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HYERS–ULAM–RASSIAS STABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS WITH A GENERALIZED ACTIONS ON THE RIGHT-HAND SIDE[1]

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Abstract: The paper considers the Hyers–Ulam–Rassias stability for systems of nonlinear differential equations with a generalized action on the right-hand side, for example, containing impulses — delta functions. The fact that the derivatives in the equation are considered distributions required a correction of the well-known Hyers–Ulam–Rassias definition of stability for such equations. Sufficient conditions are obtained that ensure the property under study.

Keywords: Hyers–Ulam–Rassias stability, Differential equations, Generalized actions, Discontinuous trajectories.

1. Introduction

The definition of the Hyers–Ulam stability appeared after Hyers gave a solution to the Ulam problem on conditions for the proximity of an additive mapping and an approximate additive mapping [1]. Then these results were interpreted for differential equations, which is reflected in many publications (see, for example, [4, 6] and the references therein). Further development of the Hyers–Ulam stability concept was developed in [5]. As a result, the concept of the Hyers–Ulam–Rassias stability arose.

The paper considers sufficient conditions for the Hyers–Ulam–Rassias stability of generalized solutions to nonlinear differential systems with a generalized action on the right-hand side. These issues for ordinary differential equations with absolutely continuous trajectories were considered, for example, in [6]. A distinctive feature of this work is that the right-hand side of the differential equation contains generalized actions — generalized derivatives of functions of bounded variation. Solutions are understood as pointwise limits of sequences of absolutely continuous solutions, which are obtained as a result of approximations of generalized actions on the right-hand side of the equation by summable functions [2, 8, 11]. The results obtained by the authors differ from [9, 10] in that [9, 10] use the solution formalization proposed in [7], while we use the solution formalization described in [8, 11].

For differential equations, the Hyers–Ulam–Rassias stability is defined as follows (see, for example, [6]).

Definition 1. *The equation*

$$\dot{x}(t) = f(t, x) \tag{1.1}$$

is Hyers–Ulam–Rassias stable with respect to a function φ (φ is a positive, continuous, nondecreasing function) if there exists a number $c_{f\varphi} > 0$ such that, for every ε and every solution $y \in C^1[a, b]$

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to the inequality

$$|y' - f(t,y)| \le \varepsilon\varphi(t), \quad t \in [a,b],$$

there exists a solution $x(t)$ to equation (1.1) satisfying the inequality

$$|y(t) - x(t)| \le c_{f\varphi}\varepsilon\varphi(t), \quad t \in [a,b].$$

Obviously, such a definition does not apply to equations with a generalized action because the right-hand side of the equation is unbounded. For linear differential equations of the first and second orders, the authors of [3] proposed a formalization of the Hyers–Ulam stability for a differential equation and obtained conditions for the presence of such stability for these equations.

2. Formulation of the problem

We will consider the following differential equation:

$$\dot{x} = f(t,x) + B(t,x)\dot{v}(t). \tag{2.1}$$

Here, $x(t)$ and $v(t)$ are n- and m-dimensional vector functions, respectively, $f(t,x)$ is an n-dimensional vector function, and $B(t,x)$ is an $n \times m$-matrix function. If the function $v(t)$ is absolutely continuous, then, under certain assumptions on $f(t,x)$ and $B(t,x)$, there exists a unique solution to equation (2.1) on the segment $[t_0, \vartheta]$ satisfying the initial condition $x(t_0) = x^0$.

If $v(t)$ is a function of bounded variation, then the derivative in equation (2.1) should be understood in the generalized sense [11]. As a result, an incorrect operation of multiplication of a discontinuous function by a generalized function occurs on the right-hand side of the equation. One of possible ways to solve this problem is based on the definition of the solution on the closure of the set of smooth solutions in the space of functions of bounded variation [2, 11]. Since the variation of a vector function can be defined in different ways, we note that, in this paper, the variation of an m-dimensional vector function $v(t)$ is understood as

$$\operatorname*{var}_{[t_0,t]} v(\cdot) = \sup_T \sum_{i=0}^{k-1} |v(t_{i+1}) - v(t_i)|,$$

where T is an arbitrary partition of the segment $[t_0, t]$.

According to [11], by an approximable solution of (2.1) corresponding to a function of bounded variation $v(t)$, we mean a function of bounded variation $x(t)$ which is the pointwise limit of a sequence $x_k(t)$ generated by a sequence of absolutely continuous functions $v_k(t)$ converging pointwise to $v(t)$ if $x(t)$ does not depend on the choice of the sequence $v_k(t)$.

Theorem 1 [11, p. 214]. *Assume that, in a domain $t \in [t_0, \vartheta]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $v(\cdot)$ is a function of bounded variation, and the components of the vector $f(t,x)$ and the elements of the matrix $B(t,x)$ are continuous in the set of variables, differentiable with respect to all variables x_i, $i \in \overline{1,n}$, and satisfy the inequalities*

$$||f(t,x)|| \le \kappa(1 + ||x||), \quad ||B(t,x)|| \le \kappa(1 + ||x||), \tag{2.2}$$

$$||f(t,x) - f(t,y)|| \le L_f|x - y|, \quad ||B(t,x) - B(t,y)|| \le L_B|x - y|, \tag{2.3}$$

where L_f, L_B, and κ are some positive constants. In addition, assume that the following equality (the Frobenius condition) holds for all admissible t and x:

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t,x)}{\partial x_\nu} b_{\nu l}(t,x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t,x)}{\partial x_\nu} b_{\nu j}(t,x), \quad i \in \overline{1,n}, \quad j,l \in \overline{1,m}.$$

Then, for every vector function $v(t)$ satisfying the above conditions, there exists an approximable solution $x(t)$ to the Cauchy problem (2.1) that satisfies the integral equation

$$x(t) = x^0 + \int_{t_0}^{t} f(\xi, x(\xi))\, d\xi + \int_{t_0}^{t} B(\xi, x(\xi))\, dv^c(\xi)$$
$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, x(t_i - 0), \Delta v(t_i - 0)\big) + \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, x(t_i), \Delta v(t_i + 0)\big), \tag{2.4}$$

where

$$S(t, x, \Delta v) = z(1) - x,$$
$$\dot z(\xi) = B(t, z(\xi))\Delta v(t), \quad z(0) = x, \tag{2.5}$$

and $\Omega_-(\Omega_+)$ is the set left-side discontinuity (right-side discontinuity) points of the vector function $v(t)$,

$$\Delta v(t - 0) = v(t) - v(t - 0), \quad \Delta v(t + 0) = v(t + 0) - v(t).$$

Definition 2. *We will say that a differential equation (2.1) is Hyers–Ulam–Rassias stable with respect to a function φ (φ is a positive, continuous, and nondecreasing function) on $[t_0, \vartheta]$ if, for every vector function $y \in BV[t_0, \vartheta]$ satisfying the inequality*

$$\Bigg| y(t) - x_0 - \int_{t_0}^{t} f(\xi, y(\xi))\, d\xi - \int_{t_0}^{t} B(\xi, y(\xi))\, dv^c(\xi)$$
$$- \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big)\Bigg| \le \epsilon\varphi(t), \tag{2.6}$$

for all $\epsilon > 0$, and every solution to the inequality (2.6), there exists a positive real number $c_{f,\varphi}$ and a solution to the equation (2.1) $x(t)$ satisfying the inequality

$$|y(t) - x(t)| < c_{f,\varphi}\epsilon\varphi(t)$$

for all $t \in [t_0, \vartheta]$.

3. Main result

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then the differential equation (2.1) is Hyers–Ulam–Rassiyas stable.*

P r o o f. Let $y(t) \in BV[t_0, \vartheta]$ be the solution to inequality (2.6), and let $x(t)$ be the solution to equation (2.4). According to (2.4),

$$|y(t) - x(t)| = \Bigg| y(t) - x^0 - \int_{t_0}^{t} f(\xi, x(\xi))\, d\xi - \int_{t_0}^{t} B(\xi, x(\xi))\, dv^c(\xi)$$
$$- \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, x(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, x(t_i), \Delta v(t_i + 0)\big)\Bigg|.$$

We add and subtract the following sum under the modulus on the right-hand side of this relation:

$$\int_{t_0}^{t} f(\xi, y(\xi))\, d\xi + \int_{t_0}^{t} B(\xi, y(\xi))\, dv^c(\xi) + \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) + \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big).$$

After grouping and taking into account the properties of the modulus, we obtain

$$\begin{aligned} |y(t) - x(t)| \le \Big| y(t) - x^0 - \int_{t_0}^{t} f(\xi, y(\xi))\, d\xi - \int_{t_0}^{t} B(\xi, y(\xi))\, dv^c(\xi) \\ S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big) \Big| \\ + \Big| \int_{t_0}^{t} \big(f(\xi, y(\xi)) - f(\xi, x(\xi))\big)\, d\xi \Big| + \Big| \int_{t_0}^{t} \big(B(\xi, y(\xi)) - B(\xi, x(\xi))\big)\, dv^c(\xi) \Big| \\ + \Big| \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) + \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big) \\ - \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, x(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, x(t_i), \Delta v(t_i + 0)\big) \Big|. \end{aligned} \tag{3.1}$$

Using the definition of the Stieltjes integral and assumption (2.3), it is not difficult to verify the validity of the inequality

$$\Big| \int_{t_0}^{t} \big(B(\xi, y(\xi)) - B(\xi, x(\xi))\big)\, dv^c(\xi) \Big| \le \int_{t_0}^{t} L_B |y(s) - x(s)|\, d \operatorname*{var}_{[t_0,\, s]} v^c(\cdot). \tag{3.2}$$

From inequality (3.1), given assumptions (2.3), and inequality (3.2), we get

$$\begin{aligned} |y(t) - x(t)| \le \Big| y(t) - x^0 - \int_{t_0}^{t} f(\xi, y(\xi))\, d\xi - \int_{t_0}^{t} B(\xi, y(\xi))\, dv^c(\xi) \\ - \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big) \Big| \\ + \Big| \int_{t_0}^{t} L_f |y(\xi) - x(\xi)|\, d\xi \Big| + \Big| \int_{t_0}^{t} L_B |y(\xi) - x(\xi)|\, d \operatorname*{var}_{[t_0,\, \xi]} v^c(\xi) \Big| \\ + \Big| \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, y(t_i - 0), \Delta v(t_i - 0)\big) - \sum_{t_i \le t,\, t_i \in \Omega_-} S\big(t_i, x(t_i - 0), \Delta v(t_i - 0)\big) \Big| \\ + \Big| \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, y(t_i), \Delta v(t_i + 0)\big) - \sum_{t_i < t,\, t_i \in \Omega_+} S\big(t_i, x(t_i), \Delta v(t_i + 0)\big) \Big|. \end{aligned}$$

From the above chain of inequalities, taking into account (2.6), we obtain

$$|y(t) - x(t)| \le \varepsilon \varphi(t) + \int_{t_0}^{t} L_f |y(\xi) - x(\xi)|\, d\xi + \int_{t_0}^{t} L_B |y(\xi) - x(\xi)|\, d \operatorname*{var}_{[t_0,\, \xi]} v^c(\cdot)$$

$$+\Big|\sum_{t_i\le t,\,t_i\in\Omega_-} S\big(t_i,y(t_i-0),\Delta v(t_i-0)\big)-\sum_{t_i\le t,\,t_i\in\Omega_-} S\big(t_i,x(t_i-0),\Delta v(t_i-0)\big)\Big|$$
$$+\Big|\sum_{t_i<t,\,t_i\in\Omega_+} S\big(t_i,y(t_i),\Delta v(t_i+0)\big)-\sum_{t_i<t,\,t_i\in\Omega_+} S\big(t_i,x(t_i),\Delta v(t_i+0)\big)\Big|. \tag{3.3}$$

According to definition (2.5) of the function $S(t,y,\Delta v)$, the following equality holds:

$$\big|S\big(t_i,y(t_i-0),\Delta v(t_i-0)\big)-S\big(t_i,x(t_i-0),\Delta v(t_i-0)\big)\big|$$
$$=\big|z_y(1)-y(t_i-0)-\big(z_x(1)-x(t_i-0)\big)\big|$$
$$=\Big|\int_0^1\big(B(t_i-0,z_y(s))-B(t_i-0,z_x(s))\big)\Delta v(t_i-0)\,ds\Big|.$$

Hence, using property (2.3), we obtain the inequality

$$\big|z_y(1)-y(t_i-0)-\big(z_x(1)-x(t_i-0)\big)\big|\le\int_0^1 L_B|\Delta v(t_i-0)|\big|z_y(s)-z_x(s)\big|\,ds.$$

Adding and subtracting $y(t_i-0)-x(t_i-0)$ under the modulus in the integral and then applying the triangle inequality to this modulus, we get

$$|z_y(1)-y(t_i-0)-(z_x(1)-x(t_i-0))|\le L_B|\Delta v(t_i-0)|\big|y(t_i-0)-x(t_i-0)\big|$$
$$+\int_0^1 L_B|\Delta v(t_i-0)|\big|z_y(s)-y(t_i-0)-(z_x(s)-x(t_i-0))\big|\,ds. \tag{3.4}$$

Using Gronwall's lemma in (3.4), we get

$$\big|z_y(1)-y(t_i-0)-\big(z_x(1)-x(t_i-0)\big)\big|\le L_B|\Delta v(t_i-0)|\big|y(t_i-0)-x(t_i-0)\big|e^{L_B|\Delta v(t_i-0)|}. \tag{3.5}$$

On the right-hand side of (3.5), we use the obvious inequality $ae^b\le e^{ab}-1$, $a>0$, $b\ge e$, which can be easily proved by means of the Taylor expansion of the exponent and the inequality $b^n>n$ for $b\ge e$. As a result, we obtain

$$\big|z_y(1)-y(t_i-0)-(z_x(1)-x(t_i-0))\big|\le|y(t_i-0)-x(t_i-0)|\big(e^{eL_B|\Delta v(t_i-0)|}-1\big). \tag{3.6}$$

It is clear that a similar inequality can also be obtained at the point (t_i+0).

Estimating the differences of the sums in (3.3) with the use of (3.6), we obtain the inequality

$$|y(t)-x(t)|\le\varepsilon\varphi(t)+\int_{t_0}^t L_f|y(\xi)-x(\xi)|\,d\xi+\int_{t_0}^t L_B|y(\xi)-x(\xi)|\,d\operatorname*{var}_{[t_0,\,\xi]}v^c(\xi)$$
$$+\sum_{t_i\le t,\,t_i\in\Omega_-}|y(t_i-0)-x(t_i-0)|\big(e^{eL_B|\Delta v(t_i-0)|}-1\big)$$
$$+\sum_{t_i<t,\,t_i\in\Omega_+}|y(t_i)-x(t_i)|\big(e^{eL_B|\Delta v(t_i+0)|}-1\big). \tag{3.7}$$

Inequality (3.7) obviously implies the inequality

$$|y(t)-x(t)| \le \varepsilon\varphi(t) + \int_{t_0}^{t} \max\{L_f; L_B\}|y(\xi)-x(\xi)|\, d\big(\xi + \operatorname*{var}_{[t_0,\,\xi]} v^c(\cdot)\big)$$

$$+ \sum_{t_i\le t,\, t_i\in\Omega_-} |y(t_i-0)-x(t_i-0)|\big(e^{eL_B|\Delta v(t_i-0)|}-1\big) + \sum_{t_i<t,\, t_i\in\Omega_+} |y(t_i)-x(t_i)|\big(e^{eL_B|\Delta v(t_i+0)|}-1\big).$$

Applying an estimate from [11, p. 192], we get

$$|y(t)-x(t)| \le \varepsilon\varphi(t)e^{H(t)},$$

where

$$H(t) = \max\{L_f; L_B\}\Big(t - t_0 + \operatorname*{var}_{[t_0,\,t]} v^c(\xi) + \sum_{t_i\le t,\, t_i\in\Omega_-} |\Delta v(t_i-0)| + \sum_{t_i\le t,\, t_i\in\Omega_+} |\Delta v(t_i+0)|\Big).$$

Taking into account that $H(t)$ is a monotonically increasing function, we set $c_{f\varphi} = H(\vartheta)$, which completes the proof of the theorem.

□

4. Conclusion

The paper presents a formalization of the concept of the Hyers–Ulam–Rassias stability for nonlinear systems of differential equations with a generalized action on the right-hand side. Sufficient conditions are obtained that ensure such stability.

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