A HAUSDORFF COMPACT SPACE IS METRIZABLE IF AND ONLY IF IT IS A CONTINUOUS OPEN IMAGE OF THE SORGENFREY LINE

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ABSTRACT. In this note we prove that a regular continuous open image of the Sorgenfrey line with an uncountable weight has a closed subspace that is homeomorphic to the Sorgenfrey line. As a corollary we deduce the theorem in the title.

1. Introduction

A continuous map is called open if the image of an open set under this map is open.

In [5] the author asked if there is a Hausdorff nonmetrizable compact space that is a continuous open image of the Sorgenfrey line? In this paper we give a negative answer to this question by proving the theorem in the title.

2. Notation and terminology

We use terminology from [3].

The symbol $:=$ means "equals by definition"; the symbol $:\longleftrightarrow$ is used to show that the expression on the left side is an abbreviation for the expression on the right side.

Notation 2.1. Let $\langle X, \tau\rangle$ be a topological space, $x \in X$, $B \subseteq X$, and $A \subseteq \mathbb{R}$. Then

- $\omega :=$ the set of finite ordinals $=$ the set of natural numbers;
- $f{\restriction}A :=$ the restriction of function f to A;
- $\mathbb{S} :=$ the Sorgenfrey line $:= \langle\mathbb{R}, \tau_{\mathbb{S}}\rangle$, where $\tau_{\mathbb{S}}$ is the topology generated by $\{[a, b) : a, b \in \mathbb{R}\}$;
- $A_{\mathbb{S}} :=$ the set A as a subspace of $\mathbb{S}$;
- $A_{\mathbb{R}} :=$ the set A as a subspace of $\langle\mathbb{R}, \tau_{\mathbb{R}}\rangle$, where $\tau_{\mathbb{R}}$ is the natural topology on the real line;
- if $p \in {}^{\omega}X$, then $p \xrightarrow{\langle X,\tau\rangle} x :\longleftrightarrow p$ converges to x in $\langle X, \tau\rangle$;
- $\mathsf{nbhds}(x, \tau) := \{U \in \tau : x \in U\}$;
- $\tau{\restriction}B := \{U \cap B : U \in \tau\} =$ the subspace topology of B;

- $\mathsf{Cl}_{\langle X,\tau\rangle}(B) :=$ the closure of B in $\langle X,\tau\rangle$;
- if $\langle Y,\sigma\rangle$ is a topological space, then $\langle X,\tau\rangle \cong \langle Y,\sigma\rangle :\longleftrightarrow \langle X,\tau\rangle$ is homeomorphic to $\langle Y,\sigma\rangle$;
- $\mathsf{w}(\langle X,\tau\rangle) :=$ the weight of $\langle X,\tau\rangle$.

3. Results

Lemma 3.1. *Let $\langle X,\tau\rangle$ be a T_1 topological space, $f : \mathbb{S} \to \langle X,\tau\rangle$ a continuous open surjection, and A at most countable subset of X. If for any $x \in X \setminus A$ the set $f^{-1}(x)$ contains a nontrivial convergent sequence, then $\mathsf{w}(\langle X,\tau\rangle) = \omega$.*

Proof. We prove that $\{f[(a,b)] : a < b \in \mathbb{Q}\}$ is a neighbourhood base at x for all $x \in X \setminus A$. Let $x \in X \setminus A$, $U \in \mathsf{nbhds}(x,\tau)$, $\langle y_n\rangle_{n\in\omega}$ a nontrivial convergent sequence in $\mathbb{S}$ such that $\{y_n : n \in \omega\} \subseteq f^{-1}(x)$, and y its limit point. There exists $c > y$ such that $[y,c) \subseteq f^{-1}[U]$. Since $\langle y_n\rangle_{n\in\omega} \xrightarrow{\mathbb{S}} y$, we see that there exists $n \in \omega$ such that $y_n \in (y,c)$. Take $a < b \in \mathbb{Q}$ such that $y_n \in (a,b) \subseteq (y,c)$, then $x = f(y_n) \in f[(a,b)] \subseteq U$.

Since $\langle X,\tau\rangle$ is a first-countable space, the Lemma is proved. □

Corollary 3.2. Let $\langle X,\tau\rangle$ be a T_1 topological space and $f : \mathbb{S} \to \langle X,\tau\rangle$ a continuous open surjection. If $\mathsf{w}(\langle X,\tau\rangle) > \omega$, then there exists an uncountable set $B \subseteq X$, such that $f^{-1}(x)$ is a closed discrete set for all $x \in B$.

Lemma 3.3. *Let F be an uncountable closed subset of the Sorgenfrey line. Then there exists a closed subset of F that is homeomorphic to the Sorgenfrey line.*

Proof. Since the Sorgenfrey line is hereditary Lindelöf, we see that there exists an uncountable closed set $C \subseteq F$ such that C has no isolated points. Then from [2, Theorem 4.3] it follows that $\langle C, \tau_{\mathbb{S}} \upharpoonright C\rangle \cong \mathbb{S}$. □

In the proof of the following theorem we use the ideas from the proof of [1, Theorem 3.7].

Theorem 3.4. *Let $\langle X,\tau\rangle$ be a T_1 regular topological space and $f : \mathbb{S} \to \langle X,\tau\rangle$ a continuous open surjection. If $\mathsf{w}(\langle X,\tau\rangle) > \omega$, then there exists a closed subset of $\langle X,\tau\rangle$ that is homeomorphic to the Sorgenfrey line.*

Proof. Suppose that $\mathsf{w}(\langle X,\tau\rangle) > \omega$. Denote $f \upharpoonright [a,b)$ by $f_{a,b}$ and $\{x \in [a,b) : f_{a,b}^{-1}(f_{a,b}(x)) = \{x\}\}$ by $P_{a,b}$ for all $a < b \in \mathbb{Q}$.

We prove that

$$P_{a,b} \text{ is a closed subset of } \mathbb{S} \text{ for all } a < b \in \mathbb{Q}. \tag{1}$$

Let $a, b \in \mathbb{Q}$ be such that $a < b$. Since $[a, b)$ is a closed subset of $\mathbb{S}$, we only need to prove that $P_{a,b}$ is a closed subset of $[a, b)_{\mathbb{S}}$. Let $x \in [a, b)$ be such that $x \in \mathsf{Cl}_{\mathbb{S}}(P_{a,b})$. Suppose that $x \notin P_{a,b}$; then there exists $y \in [a, b) \setminus \{x\}$ such that $f_{a,b}(x) = f_{a,b}(y)$. Fix $U_x \in \mathsf{nbhds}(x, \tau_{\mathbb{S}} \restriction [a, b))$ and $U_y \in \mathsf{nbhds}(y, \tau_{\mathbb{S}} \restriction [a, b))$ such that $f[U_x] = f[U_y]$ and $U_x \cap U_y = \emptyset$. Take any point $z \in P_{a,b} \cap U_x$. Since $z \in f[U_x] = f[U_y]$, there exists $z' \in U_y$ such that $z' \neq z$ and $f_{a,b}(z) = f_{a,b}(z')$, it contradicts the fact that $z \in P_{a,b}$.

Now we prove that

$$(2) \qquad f \restriction P_{a,b} \text{ is a homeomorphism for all } a < b \in \mathbb{Q}.$$

Let $a, b \in \mathbb{Q}$ be such that $a < b$. It is easy to see that $f_{a,b}$ is an open map. Since $f^{-1}_{a,b}[f_{a,b}[P_{a,b}]] = P_{a,b}$, we see that $f \restriction P_{a,b} = f_{a,b} \restriction P_{a,b}$ is a bijection and a restriction of the open map to the preimage, and so it is a homeomorphism.

Let us prove that

$$(3) \qquad \text{there exist } a, b \in \mathbb{Q} \text{ such that } a < b \text{ and } |P_{a,b}| > \omega.$$

Since $\mathsf{w}(\langle X, \tau \rangle) > \omega$, from Corollary 3.2 it follows that there exists an uncountable subset $B \subseteq X$ such that $f^{-1}(x)$ is a closed discrete set for all $x \in B$. It is enough to prove that for any $x \in B$ there exist $a, b \in Q$ such that $a < b$ and $f^{-1}(x) \cap P_{a,b} \neq \emptyset$. Let $x \in B$, since $f^{-1}(x)$ is a closed discrete set, we see that $<$ is a well ordering of this set. Take any isolated point $y \in f^{-1}(x)_{\mathbb{R}}$. There exist $a, b \in \mathbb{Q}$ such that $a < y < b$ and $[a, b) \cap f^{-1}(x) = \{y\}$, consequently $y \in P_{a,b}$.

Now we prove that

$$(4) \qquad f[P_{a,b}] \text{ is a closed subset of } f[[a, b)] \text{ for all } a < b \in \mathbb{Q}.$$

Let $a, b \in \mathbb{Q}$ be such that $a < b$. Let $x \in f[[a, b)]$ be such that $x \in \mathsf{Cl}_{\langle X, \tau \rangle}(f[P_{a,b}])$. Suppose that $x \notin f[P_{a,b}]$. Then there exist $y, z \in [a, b)$ such that $y \neq z$ and $f_{a,b}(y) = f_{a,b}(z) = x$. Fix $U_z \in \mathsf{nbhds}(z, \tau_{\mathbb{S}} \restriction [a, b))$ and $U_y \in \mathsf{nbhds}(y, \tau_{\mathbb{S}} \restriction [a, b))$ such that $f[U_z] = f[U_y]$ and $U_z \cap U_y = \emptyset$. Since f is an open map, it follows that $f[U_y] \cap f[P_{a,b}] \neq \emptyset$. Take any point $c \in f[U_y] \cap f[P_{a,b}]$. There exist $c_1 \in U_y$ and $c_2 \in U_z$ such that $f_{a,b}(c_1) = f_{a,b}(c_2) = c$, it contradicts the fact that $c \in f[P_{a,b}]$.

Since $\mathbb{S}$ is a hereditary Lindelöf space, it follows that $\langle X, \tau \rangle$ is a hereditary Lindelöf space. Also since $\langle X, \tau \rangle$ is a regular space, it follows that this space is perfectly normal. Fix $a, b \in \mathbb{Q}$ such that $a < b$ and $|P_{a,b}| > \omega$. Take $U := f[[a, b)]$. Since f is an open map, we see that $U \in \tau$. Since $\langle X, \tau \rangle$ is a perfectly normal space, it follows that $U = \bigcup_{i \in \omega} F_i$, where F_i is closed for all $i \in \omega$. From (2) it follows that

$f[P_{a,b}]$ is uncountable, and so there exists $n \in \omega$ such that

$$|F_n \cap f[P_{a,b}]| > \omega. \tag{5}$$

From (4) it follows that

$$F_n \cap f[P_{a,b}] \text{ is a closed subset of } \langle X, \tau \rangle. \tag{6}$$

From (1) and (2) it follows that

$$F_n \cap f[P_{a,b}] \text{ is homeomorphic to a closed subset of } \mathbb{S}. \tag{7}$$

From Lemma 3.3, (5) and (7) it follows that there exists a closed subset of $F_n \cap f[P_{a,b}]$ that is homeomorphic to the Sorgenfrey line. From (6) it follows that this set is closed in $\langle X, \tau \rangle$. □

Corollary 3.5. If a Hausdorff compact space is a continuous open image of the Sorgenrey line, then this space is metrizable.

Theorem 3.6. *A Hausdorff compact space is metrizable if and only if it is a continuous open image of the Sorgenfrey line.*

Proof. The theorem follows from the previous Corollary and [4, Corollary 3.8]. □

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References

[1] A. V. Arkhangel'skii, *Open and close-to-open mappings. Relations among spaces*, Tr. Mosk. Mat. Obs., 15, MSU, M., 1966, 181–223.

[2] D. K. Burke, J. T. Moore, *Subspaces of the Sorgenfrey line*, Topology Appl. **90** (1–3) (1998) 57–68.

[3] K. P. Hart, J. Nagata, and J. E. Vaughan, eds., *Encyclopedia of general topology*, Elsevier, Amsterdam, 2004.

[4] M. Patrakeev, *Metrizable images of the Sorgenfrey line*, Topology Proc. **45** (2015), 253–269.

[5] V. Smolin, *Open images of the Sorgenfrey line*, Topology Appl. **298** (2021), Paper No. 107706, 13 pp.

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