha c_k z^{n+k} \quad \text{and} \quad \widetilde{R}_{2n}^\alpha(z) = \sum_{k=-n}^{n} C_{2n}^{n+k} |k|^\alpha (\operatorname{sign} k) c_k z^{n+k}$$

are defined from the polynomials $D_R^\alpha f_n$ and $\widetilde{D}_R^\alpha f_n$, respectively. From (2.4) we obtain

$$R_{2n}^\alpha = \Lambda_{2n}^\alpha P_{2n} \quad \text{and} \quad \widetilde{R}_{2n}^\alpha = \widetilde{\Lambda}_{2n}^\alpha P_{2n},$$

where the polynomials Λ_{2n}^α and $\widetilde{\Lambda}_{2n}^\alpha$ are defined by (2.11) and (2.12), respectively. This proves Proposition 1.

In view of (1.8) we must look at inequalities (1.11) and (1.12), that is,

$$\|D_R^\alpha f_n\|_0 \leqslant B_n(\alpha, R)_0 \|f_n\|_0, \qquad f_n \in \mathscr{T}_n,$$

and

$$\|\widetilde{D}_R^\alpha f_n\|_0 \leqslant \widetilde{B}_n(\alpha, R)_0 \|f_n\|_0, \qquad f_n \in \mathscr{T}_n.$$

Using (2.8) and (2.7) we have

$$\|D_R^\alpha f_n\|_0 = \|\Lambda_{2n}^\alpha P_{2n}\|_0 \leqslant \|\Lambda_{2n}^\alpha\|_0 \|P_{2n}\|_0 = \|\Lambda_{2n}^\alpha\|_0 \|f_n\|_0, \qquad P_{2n}(e^{it}) = e^{int} f_n(t). \tag{2.13}$$

Inequality (2.13) is sharp; it becomes an equality for

$$P_{2n}(z) = I_{2n}(z) = (1+z)^{2n}. \tag{2.14}$$

From this polynomial, via (2.1) we define

$$\begin{aligned} h_n(t) &= e^{-int} I_{2n}(e^{it}) = 4^n \cos^{2n} \frac{t}{2} = 2^n (1+\cos t)^n \\ &= \sum_{k=-n}^{n} C_{2n}^{n+k} e^{ikt} = C_{2n}^n + 2\sum_{k=1}^{n} C_{2n}^{n+k} \cos kt. \end{aligned} \tag{2.15}$$

Note that $\|h_n\|_0 = \|I_{2n}\|_0 = 1$ by (2.14) and (2.3).

In view of (2.6) and (2.15) the sharp constant in (2.13) is

$$\begin{gathered} \|\Lambda_{2n}^\alpha\|_0 = \|\Lambda_{2n}^\alpha I_{2n}\|_0 = \|D_R^\alpha h_n\|_0, \\ D_R^\alpha h_n(t) = 2\sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha \cos kt. \end{gathered} \tag{2.16}$$

Thus, the polynomial (2.15) is extremal in inequality (2.13) for $p = 0$. The conjugate Riesz derivative is dealt with similarly. In this case we obtain the polynomial

$$\widetilde{D}_R^\alpha h_n(t) = 2\sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha \sin kt. \tag{2.17}$$

This establishes the following result.

Proposition 2. *The sharp constants in inequalities* (1.11) *and* (1.12) *satisfy*

$$B_n(\alpha, R)_0 = \|\Lambda_{2n}^\alpha\|_0 = \|D_R^\alpha h_n\|_0 \tag{2.18}$$

and

$$\widetilde{B}_n(\alpha, R)_0 = \|\widetilde{\Lambda}_{2n}^\alpha\|_0 = \|\widetilde{D}_R^\alpha h_n\|_0. \tag{2.19}$$

The leading coefficients of the polynomials (2.11) and (2.12) are $\lambda_{2n} = n^\alpha$. Hence by Jensen's formula (2.3) the equality $\|\Lambda_{2n}^\alpha\|_0 = n^\alpha$ (or the equality $\|\widetilde{\Lambda}_{2n}^\alpha\|_0 = n^\alpha$) holds if and only if all the $2n$ zeros of the polynomial (2.11) (or of (2.12), respectively) lie in the closed unit disc $|z| \leqslant 1$. Since $\Lambda_{2n}^\alpha(z) = z^{2n}\overline{\Lambda_{2n}^\alpha(1/\overline{z})}$ and since $\widetilde{\Lambda}_{2n}^\alpha$ has the same property, this is possible if and only if all the zeros of Λ_{2n}^α (or $\widetilde{\Lambda}_{2n}^\alpha$) lie on the unit circle. But this is equivalent to saying that all the $2n$ zeros of the polynomial $D_R^\alpha h_n$ (of $\widetilde{D}_R^\alpha h_n$, respectively) lie on the period. Thus, we have established the following result.

Proposition 3. *For parameters n and α, inequality (1.11) for the Riesz derivative in L_0 holds with constant n^α if and only if all the $2n$ zeros of the polynomial (2.16) lie on the period. The same result also holds for the conjugate Riesz derivative and the polynomial (2.17).*

The result below follows from (1.8) and Proposition 3.

Corollary. *Inequality (1.9) (inequality (1.10)) holds with the constant n^α for all $0 \leqslant p \leqslant \infty$ if and only if all the zeros of the polynomial (2.16) (of (2.17), respectively) lie on the period.*

Thus, the above problem reduces to examining the position of the zeros of the polynomials (2.16) and (2.17) (or, which is the same in view of (2.1), of the zeros of the polynomials (2.11) and (2.12)). For this investigation we invoke the polynomials

$$Q_n^\alpha(z) = \sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha z^k$$

(see (2.10)), since

$$D_R^\alpha h_n(t) = 2\,\mathrm{Re}\,Q_n^\alpha(e^{it}) \quad \text{and} \quad \widetilde{D}_R^\alpha h_n(t) = 2\,\mathrm{Im}\,Q_n^\alpha(e^{it}) \tag{2.20}$$

by (2.11) and (2.12).

§ 3. Auxiliary results

A function g is said to be *completely monotone* on the half-axis $(0,\infty)$ if it is infinitely differentiable and $(-1)^\nu g^{(\nu)}(x) \geqslant 0$ for all $\nu = 0,1,2,3,\dots$ and all $x > 0$. By the Hausdorff-Bernstein-Widder theorem a function g is completely monotone if and only if it can be expressed as

$$g(x) = \int_0^\infty e^{-tx}\,d\mu(t), \tag{3.1}$$

where μ is a nonnegative Borel measure such that the integral (3.1) converges for all $x > 0$; here the measure μ is finite if and only if $g(0) < \infty$. For the proof of this theorem, see, for example, [21], § 5.5. An example of a completely monotone function is given by $g(x) = 1/x^\beta$, $\beta > 0$; this function can be written as

$$\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-tx} t^{\beta-1}\,dt;$$

this representation can be derived from the formula

$$\Gamma(\beta) = \int_0^\infty e^{-t} t^{\beta-1}\, dt.$$

We require the following important result, which we proved in [12].

Lemma 1 (see [12], Lemma 1). *Let the polynomial*

$$Q_n(z) = \sum_{k=1}^{n} c_k z^k$$

of degree $n \in \mathbb{N}$ have real coefficients satisfying, for some $a \in \mathbb{N}$, the m conditions

$$\sum_{k=1}^{n} (-1)^k c_k g^{(a\nu)}(k) = 0, \qquad \nu = 0, 1, 2, \dots, m-1, \tag{3.2}$$

$1 \leqslant m \leqslant n-1$, *where g is a completely monotone function equal to the Laplace transform of a measure with support on a set of cardinality at least n. Then Q_n has at least m sign changes on the interval $(-1, 0)$.*

For $n \geqslant 1$ consider the function

$$S_n(\alpha) = \sum_{k=1}^{n} (-1)^k C_{2n}^{n+k} k^\alpha$$

of $\alpha \geqslant 0$. The following result holds for this function.

Lemma 2. *For any $n \in \mathbb{N}$,*

$$S_n(2r) = \sum_{k=1}^{n} (-1)^k C_{2n}^{n+k} k^{2r} = 0, \qquad r = 1, 2, 3, \dots, n-1, \tag{3.3}$$

and

$$\operatorname{sign} S_n(\alpha) = (-1)^r \quad \text{for } r = 1, 2, 3, \dots, n \text{ and } \alpha = 2r - \beta,\ 0 < \beta < 2. \tag{3.4}$$

Proof. First we prove (3.3). Consider the polynomial

$$h_n(t) = 4^n \cos^{2n} \frac{t}{2} = \sum_{k=-n}^{n} C_{2n}^{n+k} \cos kt.$$

For even $r \in \mathbb{N}$, we have

$$S_n(2r) = \frac{(-1)^{r/2}}{2} h_n^{(2r)}(\pi).$$

The polynomial h_n has a zero of multiplicity $2n$ at $t = \pi$. Hence $h_n^{(2r)}(\pi) = 0$ for $r = 1, 2, 3, \dots, n-1$. This proves (3.3).

Now let us verify (3.4). Consider the function

$$\varphi_n(x) = \frac{C_{2n}^n}{2} + \sum_{k=1}^{n} (-1)^k C_{2n}^{n+k} e^{-k^2 x}.$$

We claim that

$$\varphi_n(x) > 0 \quad \text{for any } x > 0. \tag{3.5}$$

For fixed $x > 0$ consider the function $\psi_x(y) = e^{-y^2/(4x)}/\sqrt{\pi x}$, $y \in (-\infty, \infty)$, whose Fourier transform is the Gauss-Weierstrass kernel

$$\widehat{\psi_x}(\omega) = \int_{-\infty}^{\infty} \psi_x(y) e^{-iy\omega}\, dy = e^{-x\omega^2}.$$

Now (3.5) follows since

$$\begin{aligned}\varphi_n(x) &= \frac{1}{2} \sum_{k=-n}^{n} (-1)^k C_{2n}^{n+k} \widehat{\psi_x}(k) = \frac{1}{2} \sum_{k=-n}^{n} (-1)^k C_{2n}^{n+k} \int_{-\infty}^{\infty} \psi_x(y) e^{-iky}\, dy \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \psi_x(y) \Bigg(\sum_{k=-n}^{n} (-1)^k C_{2n}^{n+k} e^{-iky} \Bigg) dy = \frac{1}{2} \int_{-\infty}^{\infty} \psi_x(y) h_n(\pi - y)\, dy > 0.\end{aligned}$$

In what follows we require the Mellin transform of φ_n,

$$g_n(s) = (\mathcal{M}\varphi_n)(s) = \int_0^{\infty} x^{s-1} \varphi_n(x)\, dx. \tag{3.6}$$

Let us find a domain where the integral in (3.6) defines an analytic function. To this end we find how $\varphi_n(x)$ behaves as $x \to 0$ and $x \to \infty$. We have

$$\varphi_n(0) = \frac{C_{2n}^n}{2} + \sum_{k=1}^{n} (-1)^k C_{2n}^{n+k} = \frac{1}{2} \sum_{k=-n}^{n} (-1)^k C_{2n}^{n+k} = \frac{1}{2}(1-1)^{2n} = 0.$$

Next, an application of (3.3) shows that $\varphi_n'(0) = \varphi_n''(0) = \cdots = \varphi_n^{(n-1)}(0) = 0$. In addition, $\lim_{x\to\infty} \varphi_n(x) = C_{2n}^n/2 = \text{const} > 0$. This implies that, for $-n < \sigma < 0$, the function $x^{\sigma-1}\varphi_n(x)$ lies in $L(0,\infty)$. Consequently, the function (3.6) is analytic in the strip $-n < \operatorname{Re} s < 0$ (see, for example, Theorem 1 in [22]).

From the definition (3.6) of g_n and the property (3.5) we obtain the following important fact, which is required below:

$$g_n(s) > 0, \qquad -n < s < 0. \tag{3.7}$$

Having the function (3.6) at our disposal, we consider the functions $g_n^r(s) = (\mathcal{M}\varphi_n^{(r)})(s)$. We claim that for each $r = 1, 2, \dots, n$ the function $g_n^r(s)$ is analytic in the half-plane $\operatorname{Re} s > -n + r$ and, in addition, can be expressed in terms of the function (3.6) in the strip $-n + r < \operatorname{Re} s < r$.

First consider g_n^1. We have

$$\begin{aligned} g_n^1(s) &= \int_0^{\infty} x^{s-1} \varphi_n'(x)\, dx = \int_0^{\infty} x^{s-1}\, d\varphi_n(x) \\ &= x^{s-1}\varphi_n(x)\big|_0^{\infty} - (s-1) \int_0^{\infty} x^{s-2} \varphi_n(x)\, dx. \end{aligned} \tag{3.8}$$

The function $\varphi_n'(x)$ behaves as $O(x^{n-1})$ as $x\to 0$ and decays exponentially as $x\to\infty$; hence the left-hand integral in (3.8) exists and is analytic for $-n+1<\operatorname{Re}s<\infty$. In the strip $-n+1<\operatorname{Re}s<1$ the term outside the integral vanishes, and the right-hand integral in (3.8) defines an analytic function. So

$$g_n^1(s)=-(s-1)g_n(s-1),\qquad -n+1<\operatorname{Re}s<1.$$

A similar analysis shows that the functions $g_n^r(s)$, $r=2,3,\dots,n$, are analytic in the half-plane $\operatorname{Re}s>-n+r$, and

$$g_n^r(s)=(-1)^r(s-r)(s-r+1)\cdots(s-1)g_n(s-r),\qquad -n+r<\operatorname{Re}s<r. \tag{3.9}$$

In the half-plane $\operatorname{Re}s>0$,

$$\begin{aligned} g_n^r(s)&=\int_0^\infty x^{s-1}\varphi_n^{(r)}(x)\,dx=\int_0^\infty x^{s-1}\biggl(\sum_{k=1}^n(-1)^kC_{2n}^{n+k}k^{2r}(-1)^re^{-k^2x}\biggr)dx\\ &=(-1)^r\Gamma(s)\sum_{k=1}^n(-1)^kC_{2n}^{n+k}k^{2r-2s}. \end{aligned}\tag{3.10}$$

For $s\in(0,1)$ we have

$$g_n^r(s)=(-1)^r\Gamma(s)\sum_{k=1}^n(-1)^kC_{2n}^{n+k}k^{\alpha}=(-1)^r\Gamma(s)S_n(\alpha),$$
$$\alpha=2r-2s=2r-\beta,\qquad \beta=2s\in(0,2).$$

Thus, for $0<\operatorname{Re}s<r$ the function $g_n^r(s)$ can be defined by (3.9) and (3.10) alike.

Let us find the sign of $g_n^r(s)$ for $1\leqslant r\leqslant n$ and $s\in(0,1)$. The sign of the product $(s-r)(s-r+1)\cdots(s-1)$ is $(-1)^r$. For such s the function $g_n(s-r)$ is positive by (3.7). This establishes (3.4) for $1\leqslant r\leqslant n$ and completes the proof of Lemma 2.

Lemma 3. *For $n\geqslant 2$ and $2n-4<\alpha<2n-2$ let Q_n^α be the polynomial defined by (2.10). Then*

1) *all the zeros of Q_n^α, save one, lie in the open unit disc;*
2) *in addition, Q_n^α has a real zero $x_0=x_0(n,\alpha)<-1$.*

Proof. To prove 1) we apply Lemma 1 to Q_n^α. Let c_k be the coefficients of the polynomial Q_n^α, that is,

$$c_k=C_{2n}^{n+k}k^\alpha=C_{2n}^{n+k}k^{2n-4+\beta},\qquad 0<\beta<2,\quad k=1,2,\dots,n.$$

We take $a=2$ in this lemma and consider the completely monotone function $g(t)=1/t^\beta$. We claim that

$$S_\nu=\sum_{k=1}^n(-1)^kc_kg^{(2\nu)}(k)=0,\qquad \nu=0,1,\dots,n-3.$$

Indeed, using (3.3) (see Lemma 2) we obtain

$$\frac{\Gamma(\beta)}{\Gamma(\beta+2\nu)}S_\nu = \sum_{k=1}^{n}(-1)^k C_{2n}^{n+k}\frac{k^{2n-4+\beta}}{k^{2\nu+\beta}}$$
$$= \sum_{k=1}^{n}(-1)^k C_{2n}^{n+k} k^{2(n-2-\nu)} = 0, \qquad \nu = 0, 1, \dots, n-3.$$

So the assumptions of Lemma 1 are met. Hence $n-2$ zeros of Q_n^α lie on $(-1, 0)$. Another zero of Q_n^α is at 0. Let us estimate the remaining zero.

To prove assertion 2) of the lemma it suffices to see that $Q_n^\alpha(-1)$ has the correct sign, that is,

$$\operatorname{sign} Q_n^\alpha(-1) = (-1)^{n-1}. \tag{3.11}$$

It is easily checked that

$$Q_n^\alpha(-1) = \sum_{k=1}^{n}(-1)^k C_{2n}^{n+k} k^\alpha = S_n(\alpha).$$

Now the required fact follows from (3.4) in Lemma 2. Lemma 3 is proved.

Lemma 4. *For $2n-4 < \alpha < 2n-2$ consider the polynomial*

$$u_n(t) = u_n^\alpha(t) = D_R^\alpha h_n(t) = 2\operatorname{Re} Q_n^\alpha(e^{it}).$$

Then

$$u_n(0) > 0, \qquad \operatorname{sign} u_n(\pi) = (-1)^{n-1} \quad \textit{and} \quad \operatorname{sign} u_n''(\pi) = (-1)^{n-1}. \tag{3.12}$$

Proof. Since

$$u_n(t) = 2\operatorname{Re} Q_n^\alpha(e^{it}) = 2\sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha \cos kt,$$

we have

$$u_n(0) = 2\sum_{k=1}^{n} C_{2n}^{n+k} k^\alpha > 0 \quad \text{and} \quad u_n(\pi) = 2\sum_{k=1}^{n}(-1)^k C_{2n}^{n+k} k^\alpha = 2S_n(\alpha).$$

Hence $\operatorname{sign} u_n(\pi) = (-1)^{n-1}$ by (3.4) in Lemma 2.

Next we have

$$u_n''(t) = -2\sum_{k=1}^{n} C_{2n}^{n+k} k^{\alpha+2} \cos kt,$$

and another appeal to (3.4) shows that $\operatorname{sign} u_n''(\pi) = (-1)^{n-1}$, which proves Lemma 4.

§ 4. Proofs of the main results

4.1. Proof of Theorem 1. Let us show that, for $2n-4<\alpha<2n-2$, for the Riesz derivative we have $B_n(\alpha,R)_0>n^\alpha$. By (2.20), $D_R^\alpha h_n(t)=2\operatorname{Re}Q_n^\alpha(e^{it})$. We denote this polynomial by $u_n(t)=u_n^\alpha(t)$. We claim that not all of its zeros lie on the period.

By Lemma 3 the polynomial $Q=Q_n^\alpha$ has precisely $n-1$ zeros in the unit disc, and it has no zeros on the unit circle. As t ranges from $-\pi$ to π, the function $Q(e^{it})$ describes a curve $Z(t)$. By the argument principle this curve makes $n-1$ circuits about the origin. Hence u_n has at least $2n-2$ zeros on the period.

By Lemma 4
$$u_n(0)>0 \quad\text{and}\quad \operatorname{sign}u_n(\pi)=(-1)^{n-1}. \tag{4.1}$$
Hence the zeros of u_n lie on the intervals $(-\pi,0)$ and $(0,\pi)$. Since the polynomial u_n is even, each of these intervals contains the same number $m\in\{n-1,n\}$ of zeros.

Assume that the polynomial u_n has ℓ distinct zeros $0<t_1<\cdots<t_\ell<\pi$ of multiplicity $\kappa_1,\dots,\kappa_\ell$, $1\leqslant\ell\leqslant m$, on $(0,\pi)$, where $\kappa_1+\cdots+\kappa_\ell=m$. If t_1 has even multiplicity κ_1, then on the interval (t_1,t_2) the polynomial $u_n(t)$ has the same sign as $u_n(0)$. If κ_1 is odd, then the sign of $u_n(t)$ on (t_1,t_2) is opposite to that of $u_n(0)$. So $\operatorname{sign}u_n(t)=(-1)^{\kappa_1}\operatorname{sign}u_n(0)$ for $t\in(t_1,t_2)$.

A similar analysis shows that
$$\operatorname{sign}u_n(t)=(-1)^{\kappa_1}\cdots(-1)^{\kappa_j}\operatorname{sign}u_n(0),\qquad t\in(t_j,t_{j+1}),\quad j=1,\dots,\ell-1.$$
For $t\in(t_\ell,\pi]$ we have
$$\operatorname{sign}u_n(t)=(-1)^{\kappa_1}\cdots(-1)^{\kappa_\ell}\operatorname{sign}u_n(0)=(-1)^m\operatorname{sign}u_n(0),\qquad t\in(t_\ell,\pi]. \tag{4.2}$$
But m is equal to either $n-1$ or n. Hence $m=n-1$ by (4.1) and (4.2). As a result, u_n has precisely $2n-2$ zeros on the period.

Now let $\alpha\in(2r,2r+2)$, $r=0,1,\dots,n-3$. We claim that the polynomial $D_R^\alpha h_n$ cannot have $2n$ zeros on the period. If $D_R^\alpha h_n$ had $2n$ zeros, then $(D_R^\alpha h_n)^{(2n-4-2r)}=(-1)^{n-2-r}D_R^{2n-4+\alpha-2r}h_n$ would have $2n$ zeros on the period, but this is not so by the above since $2n-4<2n-4+\alpha-2r<2n-2$.

Now assume that $\alpha=0$. The constant term of the polynomial D^0h_n is zero, and so D^0h_n has at least two zeros on the period. But it has at most two such zeros, because its derivative $(D^0h_n(t))'=h_n'(t)$ is positive for $t\in(-\pi,0)$ and negative for $t\in(0,\pi)$.

Thus, for $\alpha\in[0,2)\cup(2,4)\cup(4,6)\cup\cdots\cup(2n-4,2n-2)$ the polynomial $D_R^\alpha h_n$ has at most $2n-2$ zeros on the period.

For the remaining
$$\alpha\in\{2,4,6,\dots,2n-4\}\cup[2n-2,\infty),$$
we have
$$\|D_R^\alpha h_n\|_0=n^\alpha. \tag{4.3}$$
Indeed, for even $\alpha\in\mathbb{N}$, the Riesz derivative coincides, up to the sign of $(-1)^{\alpha/2}$, with the classical derivative of order α, and now (4.3) follows from the sharp Bernstein inequality (1.14), which was proved by Arestov. For $\alpha\geqslant 2n-2$ equality (4.3) also follows from this author's result in [12].

Thus, we have shown that, for $p=0$ for $n\in\mathbb{N}$ and $\alpha\geqslant 0$, the best constant (2.18) in inequality (1.11) is equal to $B_n(\alpha,R)_0=n^\alpha$ if and only if α satisfies (1.17).

Now the conclusion of Theorem 1 is secured by (1.8).

4.2. Proof of Theorem 2. Let $\alpha\in[0,1)\cup(1,3)\cup(3,5)\cup\dots\cup(2n-5,2n-3)$. We show that the polynomial $\widetilde{D}_R^\alpha h_n$ cannot have $2n$ zeros on the period. If $\widetilde{D}_R^\alpha h_n$ had $2n$ zeros on the period, then $(\widetilde{D}_R^\alpha h_n)'=D_R^{\alpha+1}h_n$ would have $2n$ zeros on the period, but this is impossible by what was proved in § 4.1, because $\alpha+1<2n-2$ and is not a natural number.

So, for $\alpha\in[0,1)\cup(1,3)\cup(3,5)\cup\dots\cup(2n-5,2n-3)$ the polynomial $\widetilde{D}_R^\alpha h_n$ has at most $2n-2$ zeros on the period.

Let us show that, for

$$\alpha\in\{1,3,5,\dots,2n-5\}\cup[2n-3,\infty), \tag{4.4}$$

we have

$$\|\widetilde{D}_R^\alpha h_n\|_0=n^\alpha. \tag{4.5}$$

For odd $\alpha\in\mathbb{N}$, the Riesz derivative coincides, up to the sign of $(-1)^{(\alpha-1)/2}$, with the classical derivative of order α, and so (4.5) follows from (1.14). For $\alpha\geqslant 2n-2$, (4.5) is secured by [12].

It remains to consider the case when $\alpha\in(2n-3,2n-2)$. Let us show that for such α all $2n$ zeros of the polynomial $v_n=v_n^\alpha=\widetilde{D}_R^\alpha h_n$ lie on the period. It is clear that $v_n^\alpha=(u_n^{\alpha-1})'$, $\alpha-1\in(2n-4,2n-3)$. The polynomial $u_n^{\alpha-1}$ has $2n-2$ zeros $-\pi<\tau_1<\dots<\tau_{2n-2}<\pi$. So the polynomial v_n has $2n-3$ zeros on the interval $[\tau_1,\tau_{2n-2}]$, and it also vanishes at π. In addition, v_n has two more zeros on the intervals $(-\pi,\tau_1)$ and (τ_{2n-2},π) because by (3.12) $u_n^{\alpha-1}(\pi)$ and $(u_n^{\alpha-1})''(\pi)$ are positive or negative simultaneously. So, all the $2n$ zeros of the polynomial v_n lie on the period.

This proves equality (4.5) for α as in (4.4).

Thus, for $p=0$, $n\in\mathbb{N}$ and $\alpha\geqslant 0$ the best constant (2.19) in inequality (1.12) is $\widetilde{B}_n(\alpha,R)_0=n^\alpha$ if and only if α satisfies condition (1.18).

Now the conclusion of Theorem 2 follows from (1.8).

§ 5. Extremal polynomials

In this section we describe the sets of extremal polynomials in inequalities (1.9) and (1.10) for α given by (1.17) and (1.18), respectively, and $0\leqslant p<\infty$. To do this we invoke some results due to Arestov [6], which provide necessary and sufficient conditions for a polynomial to be extremal in the inequality for the Szegő composition operator under some conditions on the operator.

Following [6], we denote by $\mathscr{P}_m^0$, $\mathscr{P}_m^\infty$ and $\mathscr{P}_m^1$ the subsets of the set of polynomials $\mathscr{P}_m$ such that all of their m zeros lie in the disc $|z|\leqslant 1$, in the set $|z|\geqslant 1$, or on the circle $|z|=1$, respectively.

Theorem B (Arestov; see Theorems 1, 2 and 5 in [6]). *Let $m\in\mathbb{N}$, $m\geqslant 2$, and let $\Lambda_m(z)=\sum_{k=0}^m C_m^k\gamma_k z^k\in\mathscr{P}_m^1$. In addition, let the polynomial*

$$\Lambda_m^\backprime(z)=\sum_{k=0}^{m-2}C_{m-2}^k\gamma_{k+1}z^k$$

of degree $m-2$ also lie in $\mathscr{P}^1_{m-2}$. Then for all $0 \leqslant p \leqslant \infty$,

$$\|\Lambda_m P_m\|_p \leqslant |\gamma_m| \cdot \|P_m\|_p, \qquad P_m \in \mathscr{P}_m. \tag{5.1}$$

For $p=0$ this inequality turns to equality precisely for the polynomials $P_m \in \mathscr{P}^0_m \cup \mathscr{P}^\infty_m$, while for $0<p<\infty$ it turns to equality precisely for the polynomials az^m+b, $a,b\in\mathbb{C}$.

For $\Lambda_m \in \mathscr{P}^1_m$, in view of (2.3) we have $\|\Lambda_m\|_0 = |\gamma_m|$, where γ_m is its leading coefficient, and so inequality (5.1) is contained in Theorem A.

Theorem 3. *The following results hold.*

1. *For $p=0$ the extremal polynomials for inequalities with α given by (1.17) and (1.18), respectively, are precisely the polynomials for which the polynomials P_{2n}, as defined by (2.1), lie in $\mathscr{P}^0_{2n}\cup\mathscr{P}^\infty_{2n}$.*
2. *For $0<p<\infty$ the extremal polynomials for inequalities (1.9) and (1.10) with α given by (1.17) and (1.18), respectively, are precisely the polynomials $c_{-n}e^{-int}+c_ne^{int}$ and $c_{-n}, c_n \in \mathbb{C}$, respectively.*

Proof. For the proof we use Theorem B. We will show that the polynomials

$$\Lambda^\alpha_{2n}(z) = \sum_{k=-n}^{n} C^{n+k}_{2n}\lambda_k z^{n+k}, \qquad \lambda_k = |k|^\alpha,$$

and

$$\widetilde{\Lambda}^\alpha_{2n}(z) = \sum_{k=-n}^{n} C^{n+k}_{2n}\widetilde{\lambda}_k z^{n+k}, \qquad \widetilde{\lambda}_k = |k|^\alpha(\operatorname{sign} k)$$

(see (2.11) and (2.12)), for α given by (1.17) and (1.18), respectively, satisfy

$$(\Lambda^\alpha_{2n})^\backprime(z) = \sum_{k=-n+1}^{n-1} C^{n-1+k}_{2n-2}\lambda_k z^{n-1+k} \in \mathscr{P}^1_{2n-2} \tag{5.2}$$

and

$$\big(\widetilde{\Lambda}^\alpha_{2n}\big)^\backprime(z) = \sum_{k=-n+1}^{n-1} C^{n-1+k}_{2n-2}\widetilde{\lambda}_k z^{n-1+k} \in \mathscr{P}^1_{2n-2}. \tag{5.3}$$

It is clear that $(\Lambda^\alpha_{2n})^\backprime = \Lambda^\alpha_{2(n-1)}$ and $\big(\widetilde{\Lambda}^\alpha_{2n}\big)^\backprime = \widetilde{\Lambda}^\alpha_{2(n-1)}$. If $\alpha\in\mathbb{N}$ is even, then the Riesz derivative of order α coincides, up to sign, with the classical derivative: $D^\alpha_R f_n = \pm f_n^{(\alpha)}$, $f_n \in \mathscr{T}_n$. Hence, for such α, for any $m\in\mathbb{N}$ the polynomial Λ^α_{2m} lies in $\mathscr{P}^1_{2m}$ (see [6] and [17]). In a similar way, for odd $\alpha\in\mathbb{N}$ the polynomial $\widetilde{\Lambda}^\alpha_{2m}$ lies in $\mathscr{P}^1_{2m}$.

Next, Lemma 3 in [12] asserts that for any $\alpha \geqslant 2n-2$ all zeros of the polynomials $D^\alpha_R h_n$ and $\widetilde{D}^\alpha_R h_n$ (and, *a fortiori*, of $D^\alpha_R h_{n-1}$ and $\widetilde{D}^\alpha_R h_{n-1}$) lie on the period. Hence, by (2.1) the polynomials Λ^α_{2n} and $\widetilde{\Lambda}^\alpha_{2n}$, and also Λ^α_{2n-2} and $\widetilde{\Lambda}^\alpha_{2n-2}$, lie in $\mathscr{P}^1_{2m}$. This proves (5.2) and (5.3). Now Theorem 3 follows from Theorem B.

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