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A Matrix Transform Technique for Distributed-Order Time-Fractional Advection–Dispersion Problems

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Abstract: Invoking the matrix transfer technique, we propose a novel numerical scheme to solve the time-fractional advection–dispersion equation (ADE) with distributed-order Riesz-space fractional derivatives (FDs). The method adopts the midpoint rule to reformulate the distributed-order Riesz-space FDs by means of a second-order linear combination of Riesz-space FDs. Then, a central difference approximation is used side by side with the matrix transform technique for approximating the Riesz-space FDs. Based on this, the distributed-order time-fractional ADE is transformed into a time-fractional ordinary differential equation in the Caputo sense, which has an equivalent Volterra integral form. The Simpson method is used to discretize the weakly singular kernel of the resulting Volterra integral equation. Stability, convergence, and error analysis are presented. Finally, simulations are performed to substantiate the theoretical findings.

Keywords: advection–dispersion equation; matrix transform method; convergence analysis; distributed-order; Riesz fractional derivative

MSC: 26A33; 34A08; 65D15; 35R11



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1. Introduction

In recent years, the theory and applications of derivatives and integrals of fractional order have played an important role in many fields [1–7]. Anomalous relaxation is well explained in many complex systems using multi-term fractional order models. As a result, multi-term partial differential equations (PDEs) of fractional order have found widespread application in describing real-world physical phenomena, as is the case of anomalous diffusive effects, damping, magnetic resonance imaging, and the mechanical and physical behavior of oxygen transport through capillaries [8–10]. In [11], the concept of distributed-order fractional operators and, recently, distributed-order fractional PDEs, have been generalized. However, the distributed order FDs constitute a topic that is not new. Indeed, Caputo addressed such kind of operators in 1969, in the context of properties of inelastic media. The problem was solved in 1995 and, later, in [12], a viscoelastic model involving a multi-term FD was expanded to distributed order. FDs of distributed-order are operators that are integrated, within a specified range, over the order of differentiation.

Fractional PDEs of distributed-order are an extension of single- and multi-term fractional PDEs [13–15]. Several authors have investigated various numerical approaches for solving fractional models of distributed order. Diethelm et al. [16] suggested an algorithm for approximating the solution of distributed-order differential equations. Ford et al. [17] used the implicit finite difference technique to provide an efficient numerical method to address the time distributed-order diffusion problem. For the time distributed-order Riesz-space FD model, Ye et al. [18] investigated an efficient numerical technique over bounded domains. Yang et al. [19] addressed a numerical strategy based on the WSGD-OSC technique for simulating the distributed order time fractional reaction-diffusion two-dimensional equation. Hu et al. [20] reported results of an implicit numerical approach for two-sided space- and distributed-order time-fractional ADEs. Zaky and Machado [21] proposed spectral tau approaches to solve the fractional diffusion problem with distributed-order with time and Dirichlet boundary conditions. Shi et al. [22] addressed an approach for the Riesz-space multi-term and distributed-order time-fractional wave equation based on the unstructured mesh finite element technique on an irregular convex domain. Aboelenen [23] developed a scheme based on the local discontinuous Galerkin finite element technique for solving distributed-order time- and Riesz-space-fractional PDEs. Chen et al. [24] proposed a finite difference/Laguerre spectral approximation for distributed-order time-fractional reaction-diffusion equations. Morgado et al. [25] introduced a Chebyshev collocation approach for providing approximate solutions to the distributed-order time-fractional diffusion equation. Zaky et al. [26] introduced a spectral Legendre collocation scheme for the initial fractional differential equation of distributed order. Fei and Huang [27] derived a Galerkin Legendre spectral technique for approximating the solution of time-fractional fourth-order distributed-order PDEs in two dimensions. The composite Simpson approach was used with the distributed-order integral term, whereas the $L2 - 1$ method was adopted to approximate the multi-term order Caputo FDs. Zhang et al. analyzed a numerical approach for solving the two-dimensional distributed-order Riesz-space ADE using the Galerkin–Legendre spectral with the Crank–Nicolson alternating direction implicit method.

In this paper, the following time-fractional ADE with distributed-order Riesz-space FDs is considered:

$${}^C D_t^\alpha z(x, t) = \int_1^2 \kappa(p) \frac{\partial^p z(x, t)}{\partial x^p} dp + \int_1^2 \kappa(q) \frac{\partial^q z(x, t)}{\partial x^q} dq + f(x, t, z), \quad (1)$$

with initial and boundary conditions (IBCs):

$$\begin{aligned} z(0, t) &= h_1(t), \quad z(L, t) = h_2(t), \\ z(x, 0) &= f(x), \end{aligned} \quad (2)$$

in which ${}^C D_t^\alpha z(x, t)$ is the order $\alpha \in (0, 1]$ Caputo derivative, and $\kappa(p)$ and $\kappa(q)$ are continuous weight functions satisfying

$$\int_1^2 \kappa(p) dp = k_1 > 0, \quad \int_1^2 \kappa(q) dq = k_2 > 0.$$

Also, $\frac{\partial^\zeta z(x, t)}{\partial x^\zeta}$ stands for the order $\zeta \in (1, 2]$ Riesz fractional operator, and $h_1(t)$, $h_2(t)$ and $f(x)$ are assumed continuous.

To obtain an approximation to (1), we employ stable numerical techniques based on the Simpson and finite difference methods. For this purpose, we use the finite difference scheme based on the matrix transform approach [28] to discretize the space-fractional ADE and the Simpson approach to discretize the time-fractional differential equations. To the best of our knowledge, this technique has not been used so far to approximate the solution of (1). We estimate the Riesz derivatives using the matrix transform approach, which converts (1) into a system of time-fractional differential equations. Finally, the Simpson scheme is employed for time-stepping, which avoids the need for solving nonlinear systems

at each step. The theoretical convergence and the stability of the approximate solution are assessed. The advantage of using this new approach for the introduced problem is related to the fact that the numerical method includes simple calculations. Thus, the method is easy to simulate, and it is a powerful mathematical tool to compute the approximate solution of various types of models with little additional work. The proposed method can be used reliably and effectively to obtain approximate solutions for many types of models.

The organization of this paper is as mentioned in the following. In Section 2, some preliminary notions of fractional calculus are introduced. In Section 3, the numerical schemes for approximating the solution of (1) are proposed. Moreover, the error bound and stability are discussed. In Section 4, examples are given to report the accuracy and effectiveness of the new technique. Finally, some conclusions are provided in Section 5.

2. Preliminary Notions and Used Notation

We introduce definitions of fractional differentiation and integration of order $\theta \in (n-1, n]$, with $n \in \mathbb{N}$, which are important in subsequent sections.

Definition 1 ([2]). Let $\theta \in (0, 1]$. The Riemann-Liouville integral of order θ of a function $u(x, t)$ is:

$$I_t^\theta u(x, t) = \frac{1}{\Gamma(\theta)} \int_0^t (t - \sigma)^{\theta-1} u(x, \sigma) d\sigma. \quad (3)$$

Also, the Riemann-Liouville and Caputo FDs of order θ of $u(x, t)$ are:

$$\begin{aligned} D_t^\theta u(x, t) &= \frac{d^n}{dt^n} I_t^{n-\theta} u(x, t), \quad \theta \in (n-1, n], \quad n \in \mathbb{N} \cup \{0\}, \\ {}^C D_t^\theta u(x, t) &= I_t^{n-\theta} \frac{d^n}{dt^n} u(x, t). \end{aligned} \quad (4)$$

Definition 2 ([29]). The Riesz fractional differential operator of order $\theta \in (n-1, n]$ over the finite interval $[0, L]$ is:

$$\frac{\partial^\theta u(x, t)}{\partial |x|^\theta} = -K_\theta ({}_0 D_x^\theta + {}_x D_L^\theta) u(x, t), \quad (5)$$

where

$$\begin{aligned} {}_0 D_x^\theta u(x, t) &= \frac{1}{\Gamma(n-\theta)} \frac{\partial^n}{\partial x^n} \int_0^x (x - \sigma)^{n-\theta-1} u(\sigma, t) d\sigma, \\ {}_x D_L^\theta u(x, t) &= \frac{1}{\Gamma(n-\theta)} \int_x^L (\sigma - x)^{n-\theta-1} u(\sigma, t) d\sigma, \end{aligned}$$

$$\text{and } K_\theta = \frac{1}{2 \cos(\frac{\theta\pi}{2})}.$$

Definition 3 ([30]). Let $\{\psi_n\}$ be the complete set of orthonormal eigenfunctions and λ_n^2 be the corresponding eigenvalues for the Laplacian $(-\Delta)$ over a bounded domain Ω , that is,

$$\begin{aligned} (-\Delta)\psi_n &= \lambda_n^2 \psi_n, \text{ on } \Omega, \\ B(\psi) &= 0, \text{ on } \partial\Omega, \end{aligned} \quad (6)$$

where $B(\psi)$ is the given homogeneous boundary condition. Suppose that

$$F_\rho = \left\{ f = \sum_{n=1}^{\infty} f_n \psi_n, f_n = \langle f, \psi_n \rangle : \sum_{n=1}^{\infty} |f_n|^2 |\lambda_n|^\rho < \infty, \rho = \max(\alpha, 0) \right\}. \quad (7)$$

Then, for any given function $f \in F_\rho$, $(-\Delta)^{\frac{\alpha}{2}} : F_\rho \rightarrow L_2(\Omega)$ is described by:

$$(-\Delta)^{\frac{\alpha}{2}} f = \sum_{n=1}^{\infty} f_n (\lambda_n^2)^{\frac{\alpha}{2}} \psi_n. \quad (8)$$

Definition 4 ([28]). Let $\{\psi_n\}$ be the complete set of orthonormal eigenfunctions and λ_n^2 be the corresponding eigenvalues for the Laplacian $(-\Delta)$ over a bounded domain Ω subject to the homogeneous boundary conditions. Then,

$$(-\Delta)^{\frac{\alpha}{2}} f = \begin{cases} (-\Delta)^m f, & \text{if } \alpha = 2m, m = 0, 1, 2, \dots, \\ (-\Delta)^{\frac{\alpha}{2}-m} (-\Delta)^m f, & \text{if } m-1 < \frac{\alpha}{2} < m, m = 1, 2, \dots, \\ \sum_{n=1}^{\infty} \lambda_n^\alpha \langle f, \psi_n \rangle \psi_n, & \text{if } \alpha < 0. \end{cases} \quad (9)$$

Proposition 1 ([2]). For a function $f(t)$ and $\alpha \in (n-1, n]$, with $n \in \mathbb{N}$, some practical and useful properties of the Caputo FD and Riemann–Liouville fractional integral are:

1. $I_t^\alpha ({}^C D_t^\alpha f(t)) = f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(0) t^i}{i!}, t > 0,$
2. ${}^C D_t^\alpha (I_t^\alpha f(t)) = f(t),$
3. ${}^C D_t^\alpha t^l = \begin{cases} 0, & l < [\alpha], \\ \frac{\Gamma(l+1)}{\Gamma(l-\alpha+1)} t^{l-\alpha}, & l \geq [\alpha], \end{cases}$
4. $I_t^\alpha I_t^{\alpha'} f(t) = I_t^{\alpha+\alpha'} f(t),$
5. $I_t^\alpha t^l = \frac{\Gamma(l+1)}{\Gamma(l+\alpha+1)} t^{l+\alpha}.$

Lemma 1 ([31]). For the Riesz FD of order $\theta \in (n-1, n]$, with $n \in \mathbb{N}$, and a given function $u(x, t)$ over the infinite domain $-\infty \leq x \leq \infty$, the next condition holds:

$$\frac{\partial^\theta u(x, t)}{\partial |x|^\theta} = -(-\Delta)^{\frac{\theta}{2}} u(x, t) = -K_\theta (-\infty D_x^\theta + {}_x D_\infty^\theta) u(x, t). \quad (10)$$

3. Numerical Method

This section provides a numerical approach to solve (1) by applying the finite difference method via the matrix transform.

We use the midpoint approach to estimate the supplied integrals on the right-hand side of Equation (1), yielding

$$\begin{aligned} {}^C D_t^\alpha z(x, t) &= - \int_1^2 \kappa(p) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{p}{2}} z(x, t) dp - \int_1^2 \kappa(q) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{q}{2}} z(x, t) dq + f(x, t, z) \\ &= -\frac{1}{T} \sum_{i=1}^T \kappa(p_i) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{p_i}{2}} z(x, t) - \frac{1}{T'} \sum_{j=1}^{T'} \kappa(q_j) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{q_j}{2}} z(x, t) + f(x, t, z) \\ &\quad + \mathcal{O}((\Delta\sigma)^2) + \mathcal{O}((\Delta\sigma')^2), \end{aligned} \quad (11)$$

where

$$\begin{aligned} 1 = \sigma_0 < \sigma_1 < \dots < \sigma_T = 2, \Delta\sigma &= \frac{1}{T}, p_i = \frac{\sigma_i + \sigma_{i-1}}{2}, i = 1, 2, \dots, T, \\ 1 = \sigma'_0 < \sigma'_1 < \dots < \sigma'_{T'} = 2, \Delta\sigma' &= \frac{1}{T'}, q_j = \frac{\sigma'_j + \sigma'_{j-1}}{2}, j = 1, 2, \dots, T'. \end{aligned}$$

3.1. Matrix Transform Method for Approximating the Riesz-Space Fractional Derivative

We use the matrix transform scheme introduced by Ilić [28] for approximating the Riesz-space FD, in order to spatially discretize the problem (1). First, we address the time-fractional ADE under the IBCs given in Equation (2):

$${}^C D_t^\alpha z(x, t) = -\kappa(p) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{p}{2}} z(x, t) - \kappa(q) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{q}{2}} z(x, t) + f(x, t, z). \quad (12)$$

Let $x_l = l\Delta x$, $l = 0, 1, \dots, n$, $\Delta x = \frac{L}{n}$ and $z(x_l, t) = z_l(t)$. Then, we apply the central difference method of second-order with respect to the space variable on the right side of Equation (12), to obtain

$${}^C D_t^\alpha z_l(t) = - \left[\kappa(p) \left(-\frac{\delta_x^2}{(\Delta x)^2 (1 + \frac{\delta_x^2}{12})} \right) + \kappa(q) \left(-\frac{\delta_x^2}{(\Delta x)^2 (1 + \frac{\delta_x^2}{12})} \right) \right] z_l(t) + f_l(t, z). \quad (13)$$

Thus, the matrix form of relation (13) can be obtained as:

$$\begin{aligned} {}^C D_t^\alpha \mathbb{Z}(t) &= - \left[\kappa(p) + \kappa(q) \right] \frac{\mathbb{A}^{-1} \mathbb{B}}{(\Delta x)^2} \mathbb{Z}(t) + \mathbb{F}(t, \mathbb{Z}(t)), \\ \mathbb{Z}(0) &= \mathbb{Z}_0, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbb{Z}(t) &= [z_1(t), z_2(t), \dots, z_{n-1}(t)]^T, \\ \mathbb{Z}(0) &= [z_1(0), z_2(0), \dots, z_{n-1}(0)]^T, \end{aligned}$$

and $\mathbb{A}$ and $\mathbb{B}$ are diagonally dominant matrices of size $(n-1) \times (n-1)$ given by

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} \frac{5}{6} & \frac{1}{12} & 0 & \dots & 0 & 0 \\ \frac{1}{12} & \frac{5}{6} & \frac{1}{12} & 0 & \dots & 0 \\ 0 & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & 0 & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} & 0 \\ 0 & \dots & 0 & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} \\ 0 & 0 & \dots & 0 & \frac{1}{12} & \frac{5}{6} \end{bmatrix}_{(n-1) \times (n-1)}, \\ \mathbb{B} &= \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}_{(n-1) \times (n-1)}. \end{aligned}$$

As $\mathbb{A}$ and $\mathbb{B}$ are tridiagonal Toeplitz, then by using the definition of general tridiagonal Toeplitz matrix in [32], we have

$$\begin{aligned} \mathbb{A} &= \mathbf{Q} \Theta \mathbf{Q}^{-1}, \\ \mathbb{B} &= \tilde{\mathbf{Q}} \tilde{\Theta} \tilde{\mathbf{Q}}^{-1}, \end{aligned} \quad (15)$$

and

$$\begin{aligned} \Theta &= \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}), \quad \mathbf{Q} = (\varsigma_1, \varsigma_2, \dots, \varsigma_{n-1}), \\ \tilde{\Theta} &= \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{n-1}), \quad \tilde{\mathbf{Q}} = (\tilde{\varsigma}_1, \tilde{\varsigma}_2, \dots, \tilde{\varsigma}_{n-1}), \end{aligned} \quad (16)$$

in which $\lambda_v, \tilde{\lambda}_v$ and $\zeta_v, \tilde{\zeta}_v$ are the eigenvalues and eigenvectors of the matrices $\mathbb{A}$ and $\mathbb{B}$ given by

$$\begin{aligned}\lambda_v &= 1 - \frac{1}{3} \sin^2 \frac{v\pi}{2n}, \quad \tilde{\lambda}_v = 4 \sin^2 \frac{v\pi}{2n}, \\ \zeta_v &= \tilde{\zeta}_v = \left(\sin \frac{\pi}{2n}, \sin \frac{2\pi}{2n}, \dots, \sin \frac{(n-1)\pi}{2n} \right)^T, \quad v = 1, 2, \dots, n-1.\end{aligned}\quad (17)$$

Therefore,

$$\mathbb{A}^{-1}\mathbb{B} = (\mathbf{Q}\Theta\mathbf{Q}^{-1})^{-1}(\tilde{\mathbf{Q}}\tilde{\Theta}\tilde{\mathbf{Q}}^{-1}) = \mathbf{Q}\Theta^{-1}\tilde{\Theta}\mathbf{Q}^{-1} = \mathbf{Q} \operatorname{diag} \left(\frac{\tilde{\lambda}_1}{\lambda_1}, \frac{\tilde{\lambda}_2}{\lambda_2}, \dots, \frac{\tilde{\lambda}_{n-1}}{\lambda_{n-1}} \right) \mathbf{Q}^{-1}. \quad (18)$$

Secondly, by means of Equation (9), we can write Equation (11) as

$$\begin{aligned}{}^C D_t^\alpha z(x, t) &= -\frac{1}{T} \sum_{i=1}^T \kappa(p_i) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{p_i}{2}-1} \left(-\frac{\partial^2}{\partial x^2} \right) z(x, t) - \\ &\quad \frac{1}{T'} \sum_{j=1}^{T'} \kappa(q_j) \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{q_j}{2}-1} \left(-\frac{\partial^2}{\partial x^2} \right) z(x, t) + f(x, t, z).\end{aligned}\quad (19)$$

The matrix form of the Laplacian operator under the given boundary conditions in Equation (2) is $m \left\{ -\frac{\partial^2}{\partial x^2} z(x, t) \right\} = \frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2}$. For any $p_i, i = 1, 2, \dots, T$, and $q_j, j = 1, 2, \dots, T'$, we assume that the fractional Laplacian with homogeneous boundary conditions and initial value satisfies

$$\begin{aligned}m \left\{ \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{p_i}{2}-1} \right\} &= \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right)^{\frac{p_i}{2}-1}, \\ m \left\{ \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{q_j}{2}-1} \right\} &= \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right)^{\frac{q_j}{2}-1}.\end{aligned}\quad (20)$$

Thus, the matrix form of the relation (19) with the initial condition can be obtained by

$$\begin{aligned}{}^C D_t^\alpha \mathbb{Z}(t) &= -\frac{1}{T} \sum_{i=1}^T \kappa(p_i) \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right)^{\frac{p_i}{2}-1} \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right) \mathbb{Z}(t) - \\ &\quad \frac{1}{T'} \sum_{j=1}^{T'} \kappa(q_j) \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right)^{\frac{q_j}{2}-1} \left(\frac{\mathbb{A}^{-1}\mathbb{B}}{(\Delta x)^2} \right) \mathbb{Z}(t) + \mathbb{F}(t, \mathbb{Z}(t)) \\ &= -\left[\frac{1}{T} \sum_{i=1}^T \frac{\kappa(p_i)}{(\Delta x)^{p_i}} \left(\mathbf{Q} \operatorname{diag} \left(\frac{\tilde{\lambda}_1}{\lambda_1}, \frac{\tilde{\lambda}_2}{\lambda_2}, \dots, \frac{\tilde{\lambda}_{n-1}}{\lambda_{n-1}} \right) \mathbf{Q}^{-1} \right)^{\frac{p_i}{2}} + \right. \\ &\quad \left. \frac{1}{T'} \sum_{j=1}^{T'} \frac{\kappa(q_j)}{(\Delta x)^{q_j}} \left(\mathbf{Q} \operatorname{diag} \left(\frac{\tilde{\lambda}_1}{\lambda_1}, \frac{\tilde{\lambda}_2}{\lambda_2}, \dots, \frac{\tilde{\lambda}_{n-1}}{\lambda_{n-1}} \right) \mathbf{Q}^{-1} \right)^{\frac{q_j}{2}} \right] \mathbb{Z}(t) + \mathbb{F}(t, \mathbb{Z}(t)), \\ \mathbb{Z}(0) &= \mathbb{Z}_0.\end{aligned}\quad (21)$$

Considering

$$\begin{aligned}\Psi_p &= \frac{1}{T} \sum_{i=1}^T \frac{\kappa(p_i)}{(\Delta x)^{p_i}} \left(\mathbf{Q} \operatorname{diag} \left(\frac{\tilde{\lambda}_1}{\lambda_1}, \frac{\tilde{\lambda}_2}{\lambda_2}, \dots, \frac{\tilde{\lambda}_{n-1}}{\lambda_{n-1}} \right) \mathbf{Q}^{-1} \right)^{\frac{p_i}{2}}, \\ \Phi_q &= \frac{1}{T'} \sum_{j=1}^{T'} \frac{\kappa(q_j)}{(\Delta x)^{q_j}} \left(\mathbf{Q} \operatorname{diag} \left(\frac{\tilde{\lambda}_1}{\lambda_1}, \frac{\tilde{\lambda}_2}{\lambda_2}, \dots, \frac{\tilde{\lambda}_{n-1}}{\lambda_{n-1}} \right) \mathbf{Q}^{-1} \right)^{\frac{q_j}{2}},\end{aligned}$$

we can rewrite the system (21) as:

$$\begin{aligned} {}^C D_t^\alpha \mathbb{Z}(t) &= -(\Psi_p + \Phi_q) \mathbb{Z}(t) + \mathbb{F}(t, \mathbb{Z}(t)), \\ \mathbb{Z}(0) &= \mathbb{Z}_0. \end{aligned} \quad (22)$$

3.2. Numerical Scheme Based on the Simpson Formula for Equation (22)

This subsection studies a scheme for obtaining the approximate solution of the following system:

$${}^C D_t^\alpha w(t) = f(t, w(t)), \quad w^{(n)}(0) = w_0^{(n)}, \quad n = 0, \dots, [\alpha]. \quad (23)$$

To solve (23), we apply I_t^α to both sides of (23), to obtain

$$w(t) = \sum_{r=0}^{[\alpha]} w_0^{(r)} \frac{t^r}{r!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, w(\tau)) d\tau. \quad (24)$$

To calculate the given integral on the right-hand side of Equation (24), we utilize the estimation

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, w(\tau)) d\tau = \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} \tilde{f}_{n+1}(\tau, w(\tau)) d\tau, \quad (25)$$

in which $\tilde{f}_{n+1}$ is the piecewise quadratic interpolation of f at $t_j, t_{j+\frac{1}{2}}, j = 0, 1, 2, \dots, n+1$. Applying the standard methods of Simpson theory, we obtain the integral term on the right-hand side of Equation (25) as:

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} \tilde{f}_{n+1}(\tau, w(\tau)) d\tau = \sum_{j=0}^{n+1} \vartheta_{j,n+1} f(t_j, w(t_j)) + \sum_{j=0}^n \vartheta'_{j,n+1} f(t_{j+\frac{1}{2}}, w(t_{j+\frac{1}{2}})), \quad (26)$$

in which

$$\vartheta_{j,n+1} = \begin{cases} \frac{4(\Delta t)^\alpha \left((n+2)^{\alpha+2} - (n+1)^{\alpha+2} \right)}{\alpha(\alpha+1)(\alpha+2)} - \frac{(\Delta t)^\alpha \left((n+1)^{\alpha+1} - 3(n+2)^{\alpha+1} \right)}{\alpha(\alpha+1)} + \frac{(\Delta t)^\alpha (n+2)^\alpha}{\alpha}, & j = 0, \\ \frac{4(\Delta t)^\alpha \left((n+2-j)^{\alpha+2} - (n-j)^{\alpha+2} \right)}{\alpha(\alpha+1)(\alpha+2)} - \frac{(\Delta t)^\alpha \left((n+2-j)^{\alpha+1} + 6(n+1-j)^{\alpha+1} + (n-j)^{\alpha+1} \right)}{\alpha(\alpha+1)}, & 1 \leq j \leq n, \\ \frac{2(\Delta t)^\alpha}{\alpha(\alpha+1)(\alpha+2)} \left(\frac{2-\alpha}{2} \right), & j = n+1, \end{cases} \quad (27)$$

and

$$\begin{aligned} \vartheta'_{j,n+1} &= \frac{8(\Delta t)^\alpha \left((n+2-j)^{\alpha+2} - (n+1-j)^{\alpha+2} \right)}{\alpha(\alpha+1)(\alpha+2)} - \\ &\quad \frac{4(\Delta t)^\alpha \left((n+2-j)^{\alpha+1} + (n+1-j)^{\alpha+1} \right)}{\alpha(\alpha+1)}, \quad 0 \leq j \leq n. \end{aligned} \quad (28)$$

To make the explicit method for Equation (23) to avoid iterations, we substitute the given integral on the right-hand side of (24) by the product rectangle formula, to compute the predictor of w_{n+1} . Thus, we have

$$\begin{aligned}
w_{n+1} &= \sum_{r=0}^{[\alpha]} w_0^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^n \vartheta_{j,n+1} f(t_j, w_j) + \vartheta_{n+1,n+1} f(t_{n+1}, w_{n+1}^R) + \right. \\
&\quad \left. \sum_{j=0}^n \vartheta'_{j,n+1} f(t_{j+\frac{1}{2}}, w_{j+\frac{1}{2}}) \right), \\
w_{n+1}^R &= \sum_{r=0}^{[\alpha]} w_0^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta''_{j,n+1} f(t_j, w_j),
\end{aligned} \tag{29}$$

where the symbols $\vartheta_{j,n+1}$ and $\vartheta'_{j,n+1}$ are shown in (27) and (28), respectively, and

$$\vartheta''_{j,n+1} = \frac{(\Delta t)^\alpha \left((n+1-j)^\alpha - (n-j)^\alpha \right)}{\alpha}. \tag{30}$$

Now, to compute the values $w_{j+\frac{1}{2}}$ in Equation (29). We apply the product rectangle formula to obtain the values $w_{j+\frac{1}{2}}$. Then, we have

$$w_{n+\frac{1}{2}} = \sum_{r=0}^{[\alpha]} w_0^{(r)} \frac{t_{n+\frac{1}{2}}^r}{r!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta'''_{j,n+1} f(t_j, w_j), \tag{31}$$

in which

$$\vartheta'''_{j,n+1} = \frac{(\Delta t)^\alpha \left(\left(n + \frac{1}{2} - j\right)^\alpha - \left(n - \frac{1}{2} - j\right)^\alpha \right)}{\alpha}, \quad 0 \leq j \leq n. \tag{32}$$

Therefore, a numerical method to solve Equation (23) is shown in Equations (29) and (31), with the weights $\vartheta_{j,n+1}$, $\vartheta'_{j,n+1}$, $\vartheta''_{j,n+1}$ and $\vartheta'''_{j,n+1}$, respectively. Let $t_j = j\Delta t$, $\Delta t = \frac{T}{n}$, $j = 0, 1, \dots, n$. For obtaining the approximate solutions of (22) based on the Simpson formula, we consider

$$\begin{aligned}
{}^C D_t^\alpha \mathbb{Z}(t) &= \mathbf{F}(t, \mathbb{Z}(t)), \\
\mathbb{Z}(0) &= \mathbb{Z}_0.
\end{aligned} \tag{33}$$

in which

$$\mathbf{F}(t, \mathbb{Z}(t)) = -(\Psi_p + \Phi_q)\mathbb{Z}(t) + \mathbb{F}(t, \mathbb{Z}(t)).$$

Thus, using the Simpson formula in (33), we obtain:

$$\begin{aligned}
\mathbb{Z}_{n+1} &= \sum_{r=0}^{[\alpha]} \mathbb{Z}_0^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^n \vartheta_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j) + \vartheta_{n+1,n+1} \mathbf{F}(t_{n+1}, \mathbb{Z}_{n+1}^R) + \right. \\
&\quad \left. \sum_{j=0}^n \vartheta'_{j,n+1} \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}_{j+\frac{1}{2}}) \right), \\
\mathbb{Z}_{n+1}^R &= \sum_{r=0}^{[\alpha]} \mathbb{Z}_0^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta''_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j), \\
\mathbb{Z}_{n+\frac{1}{2}} &= \sum_{r=0}^{[\alpha]} \mathbb{Z}_0^{(r)} \frac{t_{n+\frac{1}{2}}^r}{r!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta'''_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j),
\end{aligned} \tag{34}$$

Therefore, the numerical method to solve Equation (33) is shown in Equation (34), with the weights $\vartheta_{j,n+1}$, $\vartheta'_{j,n+1}$, $\vartheta''_{j,n+1}$ and $\vartheta'''_{j,n+1}$, respectively.

Theorem 1 (Error analysis). Let $F(t, \mathbb{Z}(t))$ be Lipschitz continuous. Also, suppose that the approximate solution $\mathbb{Z}$ of (33) with the initial condition satisfies

$$\left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \sum_{j=0}^n \vartheta''_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) \right| \leq \mathbb{C} t_{n+1}^\varrho (\Delta t)^\delta, \varrho \geq 0, \delta > 0. \quad (35)$$

Then, we obtain

$$|\mathbb{Z}(t_{n+1}) - \mathbb{Z}_{n+1}^R| \leq \mathbb{C}_1 (\Delta t)^\delta + \mathbb{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|, \mathbb{C}_1 > 0, \mathbb{C}_2 > 0. \quad (36)$$

Proof. According to the function structure $\mathbb{Z}_{n+1}^R$ in Equation (34), we obtain

$$\begin{aligned} |\mathbb{Z}(t_{n+1}) - \mathbb{Z}_{n+1}^R| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} \mathbf{F}(\tau, \mathbb{Z}(\tau)) d\tau - \sum_{j=0}^n \vartheta''_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \sum_{j=0}^n \vartheta''_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta''_{j,n+1} |\mathbf{F}(t_j, \mathbb{Z}(t_j)) - \mathbf{F}(t_j, \mathbb{Z}_j)| \\ &\leq \frac{\mathbb{C} t_{n+1}^\varrho (\Delta t)^\delta}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta''_{j,n+1} |\mathbb{Z}(t_j) - \mathbb{Z}_j| \\ &\leq \mathbb{C}_1 (\Delta t)^\delta + \mathbb{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j| \sum_{j=0}^n \vartheta''_{j,n+1} \\ &\leq \mathbb{C}_1 (\Delta t)^\delta + \mathbb{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|. \end{aligned} \quad (37)$$

□

Theorem 2 (Convergence analysis). Suppose that the approximate solution $\mathbb{Z}$ of Equation (33) with the initial condition satisfies

$$\begin{aligned} \left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \sum_{j=0}^n \vartheta''_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) \right| &\leq \mathbb{C}_1 t_{n+1}^{\varrho_1} (\Delta t)^{\delta_1}, \varrho_1 \geq 0, \delta_1 > 0 \\ \left| \int_0^{t_{n+\frac{1}{2}}} (t_{n+\frac{1}{2}} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \sum_{j=0}^n \vartheta'''_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) \right| &\leq \mathbb{C}_2 t_{n+1}^{\varrho_2} (\Delta t)^{\delta_2}, \varrho_2 \geq 0, \delta_2 > 0 \\ \left| \int_0^{t_{n+\frac{1}{2}}} (t_{n+\frac{1}{2}} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \sum_{j=0}^n \vartheta_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) - \vartheta_{n+1,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) \right. \\ &\quad \left. - \sum_{j=0}^n \vartheta'_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_{j+\frac{1}{2}}) \right| \leq \mathbb{C}_3 t_{n+1}^{\varrho_3} (\Delta t)^{\delta_3}, \varrho_3 \geq 0, \delta_3 > 0. \end{aligned} \quad (38)$$

Then, we obtain

$$\max_{0 \leq j \leq M} |\mathbb{Z}(t_j) - \mathbb{Z}_j| = \mathcal{O}((\Delta t)^{\theta_1}), \quad (39)$$

in which $\theta_1 = \min\{\delta_1 + \alpha, \delta_2, \delta_3\}$ and $M = \lceil \frac{T}{\Delta t} \rceil$.

Proof. By means of Theorem 1, we have

$$|\mathbb{Z}(t_{n+1}) - \mathbb{Z}_{n+1}^R| \leq \mathbb{C}_1 (\Delta t)^\delta + \mathbb{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|. \quad (40)$$

Also, by the construction of $\mathbb{Z}_{n+\frac{1}{2}}$ in Equation (34) and Theorem 1, we obtain

$$\begin{aligned} \left| \mathbb{Z}(t_{n+\frac{1}{2}}) - \mathbb{Z}_{n+\frac{1}{2}} \right| &\leq \mathbb{C}_3(\Delta t)^{\delta_2} + \\ &\mathbb{C}_4 \sum_{j=0}^n \vartheta_{j,n+1}''' |\mathbb{Z}(t_j) - \mathbb{Z}_j| \leq \mathbb{C}_3(\Delta t)^{\delta_2} + \mathbb{C}_4 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|. \end{aligned} \quad (41)$$

Then,

$$\begin{aligned} \left| \mathbb{Z}(t_{j+1}) - \mathbb{Z}_{j+1} \right| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} \mathbf{F}(\tau, \mathbb{Z}(\tau)) d\tau - \sum_{j=0}^n \vartheta_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j) - \right. \\ &\quad \left. \vartheta_{n+1,n+1} \mathbf{F}(t_{n+1}, \mathbb{Z}_{n+1}^R) - \sum_{j=0}^n \vartheta'_{j,n+1} \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}_{j+\frac{1}{2}}) \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} {}^C D_t^\alpha \mathbb{Z}(\tau) d\tau - \right. \\ &\quad \left. \sum_{j=0}^{n+1} \vartheta_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_j) - \sum_{j=0}^{n+1} \vartheta'_{j,n+1} {}^C D_t^\alpha \mathbb{Z}(t_{j+\frac{1}{2}}) \right| + \\ &\quad \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta_{j,n+1} (\mathbf{F}(t_j, \mathbb{Z}(t_j)) - \mathbf{F}(t_j, \mathbb{Z}_j)) + \\ &\quad \frac{1}{\Gamma(\alpha)} \vartheta_{n+1,n+1} (\mathbf{F}(t_{n+1}, \mathbb{Z}(t_{n+1})) - \mathbf{F}(t_{n+1}, \mathbb{Z}_{n+1}^R)) + \\ &\quad \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \vartheta'_{j,n+1} |\mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}(t_{j+\frac{1}{2}})) - \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}_{j+\frac{1}{2}})| \\ &\leq \mathbb{C}_5(\Delta t)^{\delta_3} + \mathbb{C}_6 \sum_{j=0}^n \vartheta_{j,n+1} |\mathbb{Z}(t_j) - \mathbb{Z}_j| + \\ &\quad \mathbb{C}_7 \vartheta_{n+1,n+1} (\mathbb{C}_1(\Delta t)^\delta + \mathbb{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|) + \\ &\quad \mathbb{C}_8 \sum_{j=0}^n \vartheta'_{j,n+1} (\mathbb{C}_3(\Delta t)^{\delta_2} + \mathbb{C}_4 \max_{0 \leq j \leq n} |\mathbb{Z}(t_j) - \mathbb{Z}_j|) \\ &\leq \mathbb{C}_5(\Delta t)^{\delta_3} + \mathbb{C}_7(\Delta t)^{\delta_1+\alpha} + \mathbb{C}_8(\Delta t)^{\delta_2} + \mathbb{C}_9 \max_{0 \leq j \leq n}, \\ &\quad |\mathbb{Z}(t_j) - \mathbb{Z}_j| \leq \mathbf{C}(\Delta t)^{\delta_1+\alpha, \delta_2, \delta_3}. \end{aligned} \quad (42)$$

Thus, the theorem is proven. $\square$

Theorem 3 (Stability analysis). Suppose that $\mathbb{Z}_{n+1}$ and $\mathbb{Z}'_{n+1}$ are the approximate solutions of Equation (33) with given initial condition $\mathbb{Z}_0^{(r)}$ and $\mathbb{Z}'_0^{(r)}$, respectively. Then,

$$\left| \mathbb{Z}_{n+1} - \mathbb{Z}'_{n+1} \right| \leq \mathbb{K} \left\| \mathbb{Z}_0 - \mathbb{Z}'_0 \right\|_\infty, \quad (43)$$

for any $\mathbb{K} > 0$, and the numerical method described by Equation (34) is numerically stable.

Proof. Due to Equation (33), we obtain

$$\begin{aligned}
\left| \mathbb{Z}_{n+1} - \mathbb{Z}'_{n+1} \right| &= \left| \sum_{r=0}^{[\alpha]} \mathbb{Z}_0^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^n \vartheta_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}_j) + \vartheta_{n+1,n+1} \mathbf{F}(t_{n+1}, \mathbb{Z}_{n+1}^R) + \right. \right. \\
&\quad \left. \left. \sum_{j=0}^n \vartheta'_{j,n+1} \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}_{j+\frac{1}{2}}) \right) - \right. \\
&\quad \left. \left[\sum_{r=0}^{[\alpha]} \mathbb{Z}'_0{}^{(r)} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^n \vartheta_{j,n+1} \mathbf{F}(t_j, \mathbb{Z}'_j) + \vartheta_{n+1,n+1} \mathbf{F}(t_{n+1}, \mathbb{Z}'_{n+1}{}^R) + \right. \right. \right. \\
&\quad \left. \left. \sum_{j=0}^n \vartheta'_{j,n+1} \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}'_{j+\frac{1}{2}}) \right) \right] \right| \\
&\leq \sum_{r=0}^{[\alpha]} \left\| \mathbb{Z}_0 - \mathbb{Z}'_0 \right\|_{\infty} \frac{t_{n+1}^r}{r!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^n \vartheta_{j,n+1} (\mathbf{F}(t_j, \mathbb{Z}_j) - \mathbf{F}(t_j, \mathbb{Z}'_j)) + \right. \\
&\quad \left. \vartheta_{n+1,n+1} (\mathbf{F}(t_{n+1}, \mathbb{Z}_{n+1}^R) - \mathbf{F}(t_{n+1}, \mathbb{Z}'_{n+1}{}^R)) + \sum_{j=0}^n \vartheta'_{j,n+1} \left| \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}_{j+\frac{1}{2}}) - \right. \right. \\
&\quad \left. \left. \mathbf{F}(t_{j+\frac{1}{2}}, \mathbb{Z}'_{j+\frac{1}{2}}) \right| \right) \\
&\leq \mathbb{K} \left\| \mathbb{Z}_0 - \mathbb{Z}'_0 \right\|_{\infty} + \mathbf{C}_1 \max_{0 \leq j \leq n} |\mathbb{Z}_j - \mathbb{Z}'_j| + \mathbf{C}_2 \max_{0 \leq j \leq n} |\mathbb{Z}_{n+1}^R - \mathbb{Z}'_{n+1}{}^R| + \\
&\quad \mathbf{C}_3 \max_{0 \leq j \leq n} |\mathbb{Z}_{n+\frac{1}{2}} - \mathbb{Z}'_{n+\frac{1}{2}}| \\
&\leq \mathbb{K}_1 \left\| \mathbb{Z}_0 - \mathbb{Z}'_0 \right\|_{\infty} + \mathbb{K}_2 \max_{0 \leq j \leq n} |\mathbb{Z}_j - \mathbb{Z}'_j| \\
&\leq \mathbb{K} \left\| \mathbb{Z}_0 - \mathbb{Z}'_0 \right\|_{\infty}.
\end{aligned} \tag{44}$$

Thus, the theorem is proven. $\square$

4. Illustrative Examples

This section presents numerical results computed for some examples, by applying the proposed method for a two-dimensional time-fractional ADE with distributed-order Riesz-space FDs. All calculations were carried out with the software package MATLAB 2016b on a PC with 8 GB of RAM. We illustrate the accuracy of the new method by computing the maximum absolute error $E(x, t)$:

$$E(x, t) = \max |z(x_l, t_j) - z_n|, \tag{45}$$

where $z(x, t)$ and z_n denote the exact and approximate solutions, respectively.

Example 1. We address the time-fractional and distributed-order Riesz-space fractional ADE:

$${}^C D_t^\alpha z(x, t) = \int_1^2 \kappa(p) \frac{\partial^p z(x, t)}{\partial x^p} dp + \int_1^2 \kappa(q) \frac{\partial^q z(x, t)}{\partial x^q} dq + z(x, t) + f(x, t), \tag{46}$$

under the IBCs

$$\begin{aligned}
z(0, t) &= 0, \quad z(1, t) = t^2(1 - t)^2, \\
z(x, 0) &= 0,
\end{aligned} \tag{47}$$

in which $\kappa(p) = \frac{8\Gamma(\frac{7}{2}-p)\cos(\frac{p\pi}{2})}{15\sqrt{\pi}}$, $\kappa(q) = \frac{8\Gamma(\frac{7}{2}-q)\cos(\frac{q\pi}{2})}{15\sqrt{\pi}}$ and

$$f(x, t) = \left(\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{24}{\Gamma(5-\alpha)} t^{4-\alpha} - \frac{12}{\Gamma(4-\alpha)} t^{3-\alpha} \right) x^{\frac{5}{2}} + 2t^2(1-t)^2 \left[-\frac{\sqrt{1-x}}{\ln(1-x)} + \frac{\sqrt{xx-1}}{\ln(x)} \right] - x^{\frac{5}{2}} t^2 (1-t)^2.$$

The exact solution for this Example 1 is $z(x, t) = x^{\frac{5}{2}} t^2 (1-t)^2$. Figure 1 depicts the solution for Example 1 using the proposed method on the interval $[0, 1] \times [0, 1]$ with various choices of n for $\alpha = 0.75$ and $T = T' = 10$. Figure 2 compares the exact and numerical solutions at $t = 0.5$. Figure 3 illustrates the absolute error for different choices of n for $\alpha = 0.75$ with $T = T' = 10$. Figure 4 depicts the absolute error at $t = 0.5$. In Table 1, we compare the exact and numerical solutions at $t = 0.5$ for different values of n for $\alpha = 0.75$ with $T = T' = 10$.

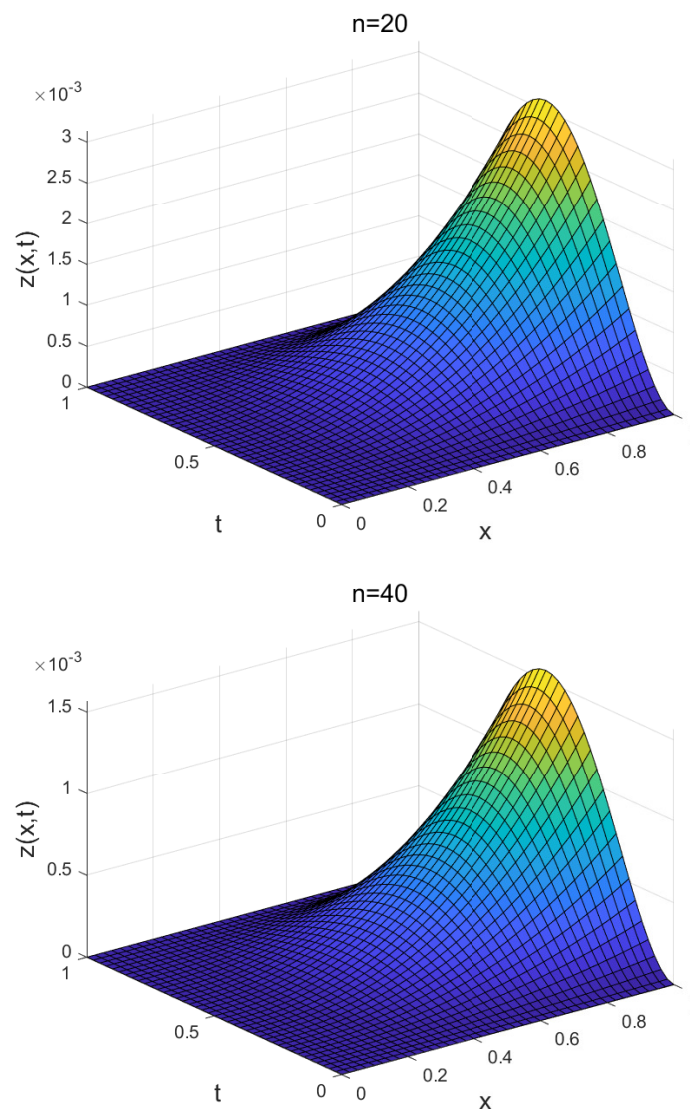


Figure 1. Cont.

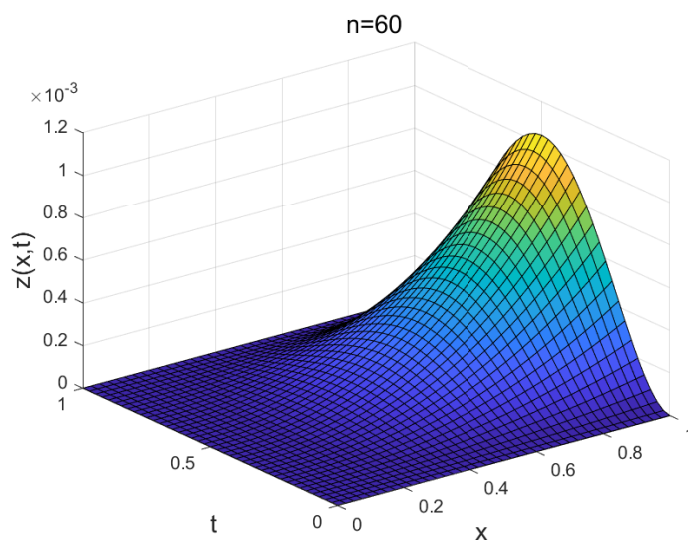


Figure 1. Solution for Example 1 with different values of n for $\alpha = 0.75$ with $T = T' = 10$.

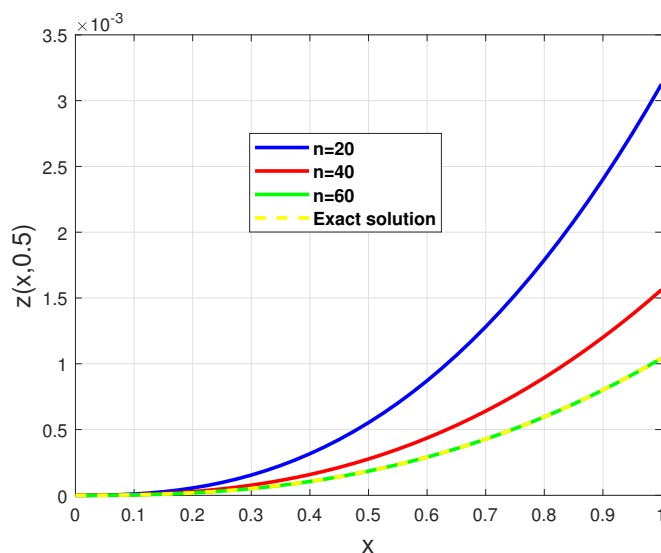


Figure 2. Numerical and exact solutions for Example 1 with different values of n for $\alpha = 0.75$ with $T = T' = 10$, at $t = 0.5$.

Table 1. The absolute errors with various values of n for $\alpha = 0.75$ with $T = T' = 10$ for Example 1.

(x, t)	$E(x, t)$		
	$n = 20$	$n = 40$	$n = 60$
$(0.1, 0.1)$	7.7495×10^{-10}	7.9413×10^{-18}	8.2919×10^{-30}
$(0.2, 0.2)$	3.3200×10^{-10}	3.3419×10^{-18}	3.3898×10^{-30}
$(0.3, 0.3)$	5.4883×10^{-10}	5.3308×10^{-18}	5.4776×10^{-30}
$(0.4, 0.4)$	8.2341×10^{-10}	8.3890×10^{-18}	8.4384×10^{-30}
$(0.5, 0.5)$	6.6285×10^{-10}	5.7829×10^{-18}	5.7829×10^{-30}
$(0.6, 0.6)$	5.2021×10^{-10}	4.9610×10^{-18}	5.1242×10^{-30}
$(0.7, 0.7)$	9.4310×10^{-10}	8.9003×10^{-18}	7.2027×10^{-30}
$(0.8, 0.8)$	4.4560×10^{-10}	2.2477×10^{-18}	6.6203×10^{-31}
$(0.9, 0.9)$	9.3799×10^{-10}	9.0139×10^{-18}	7.0545×10^{-30}
CO	1.61	1.65	1.11
CPU – time (s)	57.56	252.75	5458.33

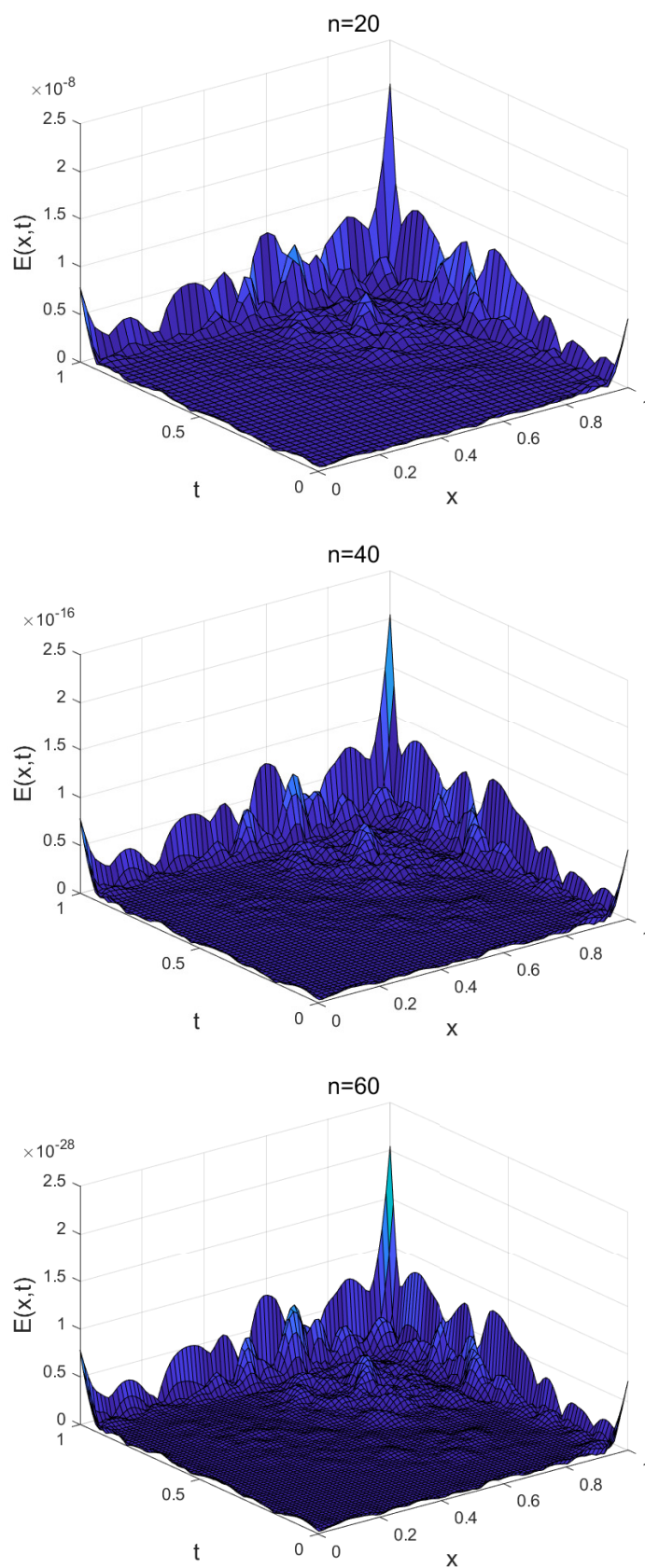


Figure 3. Absolute errors with various values of n for $\alpha = 0.75$ with $T = T' = 10$ for Example 1.

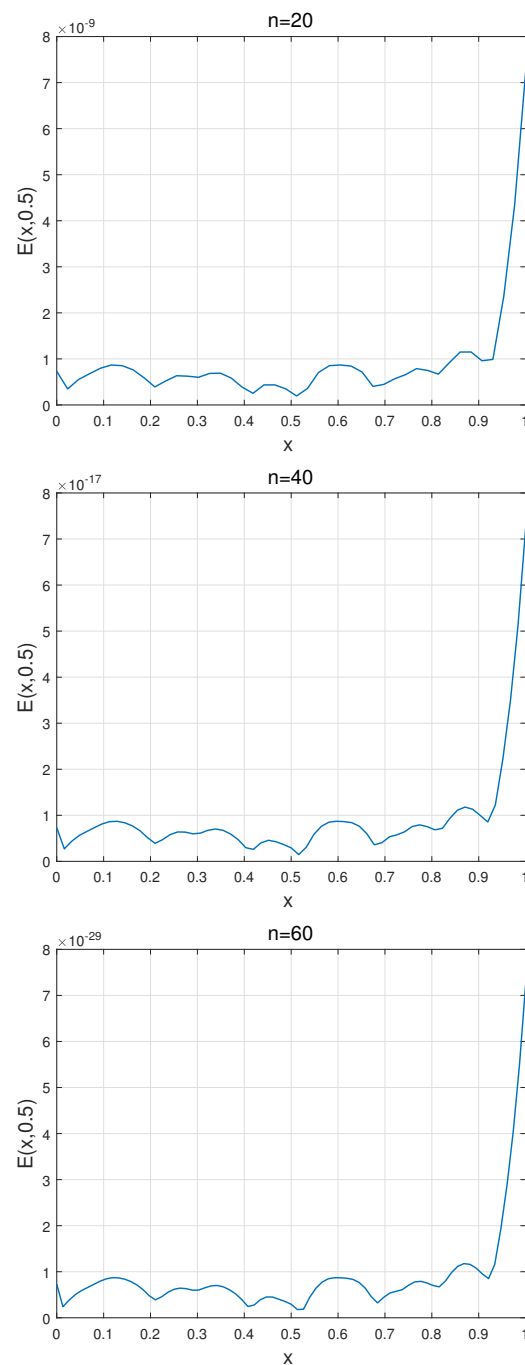


Figure 4. Absolute errors with various values of n for $\alpha = 0.75$ with $T = T' = 10$, at $t = 0.5$ for Example 1.

Example 2. We solve the time-fractional ADE with distributed-order Riesz-space FDs:

$${}^C D_t^\alpha z(x, t) = \int_1^2 \kappa(p) \frac{\partial^p z(x, t)}{\partial x^p} dp + \int_1^2 \kappa(q) \frac{\partial^q z(x, t)}{\partial x^q} dq + z^2(x, t) + f(x, t), \quad (48)$$

under the IBCs

$$\begin{aligned} z(0, t) &= 0, \quad z(2, t) = 4t(t - 2), \\ z(x, 0) &= 0, \end{aligned} \quad (49)$$

in which $\kappa(p) = \Gamma(3 - p) \cos(\frac{p\pi}{2})$, $\kappa(q) = \Gamma(3 - q) \cos(\frac{q\pi}{2})$ and

$$f(x, t) = \left(\frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} - \frac{2}{\Gamma(2 - \alpha)} t^{1-\alpha} \right) x^2 + 2t(t - 2) \left[\frac{1 - x}{\ln(2 - x)} + \frac{x - 1}{\ln(x)} \right] - x^4 t(t - 2)^2.$$

The exact solution for this Example 2 is $z(x, t) = x^2 t(t - 2)$. Figure 5 shows the approximate solution for different values of n for $\alpha = 0.75$ with $T = T' = 10$. Figure 6 compares the exact and numerical solutions at $t = 0.5$. Figure 7 illustrates the absolute error for different values of n for $\alpha = 0.75$ with $T = T' = 10$. Figure 8 depicts the absolute error at $t = 0.5$. In Table 2, we compare the exact and numerical solutions at $t = 0.5$ for different values of n for $\alpha = 0.75$ with $T = T' = 10$.

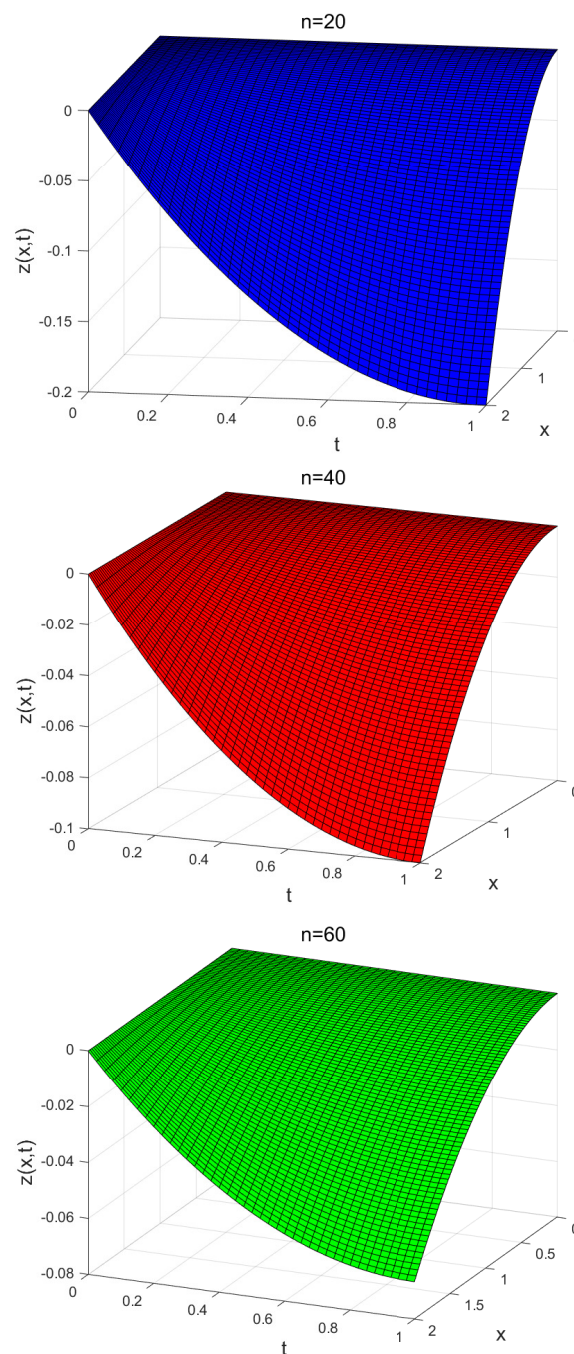


Figure 5. Approximations for Example 2 with different values of n for $\alpha = 0.75$ with $T = T' = 10$.

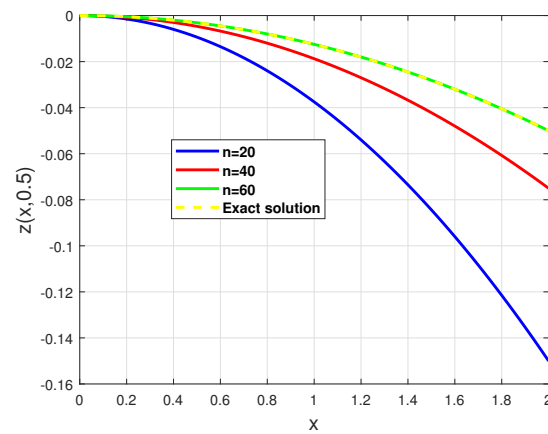


Figure 6. Approximate and exact solutions for Example 2 with different values of n for $\alpha = 0.75$ with $T = T' = 10$, at $t = 0.5$.

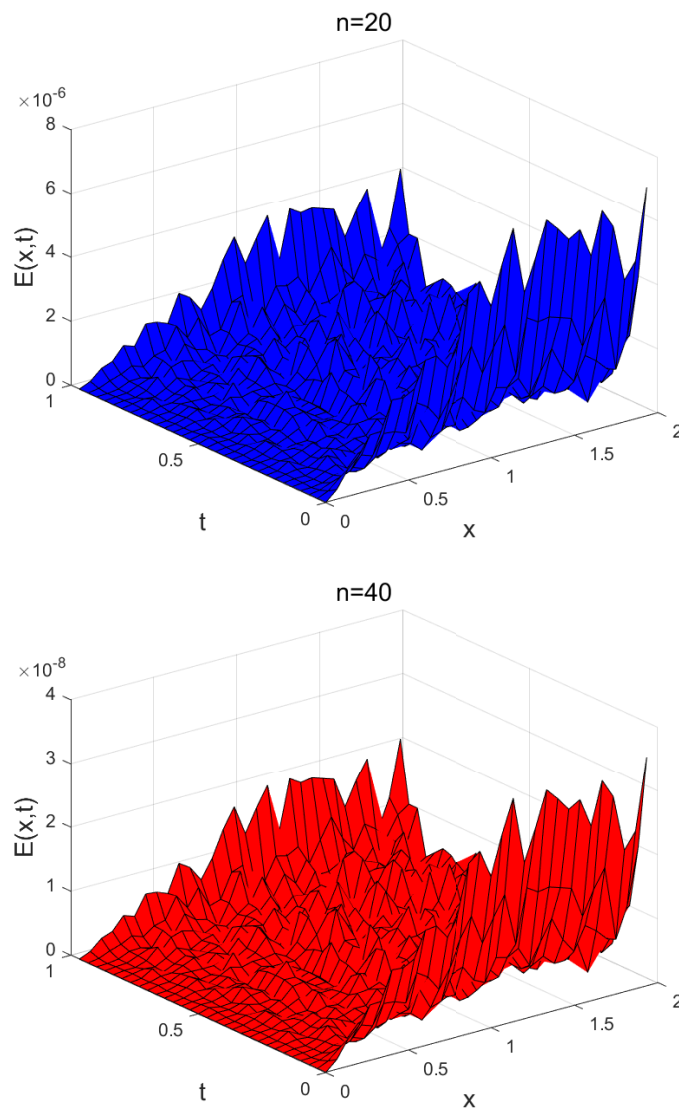


Figure 7. Cont.

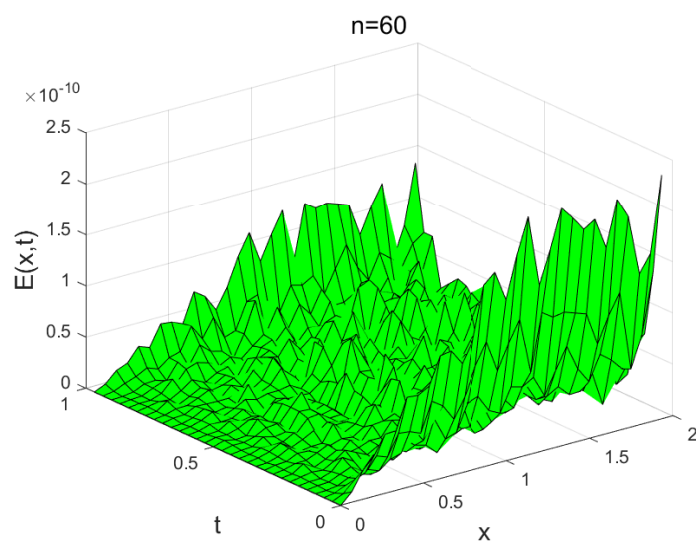


Figure 7. Absolute errors for Example 2 with different values of n for $\alpha = 0.75$ with $T = T' = 10$.

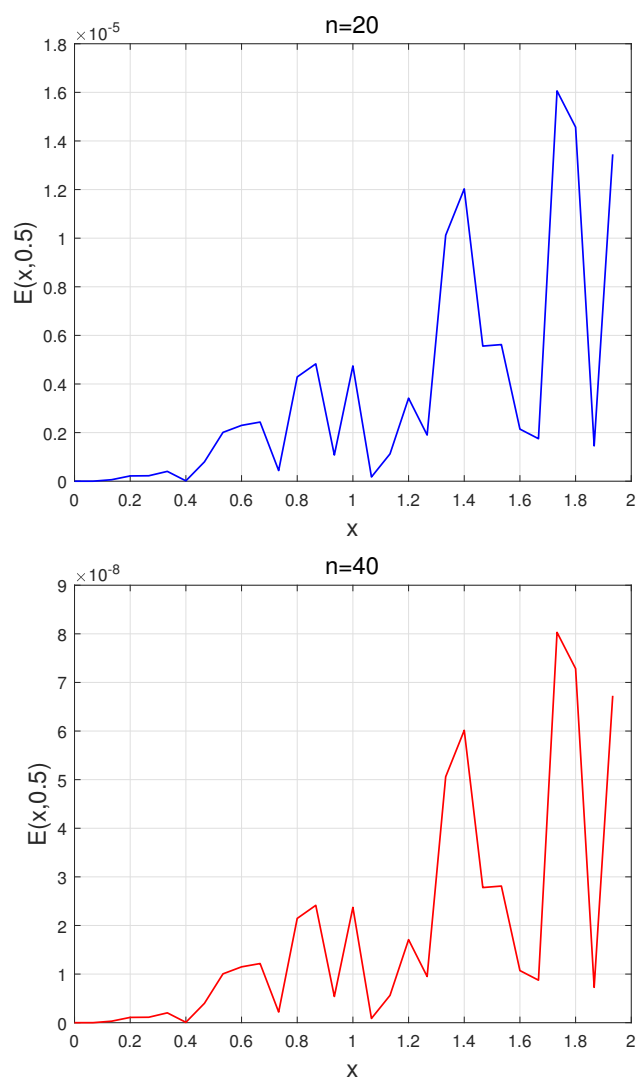


Figure 8. Cont.

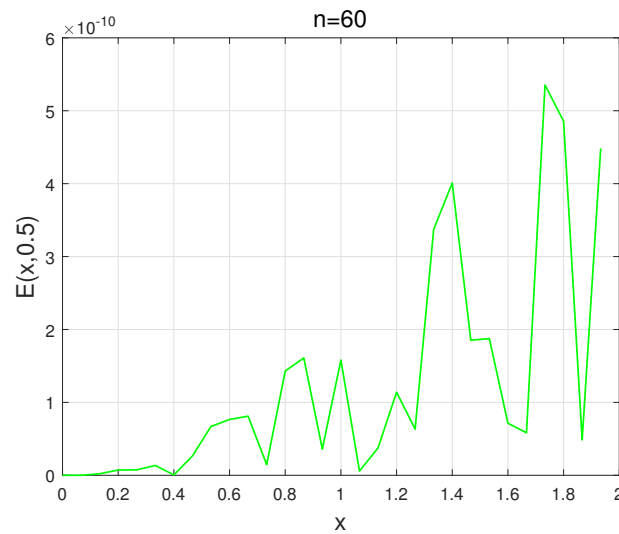


Figure 8. Absolute errors for Example 2 with different values of n for $\alpha = 0.75$ with $T = T' = 10$, at $t = 0.5$.

Table 2. The absolute errors for various values of n for $\alpha = 0.75$ with $T = T' = 10$ for Example 2.

(x, t)	$E(x, t)$		
	$n = 20$	$n = 40$	$n = 60$
$(0.1, 0.2)$	1.1668×10^{-11}	5.8340×10^{-14}	3.8893×10^{-16}
$(0.2, 0.4)$	2.0264×10^{-11}	1.0132×10^{-13}	6.7547×10^{-16}
$(0.3, 0.6)$	2.4493×10^{-10}	1.2247×10^{-12}	8.1645×10^{-15}
$(0.4, 0.8)$	8.7688×10^{-11}	4.3844×10^{-13}	2.9229×10^{-15}
$(0.5, 1)$	1.0531×10^{-09}	5.2657×10^{-12}	3.5105×10^{-14}
$(0.6, 1.2)$	9.8788×10^{-10}	4.9394×10^{-12}	3.2929×10^{-14}
$(0.7, 1.4)$	6.0759×10^{-10}	3.0380×10^{-12}	2.0253×10^{-14}
$(0.8, 1.6)$	3.9332×10^{-09}	1.9666×10^{-11}	1.3111×10^{-13}
$(0.9, 1.8)$	6.7664×10^{-10}	3.3832×10^{-12}	2.2555×10^{-14}
CO	1.044	0.997	0.989
CPU – time (s)	38	78	159

5. Conclusions

We investigated and validated the numerical solution of the nonlinear time-fractional ADE with distributed-order Riesz-space FDs under IBCs. We employed two numerical approaches to estimate the proposed equation. For approximating the Riesz FD in space, we employed the finite difference approach based on the matrix transform algorithm, and for estimating the time-fractional ADE, we used the compound Simpson method. The stability and convergence of the proposed method were theoretically proven. Finally, some numerical experiments were performed to demonstrate and validate the accuracy of the new technique. In future work, numerical schemes based on alternating direction implicit methods for the two-dimensional nonlinear time-fractional ADE with distributed-order Riesz-space FDs under the IBCs will be investigated, and a high-order method for the two-dimensional nonlinear time-fractional ADE will be presented.

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