ON THE PRONORMALITY OF SUBGROUPS OF ODD INDEX IN SOME DIRECT PRODUCTS OF FINITE GROUPS

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Abstract. A subgroup H of a group G is said to be *pronormal* in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for each $g \in G$. Some problems in Finite Group Theory, Combinatorics, and Permutation Group Theory were solved in terms of pronormality, therefore, the question of pronormality of a given subgroup in a given group is of interest. Subgroups of odd index in finite groups satisfy a native necessary condition of pronormality. In this paper we continue investigations on pronormality of subgroups of odd index and consider the pronormality question for subgroups of odd index in some direct products of finite groups.

In particular, in this paper we prove that the subgroups of odd index are pronormal in the direct product G of finite simple symplectic groups over fields of odd characteristics if and only if the subgroups of odd index are pronormal in each direct factor of G. Moreover, deciding the pronormality of a given subgroup of odd index in the direct product of simple symplectic groups over fields of odd characteristics is reducible to deciding the pronormality of some subgroup H of odd index in a subgroup of $\prod_{i=1}^t \mathbb{Z}_3 \wr Sym_{n_i}$, where each Sym_{n_i} acts naturally on $\{1, \dots, n_i\}$, such that H projects onto $\prod_{i=1}^t Sym_{n_i}$. Thus, in this paper we obtain a criterion of pronormality of a subgroup H of odd index in a subgroup of $\prod_{i=1}^t \mathbb{Z}_{p_i} \wr Sym_{n_i}$, where each p_i is a prime and each Sym_{n_i} acts naturally on $\{1, \dots, n_i\}$, such that H projects onto $\prod_{i=1}^t Sym_{n_i}$.

Keywords: finite group, pronormal subgroup, odd index, direct product, simple symplectic group, wreath product.

1. Introduction

Throughout the paper we consider only finite groups, and henceforth the term group means finite group. Our further terminology and notation are mostly standard and can be found in [10]. However, we will denote by Sym_n and Alt_n the symmetric group and the alternating group of degree n, respectively.

A subgroup H of a group G is said to be *pronormal* in G (notation $H \, prn \, G$) if H and H^g are conjugate in $\langle H, H^g \rangle$ for each $g \in G$. Some of well-known examples of pronormal subgroups are the following: normal subgroups; maximal subgroups; Sylow subgroups; Sylow subgroups of proper normal subgroups; Hall subgroups of solvable groups; Hall subgroups of proper normal subgroups of solvable groups. The following assertion by Ph. Hall demonstrates a close connection between properties of permutation representations of finite groups and pronormality of their subgroups.

Proposition 1 ([9, Theorem 6.6])**.** *Let G be a group and $H \leq G$. Then H is pronormal in G if and only if in any transitive permutation representation of G, the subgroup $N_G(H)$ acts transitively on the set $fix(H)$ of fixed points of H.*

Some problems in Finite Group Theory as well as in Combinatorics and in Permutation Group Theory are solved in terms of pronormality. For example, consider the well-known Frattini Argument: *if G is a finite group with normal subgroup H, and if P is a Sylow subgroup of H, then $G = N_G(P)H$.* It is easy to see that the condition "P is a Sylow subgroup of H" can be replaced to the following more general condition "P is a pronormal subgroup of G", and the implication remains true. Furthermore, the concept of pronormal subgroup in some sense is universal with respect to the Frattini Argument: in the introduced notation, a subgroup P is pronormal in G if and only if P is pronormal in H and $G = N_G(P)H$ (see [7, Lemma 4]). In 1971, T. Peng [21] showed that if G is solvable, then a subgroup P is pronormal in G if and only if P possesses the *Frattini property* in G: $L = N_L(P)H$ for all subgroups $L, H \leq G$ such that $H \trianglelefteq L$ and $P \leq H$. In particular, solvable groups have Frattini factorizations with respect to Hall subgroups of their normal subgroups. Recently E. P. Vdovin and the second author [24] have showed that the existence of a π-Hall subgroup in a group G for some set π of primes is equivalent to the existence of a pronormal π-Hall subgroup in each normal subgroup of G. Thus, the E_π-groups possess corresponding Frattini factorizations.

Moreover, the concept of pronormal subgroup is closely connected to the Cayley isomorphism problem as follows. According to L. Babai [1], a finite group G is a *CI-group* (abbreviation of Cayley isomorphism property) if between any two isomorphic relational structures on the group G as underlying set which are invariant under the group

$$G_R = \{gR \mid g \in G\}$$

of right multiplications $gR : x \to xg$ (where $g, x \in G$), there exists an isomorphism which is at the same time an automorphism of G. L. Babai [1] proved that a group G is a CI-group if and only if the subgroup G_R is pronormal in $Sym(G)$, in particular, if G is a CI-group, then G is abelian; with using this Babai's result, P. Palfy [20] has obtained the complete classification of CI-groups. Moreover, in [1], L. Babai characterized combinatorial CI-objects (in particular, Cayley graphs of groups) in terms of concepts close to pronormality. These characterizations are useful tools, for example, in researches of Cayley isomorphism problem concerning undirected graphs (see, for example, [5]).

Thus, the following problem is of interest.

General Problem. Is a given subgroup H pronormal in a given group G?

Ch.E. Praeger [22] investigated pronormal subgroups of permutation groups. She has obtained the following result.

Proposition 2. *Let G be a transitive permutation group on a set Ω of n points, and let K be a non-trivial pronormal subgroup of G. Suppose that K fixes exactly f points of Ω. Then $f \leq \frac{1}{2}(n-1)$, and if $f = \frac{1}{2}(n-1)$, then K is transitive on its support in Ω, and either $G \geq Alt(n)$, or $G = GL_d(2)$ acting on the $n = 2^d - 1$ non-zero vectors, and K is the pointwise stabilizer of a hyperplane.*

Thus, if in some transitive permutation representation of G, $|fix(H)|$ is too big, then H is not pronormal in G. Therefore, first of all, it is important to consider

General Problem for a subgroup H of a group G such that H contains a subgroup S which is pronormal in G since in this case H is already satisfying a necessary condition of pronormality in G. Thus, it is interesting solve General Problem for overgroups of Sylow subgroups in finite groups, in particular, for subgroups of odd index in a finite group G which are exactly overgroups of Sylow 2-subgroups of G.

In 2012, E. P. Vdovin and the second author [23] proved that the Hall subgroups are pronormal in all simple groups and, guided by the analysis in their proof, they conjectured that any subgroup of odd index of a simple group is pronormal in this group. This conjecture was disproved in [13, 14]. The following problem naturally arose.

Problem A. Determine finite simple groups in which the subgroups of odd index are pronormal.

Problem A was investigated in [12, 13, 14, 15, 17], and in this moment, Problem A is still open only for some linear and unitary simple groups over fields of odd characteristics. More detailed surveys of investigations on pronormality of subgroups of odd index in finite (not necessary simple) groups can be found in survey papers [8, 16]. These surveys contain new results and some conjectures and open problems. In particular, in [16, Section 10] we have provided ideas how to reduce solving General Problem for a subgroup of odd index in a non-simple group to solving General Problem for some subgroup of odd index in a group of smaller order. In connection with this, the following problem arose.

Problem B. Determine direct products of finite simple groups in which the subgroups of odd index are pronormal.

A detailed motivation for Problem B was provided in [7] and in the survey paper [16]. In general, the question of pronormality of a subgroup in a direct product of finite groups is natural and was studied in some special cases. For example, B. Brewster, A. Martínez-Pastor, and M. D. Pérez-Ramos [2] have given criteria to characterize abnormal, pronormal and locally pronormal subgroups of a direct product of two finite groups $A \times B$, under hypotheses of solvability for at least one of the factors, either A or B (see for some details Lemma 9 in Section 2).

Note that the subgroups of odd index are pronormal in groups with self-normalizing Sylow 2-subgroups (see [12, Lemma 5]), and Sylow 2-subgroups are self-normalizing in the direct product of groups with self-normalizing Sylow 2-subgroups (see, for example, [14, Lemma 2]). Taking into account that the Sylow 2-subgroups are self-normalizing in many nonabelian simple groups [11], we conclude that the situation when the subgroups of odd index are pronormal in a direct product of finite simple groups occurs rather often. However, there are examples of nonabelian simple groups G (in which the Sylow 2-subgroups are not self-normalizing) such that all the subgroups of odd index are pronormal in G, but the group $G \times G$ contains a non-pronormal subgroups of odd index (see [7, Proposition 1]).

In this paper we consider direct products of finite simple symplectic groups. If $q \equiv \pm 3 \pmod 8$, then the Sylow 2-subgroups are not self-normalizing in the group $PSp_{2n}(q)$ for any n (see [11]). However, we prove the following theorem.

Theorem 1. *Let* $G = \prod_{i=1}^{t} G_i$, *where for each* $i \in \{1, ..., t\}$, $G_i \cong Sp_{n_i}(q_i)$ *for odd* q_i. *Then the following statements are equivalent*:

(1) *all the subgroups of odd index are pronormal in* G;
(2) *for each* $i \in \{1, \ldots, t\}$, *all the subgroups of odd index are pronormal in* G_i;
(3) *for each* $i \in \{1, \ldots, t\}$, *if* $q_i \equiv \pm 3 \pmod 8$, *then* n_i *is either a power of* 2 *or is a number of the form* $2^w(2^{2k}+1)$ *for non-negative integers* w *and* k.

Note that if q is odd, then $|Z(Sp_{2n}(q))| = 2$ for any n. Therefore, if H is a subgroup of odd index in the group $G = \prod\limits_{i=1}^{t} Sp_{2n_i}(q_i)$, where all q_i are odd, then $Z(G) \leq H$. Thus, applying Lemma 3 from Section 2 below, we obtain the following corollary.

Corollary 1. *Let* $G = \prod\limits_{i=1}^{t} G_i$, *where for each* $i \in \{1, ..., t\}$, $G_i \cong PSp_{n_i}(q_i)$ *for odd* q_i. *Then the following statements are equivalent*:
(1) *all the subgroups of odd index are pronormal in* G;
(2) *for each* $i \in \{1, \ldots, t\}$, *all the subgroups of odd index are pronormal in* G_i;
(3) *for each* $i \in \{1, \ldots, t\}$, *if* $q_i \equiv \pm 3 \pmod 8$, *then* n_i *is either a power of* 2 *or is a number of the form* $2^w(2^{2k}+1)$ *for non-negative integers* w *and* k.

Moreover, guided by the analysis in the proof of Theorem 1 (see Remark 1 in the end of Section 3), we conclude that solving General Problem for a given subgroup H of odd index in the direct product of symplectic groups over fields of odd characteristics is reducible to solving General Problem for some subgroup H^* (depending on H) of odd index in some group

$$K \leq \prod_{i=1}^{t} \mathbb{Z}_3 \wr Sym_{m_i}$$

such that $H \leq K$ and H projects onto each Sym_{m_i}. In this paper, we obtain a criterion of pronormality of such a subgroup H in such a group K, see Theorem 2 formulated in Section 4.

Thus, General Problem for a subgroup of odd index in a direct product of simple symplectic groups over fields of odd characteristics (in particular, in a simple symplectic group over a field of odd characteristic) can be formally solved with using inductive reasonings. As an example, the detailed solution of General Problem for an arbitrary subgroup of odd index in the group $PSp_6(3)$ is presented in Example 1, see Section 4 after Theorem 2.

The following problems naturally arise.

Problem 1. Find a criterion of pronormality of a subgroup of odd index in the direct product of simple symplectic groups over fields of odd characteristics (in particular, in a simple symplectic group over a field of odd characteristic).

Problem 2. Provide an effective algorithm which solves General Problem for a subgroup of odd index in the direct product of simple symplectic groups over fields of odd characteristics (in particular, in a simple symplectic group over a field of odd characteristic).

2. Preliminaries and auxiliary results

The largest integer power of a prime p dividing a positive integer k is called the *p-part* of k and is denoted by k_p.

Let m and n be non-negative integers with the binary expansions

$$m = \sum_{i=0}^{\infty} a_i \cdot 2^i \text{ and } n = \sum_{i=0}^{\infty} b_i \cdot 2^i,$$

where $a_i, b_i \in \{0, 1\}$ for every i. We write $m \preceq n$ if $a_i \leq b_i$ for each i and $m \prec n$ if additionally $m \neq n$. It is clear that $m \preceq n$ if and only if $n - m \preceq n$.

Let G be a group and p be a prime. We write $H \leq_p G$ if $H \leq G$ and $|G : H|_p = 1$.

Let $\mathbb{X}_p$ be the class of groups with self-normalizing Sylow p-subgroups.

Let $\mathbb{Y}_p$ be the class of groups in which the overgroups of Sylow p-subgroups are pronormal. Lemma 5 below shows that $\mathbb{X}_p \subset \mathbb{Y}_p$. Note that $\mathbb{Y}_2$ is exactly the class of all groups in which the subgroups of odd index are pronormal.

The following two lemmas deal with overgroups of Sylow subgroups in direct products of finite groups and are of independent interest.

Lemma 1. *Let p be a prime and Q be a subgroup of the group $L = L_1 \times L_2 \times \ldots \times L_n$, and let $\pi_i : L \to L_i$ be the projection for each $i \in \{1, \ldots, n\}$. Assume that $Q \leq_p L$ and for some i, $\pi_i(Q) = L_i$ and $L_i/O_p(L_i)$ is an almost simple group such that $(L_i/O_p(L_i))/Soc(L_i/O_p(L_i))$ is a p-group. Then $L_i \leq Q$.*

Proof. Note that $O_p(L_i) \leq O_p(L)$. Therefore, $O_p(L_i) \leq Q$ since $Q \leq_p L$. Thus, we can assume assume that $O_p(L_i) = 1$, and so L_i is almost simple.

Now one can prove this assertion repeating all the reasonings from the proof of [7, Lemma 9] and replacing the prime 2 into an arbitrary prime p as follows. Since $L_i \trianglelefteq L$, we have $Q \cap L_i \trianglelefteq Q$, and therefore, $\pi_i(Q \cap L_i)$ is a normal subgroup of $\pi_i(Q) = L_i$. Choose $S \in Syl_p(L)$ such that $S \leq Q$. Then $S \cap L_i \in Syl_p(L_i)$ and

$$S \cap L_i = \pi_i(S \cap L_i) \leq \pi_i(Q \cap L_i).$$

Therefore, $\pi_i(Q \cap L_i) \trianglelefteq L_i$ and $\pi_i(Q \cap L_i) \leq_p L_i$. The group L_i is almost simple and $L_i/Soc(L_i)$ is a p-group, so $\pi_i(Q \cap L_i) = L_i$, whence $L_i \leq Q$. □

Lemma 2. *Let $G = X \times Y$ for groups X and Y and let*

$$\xi : G \to X \quad \text{and} \quad \eta : G \to Y$$

be the projections. Assume that $X \in \mathbb{X}_p$ and $H \leq_p G$. Then $\xi(H) = H \cap X$ and $\eta(H) = H \cap Y$. In particular,

$$H = \langle \xi(H), \eta(H) \rangle = \xi(H) \times \eta(H).$$

Proof. Since $H \cap Y \leq \eta(H)$ and $h = \xi(h)\eta(h)$ for every $h \in H$, it is sufficient to establish that $H \cap X = \xi(H)$. Now $X \trianglelefteq G$ implies that $H \cap X \trianglelefteq H$ and

$$H \cap X = \xi(H \cap X) \trianglelefteq \xi(H).$$

On the other hand, $H \leq_p G$ means $H \cap X \leq_p X$. If $P \in Syl_p(H \cap X)$, then $P \in Syl_p(X)$ and $N_X(P) = P$ since $X \in \mathbb{X}_p$. The Frattini Argument implies that

$$\xi(H) = (H \cap X)N_{\xi(H)}(P) \leq (H \cap X)N_X(P) = (H \cap X)P = H \cap X.$$

□

In the following series of lemmas we provide some general properties of pronormal subgroups in finite groups.

Lemma 3 (See [12, Lemma 3] and [6, Ch. 1, Prop. 6.4])**.** *Let $A \trianglelefteq G$ and $H \leq G$. Then the following statements hold*:

(1) *if $H\,prn\,G$, then $HA/A\,prn\,G/A$*;

(2) *$H\,prn\,G$ if and only if $HA/A\,prn\,G/A$ and $H\,prn\,N_G(HA)$*;

(3) *if $A \leq H$, then $H\,prn\,G$ if and only if $H/A\,prn\,G/A$*;

(4) *if p is a prime and $H \leq_p G$, then $H\,prn\,G$ if and only if $H/O_p(G)\,prn\,G/O_p(G)$.*

The following assertion is a direct corollary of Proposition 1, however, the assertion was independently proved in [23, Lemma 5].

Lemma 4. *Suppose that G is a group and $H \leq G$. Assume also that H contains a Sylow subgroup S of G. Then H is pronormal in G is and only if H and H^g are conjugate in $\langle H, H^g\rangle$ for each $g \in N_G(S)$.*

Lemma 5 (See [12, Lemma 5])**.** *Suppose that H and M are subgroups of a group G and $H \leq M$.*

(1) *If $H\,prn\,G$, then $H\,prn\,M$*;

(2) *If $S \leq H$ for some Sylow subgroup S of G, $N_G(S) \leq M$, and $H\,prn\,M$, then $H\,prn\,G$.*

Lemma 6 (See [13, Theorem 1])**.** *Let H and V be subgroups of a group G such that V is an abelian normal subgroup of G and $G = HV$. Then H is pronormal in G if and only if $U = N_U(H)[H,U]$ for any H-invariant subgroup U of V.*

Lemma 7 (See [7, Lemma 11])**.** *Let $G = V \rtimes B$, where V is an abelian normal subgroup of G and $B \leq G$, and let $H \leq G$. Define $\overline{} : G \to B$ such that $\overline{g} = b$, where $g = vb$ for $v \in V$ and $b \in B$. Then $\overline{H}\,prn\,\overline{H}V$ implies $H\,prn\,HV$.*

Lemma 8 (See [7, Lemma 6])**.** *Let $N \trianglelefteq G$, $G/N \in \mathbb{X}_p$, and $H \leq_p G$. Then $H\,prn\,G$ if and only if $H\,prn\,HN$.*

Lemma 9. [2, Propositions 2.1, 4.3, 4.4 and Corollary 4.7] *Let $G = G_1 \times G_2$ and $H \leq G$. For $i \in \{1,2\}$, denote by π_i the projection $G \to G_i$ and set*

$$C_i = \{x \in G_i \mid [x, \pi_i(H)] \leq G_i \cap H\}.$$

Then the following statements hold.

(1) *$C_i = N_G(H) \cap G_i$.*

(2) *If $\pi_i(H)\,prn\,G_i$ for every $i \in \{1,2\}$ and*

$$N_G(H) = \langle N_{G_i}(\pi_i(H)) \mid i \in \{1,2\}\rangle = N_{G_1}(\pi_1(H)) \times N_{G_2}(\pi_2(H)),$$

then $H\,prn\,G$.

(3) *If one of G_i, where $i \in \{1,2\}$, is solvable, then $H\,prn\,G$ if and only if $\pi_i(H)\,prn\,G_i$ for every $i \in \{1,2\}$ and*

$$N_G(H) = \langle N_{G_i}(\pi_i(H)) \mid i \in \{1,2\}\rangle = N_{G_1}(\pi_1(H)) \times N_{G_2}(\pi_2(H)).$$

(4) *If one of G_i, where $i \in \{1,2\}$, is solvable, then $H\,prn\,G$ if and only if $\pi_i(H)\,prn\,G_i$ and $N_{G_i}(\pi_i(H)) \leq C_i$ for each $i \in \{1,2\}$.*

Two further assertions give criteria of pronormality of overgroups of Sylow subgroups in extensions of finite groups with special properties.

Lemma 10. *Assume that p is a prime and X and Y are groups such that $X \in \mathbb{X}_p$. Let $H \leq_p X \times Y$ and let*

$$\xi : G \to X \quad \text{and} \quad \eta : G \to Y$$

be the projections. Then the following statements hold:

(*i*) *H is pronormal in $X \times Y$ if and only if $\eta(H)$ is pronormal in Y.*

(*ii*) *$X \times Y \in \mathbb{Y}_p$ if and only if $Y \in \mathbb{Y}_p$.*

Proof. Prove Statement (i). By Lemma 3, if H is pronormal in $X \times Y$, then $\xi(H)$ is pronormal in $\xi(X \times Y) = X$ and $\eta(H)$ is pronormal in $\eta(X \times Y) = Y$.

Show the converse. Suppose that $\eta(H)$ is pronormal in Y and note that $\xi(H)$ is pronormal in X because $X \in \mathbb{X}_p \subset \mathbb{Y}_p$ and $\xi(H) \leq_p \xi(X \times Y) = X$. Since $H \leq_p X \times Y$ and $X \in \mathbb{X}_p$, Lemma 2 implies that $H = \xi(H) \times \eta(H)$. Therefore,

$$N_{X \times Y}(H) = N_X(\xi(H)) \times N_Y(\eta(H)).$$

Now H is pronormal in $X \times Y$ by Lemma 9 part (2).

Statement (ii) follows directly from Statement (i).

□

Proposition 3. *Let G be a group, $A \trianglelefteq G$, where $G/A \in \mathbb{X}_p$ and $A \in \mathbb{Y}_p$. Let T be a Sylow p-subgroup of A. Then*

(*i*) *if $H \geq S$ for some $S \in Syl_p(G)$, $T = A \cap S$, $Y = N_A(H \cap A)$ and $Z = N_{H \cap A}(T)$, then Z is normal in both $N_H(T)$ and $N_Y(T)$ and the following conditions are equivalent*:

(1) *H is pronormal in G;*

(2) *$N_H(T)/Z$ is pronormal in $(N_H(T)N_Y(T))/Z$;*

(3) *$N_H(T)$ is pronormal in $N_H(T)N_Y(T)$.*

(*ii*) *$G \in \mathbb{Y}_p$ if and only if $N_G(T)/T \in \mathbb{Y}_p$.*

Proof. Statement (ii) is [7, Theorem 1].

To prove Statement (i) we need to go throw the proof of [7, Lemma 15] and generalize the reasonings. Define

$$X = N_{HA}(H \cap A).$$

Note that $H \leq X$, therefore,

$$X = N_{HA}(H \cap A) = HN_A(H \cap A) = HY.$$

By the Frattini Argument,

$$H = N_H(T)(H \cap A) \text{ and } Y = N_Y(T)(H \cap A).$$

Moreover, $N_H(T)$ normalizes $N_Y(T)$ and

$$N_H(T)\cap(H\cap A) = N_Y(T)\cap(H\cap A) = N_{H\cap A}(T) = N_G(T)\cap(H\cap A) = (N_H(T)N_Y(T))\cap(H\cap A).$$

Note that by Lemma 3,

$$H = N_H(T)(H \cap A)\, prn\, X = N_{HA}(H \cap A) = (H \cap A)N_Y(T)N_H(T)$$

if and only if

$$N_H(T)(H \cap A)/(H \cap A)\, prn\, (N_Y(T)N_H(T)(H \cap A))/(H \cap A)$$

if and only if

$$N_H(T)/N_{H\cap A}(T)\, prn\, N_H(T)N_Y(T)/N_{H\cap A}(T)$$

if and only if

$$N_H(T)\,prn\,N_H(T)N_Y(T).$$

Thus, we have proved that the following conditions are equivalent:
(1) H is pronormal in $N_{HA}(H\cap A)$;
(2) $N_H(T)/Z$ is pronormal in $(N_H(T)N_Y(T))/Z$;
(3) $N_H(T)$ is pronormal in $N_H(T)N_Y(T)$.

Now the condition $A\in\mathbb{Y}_p$ and [7, Lemma 15, part (2)] imply that H is pronormal in $N_{HA}(H\cap A)$ if and only if H is pronormal in HA.

The condition $G/A\in\mathbb{X}_p$ and Lemma 8 imply that H is pronormal in G if and only if H is pronormal in HA.

□

In the further series of lemmas we provide some properties of subgroups of odd index in finite groups of special type.

Lemma 11 (See [11])**.** *Let $G = PSp_n(q)$, where q is odd, and $S\in Syl_2(G)$.*
(1) *If $q\equiv\pm1 \pmod 8$, then $N_G(S)=S$.*
(2) *If $q\equiv\pm3 \pmod 8$ and $n=2^{s_1}+\dots+2^{s_t}$ for $s_1>\dots>s_t\geq 0$, then $N_G(S)/S$ is elementary abelian of order 3^t.*

Lemma 12 (See [18] and [19])**.** *Let $G=Sp_{2n}(q)$, where $n\geq 1$ and q is odd; let V be the natural module of G. A subgroup H is a maximal subgroup of odd index in G if and only if one of the following statements holds:*
(1) *$H\cong Sp_{2n}(q_0)$, where $q=q_0^r$ and r is an odd prime, is the centralizer in G of a field automorphism of order r;*
(2) *$H\cong Sp_{2m}(q)\times Sp_{2(n-m)}(q)$ is the stabilizer of a non-degenerate subspace of dimension $2m$ of V, and $n\succ m$;*
(3) *$H\cong Sp_{2m}(q)\wr Sym_t$ is the stabilizer of an orthogonal decomposition $V=\bigoplus V_i$ into a sum of pairwise isometric non-degenerate subspaces V_i of dimension $2m$, $m=2^w$, w is a non-negative integer, and $n=mt$;*
(4) *$n=1$ and $H\cong SL_2(q_0).2$ is the centralizer in G of a field automorphism of order 2;*
(5) *$n=1$, $H/Z(G)\cong Alt_4$, q is prime, and $q=5$ or $q\equiv\pm3,\pm13 \pmod{40}$;*
(6) *$n=1$, $H/Z(G)\cong Sym_4$, q is prime, and $q\equiv\pm7 \pmod{16}$;*
(7) *$n=1$, $H/Z(G)\cong Alt_5$, q is prime, and $q\equiv 11,19,21,29 \pmod{40}$;*
(8) *$n=1$, $H/Z(G)\cong D_{q+1}$, and $7<q\equiv 3 \pmod 4$;*
(9) *$n=2$, $H/Z(G)\cong 2^4.Alt_5$, q is prime, and $q\equiv\pm3 \pmod 8$.*

Lemma 13. *Let $L=Sp_{2^w}(q)$ for odd q and $w\geq 1$, $P\in Syl_2(L)$, and $G=L\wr Sym_n$, where Sym_n acts naturally on $\{1,\dots,n\}$. Let $K=\prod_{i=1}^{n}K_i$, where each K_i is isomorphic to L, be a normal subgroup of G, which coincides with the base of this wreath product. Let $S\in Syl_2(G)$, $T=K\cap S$, and $T_i=K_i\cap T$. Then the following statements hold:*
(1) *$T_i\cong P$ for each i;*
(2) *$T=\prod_{i=1}^{n}T_i\cong\underbrace{P\times\dots\times P}_{n\text{ times}}$;*

(3) $N_K(T) = \prod_{i=1}^{n} N_{K_i}(T_i) \cong \underbrace{N_L(P) \times \cdots \times N_L(P)}_{n \text{ times}}$;

(4) $N_G(T) \cong N_L(P) \wr Sym_n$, *where* Sym_n *acts naturally on* $\{1, \ldots, n\}$;

(5) $N_G(T)/T \cong (N_L(P)/P) \wr Sym_n \cong Z \wr Sym_n$, *where* Sym_n *acts naturally on* $\{1, \ldots, n\}$, $Z \cong \mathbb{Z}_3$ *if* $q \equiv \pm 3 \pmod 8$, *and* Z *is trivial if* $q \equiv \pm 1 \pmod 8$.

Proof. Assertions (1)–(3) are obvious. Assertion (4) easily follows from (3) and the Frattini Argument. Assertion (5) follows from (4) and Lemma 11. □

In the following series of lemmas we provide some results on pronormality of subgroups of odd index in finite groups.

Lemma 14 (See [13, Corollary])**.** *Let* $G = A \wr Sym_n = HV$ *be the wreath product of an abelian group* A *by the symmetric group* $H = Sym_n$ *acting naturally on the set* $\{1, \ldots, n\}$, *where* V *denotes the base of the wreath product. Then the subgroup* H *is pronormal in* G *if and only if* $(|A|, n) = 1$.

Lemma 15 (See [7, Theorem 2])**.** *Let* A *be an abelian group and*

$$G = \prod_{i=1}^{t} (A \wr Sym_{n_i}),$$

where each Sym_{n_i} *acts naturally on* $\{1, \ldots, n_i\}$. *Then the subgroups of odd index are pronormal in* G *if and only if for any positive integer* m, *the inequality* $m \preceq n_i$ *for some* i *implies that* $(|A|, m)$ *is a power of* 2.

Lemma 16 (See [15])**.** *Let* $G = PSp_{2n}(q)$. *Then* $G \in \mathbb{Y}_2$ *if and only if one of the following statements holds:*

(1) $q \not\equiv \pm 3 \pmod 8$;

(2) n *is of the form* 2^w *or* $2^w(2^{2k}+1)$, *where* k *and* w *are non-negative integers.*

Lemma 17 (See [7, Theorem 3])**.** *Let* $G = \prod_{i=1}^{t} G_i$, *where for each* $i \in \{1, \ldots, t\}$, $G_i \cong Sp_{2n_i}(q_i)$, *each* q_i *is odd, and each* n_i *is a power of* 2. *Then all the subgroups of odd index are pronormal in* G.

3. Proof of Theorem 1

(3) $\Rightarrow$ (1) Let us tell that a group G satisfy *condition* $(*)$ if the following statements hold:

— $G = \prod_{i=1}^{t} G_i$, where $G_i \cong Sp_{2n_i}(q_i)$ for each $i \in \{1, ..., t\}$;

— each q_i is odd;

— if $q_i \equiv \pm 3 \pmod 8$ for some i, then n_i is either a power of 2 or is a number of the form $2^{w_i}(2^{2k_i}+1)$, where w_i and k_i are non-negative integers.

Assume that G is a group of the smallest order satisfying $(*)$, such that G contains a non-pronormal subgroup $H \leq_2 G$, and take some $S \in Syl_2(G)$ with $S \leq H$. By Lemma 11, $Sp_{2n_i}(q) \in \mathbb{X}_2$ if $q_i \equiv \pm 1 \pmod 8$, therefore, Lemma 10 and the minimality of G implies that $q_i \equiv \pm 3 \pmod 8$ for each $i \in \{1, \ldots t\}$.

Let $\pi_i : G \to G_i$ be the projection for each $i \in \{1, \ldots t\}$. If $\pi_i(H) = G_i$ for some i, then $G_i \leq H$ by Lemma 1. Thus, H/G_i is a non-pronormal subgroup of odd index in $G/G_i \cong \prod_{j\neq i} G_i$ by Lemma 3. But the group G/G_i satisfy condition $(*)$, a contradiction to the minimality of G.

So, for each i, there exists a maximal subgroup $M_i < G_i$ such that $\pi_i(H) \leq M_i$. Thus, for any i,

$$H \leq M(i) = \prod_{j\neq i} G_j \times M_i.$$

Possibilities for M_i are listed in Lemma 12.

Assume that for some i, $M_i \cong Sp_{2n_i}(\tilde{q}_i)$, where $q_i = \tilde{q}_i^{r_i}$ and r_i is an odd prime. Note that $q_i \equiv \pm 3 \pmod 8$ implies $\tilde{q}_i \equiv \pm 3 \pmod 8$. It is easy to see that in this case $M(i)$ satisfies condition $(*)$, and H is pronormal in $M(i)$ by the minimality of G. Moreover, $N_G(S) = N_{M(i)}(S)$ by Lemma 11. Thus, $H \, prn \, G$ by Lemma 5, a contradiction.

Assume that for some i, M_i is the stabilizer of a non-degenerate subspace of dimension $2m_i$ of the natural module of G_i, and $n_i \succ m_i$. Note that

$$M_i \cong Sp_{2m_i}(q_i) \times Sp_{2(n_i-m_i)}(q_i).$$

In this case $M(i)$ satisfies condition $(*)$ since $n_i \succ m_i$. By the minimality of G, $H \, prn \, M$. Moreover, since $n_i \succeq m_i$, we have $N_G(S) = N_{M(i)}(S)$ by Lemma 11. Thus, $H \, prn \, G$ by Lemma 5, a contradiction.

So, by Lemma 12, for each i such that n_i is not a power of 2, M_i is the stabilizer of an orthogonal decomposition of the corresponding natural module V_i of G_i into a sum of pairwise isometric non-degenerate subspaces of dimension $2s_i$, and $s_i = 2^{w_i}$ for a non-negative integer w_i. Note that by [10, Proposition 4.9.2], in this case we have

$$M_i \cong Sp_{2s_i}(q_i) \wr Sym_{m_i},$$

where $n_i = s_i m_i$ and Sym_{m_i} acts naturally on $\{1, \dots, m_i\}$. Put

$$R_i = \begin{cases} M_i, & \text{if } n_i \text{ is not a power of 2,} \\ G_i, & \text{otherwise.} \end{cases}$$

If n_i is a power of 2, then put $s_i = n_i$ and $m_i = 1$. Thus,

$$H \leq_2 R = \prod_{i=1}^{t} R_i \cong \prod_{i=1}^{t} Sp_{2s_i}(q_i) \wr Sym_{m_i} = N \rtimes C,$$

where $N = \prod_{i=1}^{t} (Sp_{2s_i}(q_i))^{m_i}$ and $C = \prod_{i=1}^{t} Sym_{m_i}$.

Note that $N \in \mathbb{Y}_2$ by Lemma 17. Moreover, $R/N \cong C \in \mathbb{X}_2$ (see, for example, [3, Lemma 4]).

Let $T \in Syl_2(N)$. Then by Lemmas 11 and 13,

$$N_R(T)/T \cong \prod_{i=1}^{t} \mathbb{Z}_3 \wr Sym_{m_i}.$$

Note that for each $i \in \{1, \dots, t\}$, 3 does not divide m if $m \preceq m_i$. Indeed, since each $q_i \equiv \pm 3 \pmod 8$, $(*)$ implies that $m_i = 2^{w_i}$ or $m_i = 2^{w_i}(2^{2k_i} + 1)$ for some non-negative integers w_i and k_i. Therefore, if $m \preceq m_i$, then m is ether a power of 2 itself or

$$m = 2^{w_i}(2^{2k_i} + 1) \equiv (-1)^{w_i+1} \pmod 3.$$

Thus, by Lemma 15, $N_R(T)/T \in \mathbb{Y}_2$. Hence, $R \in \mathbb{Y}_2$ by Proposition 3 and H is pronormal in R. Moreover, $N_G(S) = N_R(S)$ by Lemma 11. Thus, H is pronormal in G by Lemma 5, a contradiction.

(1) $\Rightarrow$ (2) It is easy to see that if for some i, G_i contains a non-pronormal subgroups H_i of odd index, then G contains a non-pronormal subgroup $H_i \times \prod_{j\neq i} G_j$ of odd index. Thus, $G_i \in \mathbb{Y}_2$ for each $i \in \{1,\ldots,t\}$.

(2) $\Rightarrow$ (3) follows from Lemma 16. □

Remark 1. Theorem 1 provides a criterion when all the subgroups of odd index are pronormal in the direct product of symplectic groups over fields of odd characteristics. However, it easy follows form the proof of Theorem 1 that solving General Problem for a subgroup H of odd index in the direct product G of symplectic groups over fields of odd characteristics is reducible to solving General Problem for the subgroup H in some subgroup

$$R = \prod_{i=1}^{t} Sp_{2s_i}(q_i) \wr Sym_{m_i},$$

where each s_i is a power of 2, each $q_i \equiv \pm 3 \pmod 8$, and each Sym_{m_i} acts naturally on $\{1,\ldots,m_i\}$, moreover, we can assume that this action is primitive. Proposition 3 allows to reduce solving General Problem for H in R to solving General Problem for a subgroups H^* (which depends on H) of odd index in the factor-group $N_R(T)/T$, where T is a Sylow 2-subgroup of $H \cap A$ and A is the base of the wreath product R, and

$$N_R(T)/T \cong \prod_{i=1}^{t} \mathbb{Z}_3 \wr Sym_{m_i}.$$

Moreover, H^* projects naturally to each Sym_{m_i}, and we can assume that the image of H^* is a primitive subgroup of Sym_{m_i} for each i. Since H^* is a subgroup of odd index in $N_R(T)/T$, its image in Sym_{m_i} contains a transposition and, therefore, by the well-known Jordan theorem [25, Theorem 13.3], the image of H^* coincides with Sym_{m_i} for each $i \in \{1,\ldots,t\}$. In Section 4 we obtain a criterion which allows to answer the question, is H^* pronormal in $N_R(T)/T$?

4. Pronormality of a subgroup of odd index in the direct product of groups of the type $\mathbb{Z}_p \wr Sym_n$

Let

$$G = \prod_{i=1}^{t} G_i = V \rtimes B, \text{ where } G_i \cong \mathbb{Z}_{p_i} \wr Sym_{n_i} \text{ for each } i,$$

each p_i is an odd prime, and each Sym_{n_i} acts naturally on $\{1,\ldots,n_i\}$;

$$V = \sum_{i=1}^{t} V_i, \text{ where } V_i = (\mathbb{Z}_{p_i})^{n_i} \text{ for each } i,$$

and

$$B = \prod_{i=1}^{t} B_i, \text{ where } B_i = Sym_{n_i} \text{ for each } i.$$

Define the map $\overline{} : G \to B$ such that $\overline{g} = b$ for $g = vb$, where $v \in V$ and $b \in B$.

Let $\pi_i : G \to G_i$ be the projection for each $i \in \{1,\ldots,t\}$.

It is easy to see that $G_i = V_i \rtimes B_i$ for each i. Let $\overline{}^i : G_i \to B_i$ be the corresponding natural epimorphism for each $i \in \{1, \dots, t\}$.

Let $\overline{\pi_i} : B \to B_i$ be the corresponding projection for each $i \in \{1, ..., t\}$.

It is easy to see that for each $i \in \{1, ..., t\}$, the corresponding diagram

$$\begin{array}{ccc} G & \xrightarrow[\pi_i]{} & G_i \\ \big\downarrow{\scriptstyle \overline{}} & & \big\downarrow{\scriptstyle \overline{}^i} \\ B & \xrightarrow[\overline{\pi_i}]{} & B_i \end{array}$$

is commutative.

In this section, we obtain a criterion of pronormality of a subgroup $H \leq_2 G$ such that $\overline{\pi_i(H)}^i$ is a primitive subgroup of Sym_{n_i} for each $i \in \{1, \dots, t\}$ or, equivalently, $\overline{H} = B$. Let us explain that $\overline{H} = B$ if each $\overline{\pi_i(H)}^i$ is a primitive subgroup of Sym_{n_i}. Note that for each $i \in \{1, \dots, t\}$, $\pi_i(H) \leq_2 G_i$, therefore, $\overline{\pi_i(H)}^i \leq_2 B_i = Sym_{n_i}$ and so, $\overline{\pi_i(H)}^i$ contains a transposition. Thus, by [25, Theorem 13.3],

$$\overline{\pi_i(H)}^i = B_i = Sym_{n_i} \text{ for each } i \in \{1, \dots, t\}.$$

Recall that for each i, $B_i = Sym_{n_i} \in \mathbb{X}_2$ by [3, Lemma 4]. Thus, $\overline{H} = \prod\limits_{i=1}^{t} \overline{\pi_i(H)}^i = B$ by Lemma 2.

First, consider the case $t = 1$ and specify our notation for this situation.

Let $G = \mathbb{Z}_p \wr Sym_n = V \rtimes B$, where p is an odd prime and $B = Sym_n$ acts naturally on $\{1, \dots, n\}$, and let $V = (\mathbb{Z}_p)^n$ coincides with the base of the wreath product. Define

$$V^+ = \{(x, x, \dots, x) \mid x \in \mathbb{Z}_p\} \leq V$$

and

$$V^- = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{Z}_p \text{ and } \sum_{i=1}^{n} x_i = 0\} \leq V.$$

Since V is abelian, $\overline{G}$ acts on V by the natural way.

Lemma 18. *In the introduced notation, let $G = \mathbb{Z}_p \wr Sym_n$, where p is an odd prime. Assume that $\overline{H} = Sym_n$. Then the following statements hold*:

(1) *all the $\overline{H}$-invariant subgroups of V are 0, V^+, V^-, and V;*

(2) $[\overline{H}, V^-] = [\overline{H}, V] = [H, V^-] = [H, V] = V^-$;

(3) *the only proper normal subgroup of odd index in G is BV^-;*

(4) *BV^- does not contain a proper normal subgroup of odd index*;

(5) *if $V^- \leq H$, then $H \in \{BV^-, G\}$ and $H \trianglelefteq G$.*

Proof. (1) Follows from [10, Proposition 5.3.4], for example.

(2) It is easy to see that

$$[V^-, H] = [V^-, \overline{H}] = [V^-, B].$$

Moreover, it is clear that

$$[\overline{H}, V^-] = [H, V^-] \leq [\overline{H}, V] = [H, V] \leq V^-.$$

Prove that $V^- \leq [H, V^-]$. Let $i, j \in \{1, \dots, t\}$ with $i < j$ and $\langle a \rangle = \mathbb{Z}_p$. Define $w_{ij}(a) \in V$ as follows:

$$w_{ij}(a) = (x_1, x_2, \dots, x_n), \text{ where } x_i = a, x_j = -a, \text{ and } x_k = 0 \text{ for } k \notin \{i, j\}.$$

Recall that

$$V^- = \langle w_{ij}(a) \mid 1 \leq i < j \leq t \rangle.$$

Moreover, $[w_{ij}(a), (i, j)] = w_{ij}(-2a)$, $\langle -2a \rangle = \mathbb{Z}_p$ (since p is odd), and $\overline{H} = B$ contains all the transpositions. Now it is clear that $V^- \leq [V^-, B] = [V^-, H]$.

(3), (4) It is easy to see that BV^- is a proper normal subgroup of odd index in G. Let K be a normal subgroup of odd index in G (of BV^-, respectively). Then by the Sylow theorems, K contains each Sylow 2-subgroup of G (of BV^-, respectively). In particular, K contains each Sylow 2-subgroup of B and each transposition of B. Therefore, K contains B (which is generated by these transpositions). Thus, by part (2) of this lemma, K contains $[V^-, B] = V^-$, and so, $K \geq BV^-$.

(5) Assume that $V^- \leq H$. Show that $B \leq H$. Let s be a transposition from B. Since $\overline{H} = B$, there exists $h \in H$ such that $h = sv$ for some $v \in V$. Note that $|s| = 2$ and $|v| = p$ is odd. Using elementary calculations, it is easy to show that $h^p = sv^-$ for some $v^- \in [V, B]$. By part (2) of this lemma, $[V, B] = V^-$. Since $h^p \in H$ and $V^- \leq H$, we have $s \in H$. Thus, H contains each transposition from B. Therefore, $B \leq H$ and $BV^- \leq H$. Now $|G : BV^-| = p$ implies $H \in \{BV^-, G\}$ and $H \trianglelefteq G$ by part (3) of this lemma.

□

Proposition 4. *In the introduced notation, let $G = \mathbb{Z}_p \wr Sym_n$, where p is an odd prime, and $H \leq G$ such that $\overline{H} = B = Sym_n$. Then the following statements hold:*

(1) *if p does not divide n, then H is pronormal in G;*

(2) *if p divides n, then H is pronormal in G if and only if $V^- \leq H$;*

(3) *if $H \leq K < G$, then H is pronormal in K.*

Proof. Note that $H \cap V$ is an $\overline{H}$-invariant subgroup of V, therefore, by Lemma 18 part (1), $H \cap V \in \{0, V^+, V^-, V\}$.

If p does not divide n, then by Lemma 14, $\overline{H} = B$ is pronormal in $G = VB = V\overline{H}$. Now, Lemma 7 implies that H is pronormal in $G = VH$. Therefore, for each K such that $H \leq K \leq G$ we have that H is pronormal in K by Lemma 5.

Let p divides n. Then $n > 2$ and $V^+ < V^-$. Suppose that H is pronormal in G. Show that $V^- \leq H$. By Lemma 18 part (1), it is sufficient to understand that $H \cap V \neq 0$ and $H \cap V \neq V^+$.

Assume that $H \cap V = 0$. Then

$$N_V(H) = C_V(H) = C_V(\overline{H}) = V^+.$$

By Lemma 18 part (2), we have $[H, V] = V^-$. Recall that $V^+ < V^-$. So,

$$N_V(H) + [H, V] = V^- < V.$$

Lemma 6 implies that H is not pronormal in $VH = G$.

Assume that $H \cap V = V^+$. Let $u = (u_1, \dots, u_n) \in N_V(H)$. Then $[u, h] \in H \cap V = V^+$ for each $h \in H$. So, if we take h such that $\overline{h}$ is a transposition (i, j), then we obtain that $u_i - u_j = u_j - u_i$. The oddness of p implies that $u_i = u_j$. Thus, $N_V(H) \leq V^+$. By Lemma 18 part (2), $[H, V] = V^-$. Thus,

$$N_V(H) + [H, V] \leq V^+ + V^- = V^- < V.$$

Lemma 6 implies that H is not pronormal in G.

Assume that $V^- \leq H$. By Lemma 18 part (5), $H \in \{BV^-, G\}$ and $H \trianglelefteq G$. Thus, $H\,prn\,G$.

Let K be a subgroup of G such that $H \leq K$ and $K < G$. Then $V \not\leq K$ and all the possibilities for H-invariant subgroups from $K \cap V$ are 0, V^+, and V^-. Now recall that by Lemma 18 part (2), $[H, V^-] = V^-$, and $V^+ = C_{V^+}(H) \leq N_{V^+}(H)$ since $V^+ \leq Z(G)$. Therefore, by Lemma 6, H is pronormal in $(K \cap V)H = K$. □

Now consider the case $t > 1$. For each $i \in \{1, \dots, t\}$, define corresponding subgroups V_i^+ and V_i^- of V_i as above.

Let K be a subgroup of G such that $H \leq K \leq G$. It follows from Lemma 3 part (1) that if for some i, $\pi_i(H)$ in not pronormal in $\pi_i(K)$, then H is not pronormal in K.

Suppose $\pi_i(H)$ is pronormal in $\pi_i(K)$ for each $i \in \{1, \dots, t\}$ and show that H is pronormal in K. Assume that p_i divides n_i for some i. By Proposition 4, if $\pi_i(K) = G_i$, then $V_i^- \leq \pi_i(H)$. By Lemma 18 part (5), we have $\pi_i(H) \in \{B_iV_i^-, G_i\}$. Note that $H \cap G_i$ is a normal subgroup in H. Consiquently,

$$H \cap G_i = \pi_i(H \cap G_i) \trianglelefteq \pi_i(H) \text{ and } |\pi_i(H) : H \cap G_i| \text{ is odd.}$$

Parts (3) and (4) of Lemma 18 imply that $B_iV_i^- \trianglelefteq H \cap G_i$. Moreover, $B_iV_i^- \trianglelefteq G$ and by Lemma 3, H is pronormal in K if and only if $H/(B_iV_i^-)$ is pronormal in $K/(B_iV_i^-)$. Note that $G_i/B_iV_i^- \cong \mathbb{Z}_{p_i} = \mathbb{Z}_{p_i} \wr Sym_1$ and p_i does not divide 1, of course. Thus, replacing K and H with the corresponding quotients by the normal subgroup of G generated by all $B_iV_i^-$ such that p_i divides n_i and $\pi_i(K) = G_i$, we can assume that for each i, if p_i divides n_i, then $\pi_i(K) \leq B_iV_i^-$. Prove the following assertion.

Proposition 5. *In the introduced notation, let*

$$R = \prod_{i=1}^{t_0} B_iV_i \times \prod_{i=t_0+1}^{t} B_iV_i^-$$

be a subgroup of G. Additionally assume that p_i does not divide n_i if $1 \leq i \leq t_0$, and p_i divides n_i if $t_0 + 1 \leq i \leq t$. Suppose that $H \leq K \leq R$ and $\overline{H} = \prod_{i=1}^{t} B_i$. Then H is pronormal in K.

Proof. By Lemma 5, it is sufficient to prove that H is pronormal in R. Note that $R = (R \cap V)H$. Thus, we use Lemma 6.

Let U be an H-invariant (equivalently, $\overline{H}$-invariant) subgroup of $R \cap V$. For each $i \in \{1, \dots, t\}$, denote by $\sigma_i : V \to V_i$ the corresponding projection. Furthermore, consider the restriction of $^-$ to H and denote by H_i the complete preimage of B_i under the epimorphism $H \to B$. By Lemma 18 part (2), for each i we have

$$V_i^- \geq \left[V_i^-, H\right] \geq \left[V_i^-, H_i\right] = \left[V_i^-, \overline{H_i}\right] = \left[V_i^-, B_i\right] = V_i^-.$$

Therefore, $V_i^- = \left[V_i^-, H\right]$ for each i.

Assume that $\sigma_j(U) \not\leq V_j^+$ for some j. Then by Lemma 18 part (1), we have $\sigma_j(U) \geq V_j^-$. Thus,

$$V_j^- = \left[V_j^-, \overline{H_j}\right] = \left[V_j^-, H_j\right] \leq [\sigma_j(U), H_j] = [U, H_j] \leq [U, H] \leq U.$$

So, if $\sigma_j(U) \not\leq V_j^+$, then $V_j^- \leq U$.

Show that each $u \in U$ can be presented in the form $u = u^+ + u^-$, where $u^- \in [U, H]$ and $u^+ \in C_U(H)$.

Let $u = \sum_{i=1}^{t} u_i$, where $u_i \in V_i$.

Assume $i \leq t_0$. Then

$$V_i = V_i^+ \oplus V_i^-.$$

So, there exists a unique decomposition

$$u_i = u_i^+ + u_i^-, \text{where } u_i^+ \in V_i^+ \text{ and } u_i^- \in V_i^-.$$

Moreover, if $u_i^- \neq 0$, then $\sigma_i(U) \not\leq V_i^+$. Therefore $V_i^- \leq U$ and we have

$$u_i^- \in U \cap V_i^- \text{ and } u_i^+ \in U \cap V_i^+.$$

Let $i > t_0$. In this case $V_i^+ < V_i^-$. Define

$$u_i^+ = \begin{cases} u_i, & \text{if } u_i \in V_i^+; \\ 0, & \text{if } u_i \in V_i^- \setminus V_i^+; \end{cases} \quad \text{and} \quad u_i^- = \begin{cases} 0, & \text{if } u_i \in V_i^+; \\ u_i, & \text{if } u_i \in V_i^- \setminus V_i^+. \end{cases}$$

Let

$$u^+ = \sum_{i=1}^{t} u_i^+ \text{ and } u^- = \sum_{i=1}^{t} u_i^-.$$

So, $u^+, u^- \in U$ and $u = u^+ + u^-$. Now $u_i^- \in V_i^-$ for each i, and if $u_i^- \neq 0$ for some i, then

$$\left[V_i^-, B_i\right] = \left[V_i^-, \overline{H_i}\right] = V_i^- \leq U.$$

Thus, if $u_i^- \neq 0$, then there are $w_i \in V_i^- \leq U$ and $h_i \in H_i$ such that $u_i^- = \left[w_i, \bar{h}_i\right] = [w_i, h_i]$. In the case $u_i^- = 0$, we put $h_i = 1$ and $w_i = 0$. Let

$$h = \prod_{i=1}^{t} h_i \quad \text{and} \quad w = \sum_{i=1}^{t} w_i.$$

We have

$$u^- = [w, \bar{h}] = [w, h] \in [U, H].$$

Taking into account that $V_i^+ \leq Z(R)$ for each $i \in \{1, \ldots, t\}$, we obtain that

$$u^+ \in C_U(H).$$

So, for each $u \in U$ we have the decomposition

$$u = u^+ + u^- \in C_U(H) + [U, H].$$

Therefore,

$$U \leq C_U(H) + [U, H] \leq N_U(H) + [U, H].$$

The inclusion $N_U(H) + [U, H] \leq U$ is clear. Application of Lemma 6 completes the proof. Thus, H is pronormal in R. □

So, we have proved the following theorem.

Theorem 2. *Let* $G = \prod_{i=1}^{t} G_i$*, where* $G_i \cong \mathbb{Z}_{p_i} \wr Sym_{n_i}$ *for each* i*, each* p_i *is an odd prime, and each* Sym_{n_i} *acts naturally on* $\{1, \dots, n_i\}$*. In the introduced notation, assume that* H *is a (non-trivial) subgroup of odd index of* G *such that* $\overline{\pi_i(H)}^{\,i}$ *is a primitive subgroup of* Sym_{n_i} *for each* $i \in \{1, \dots, t\}$*. Then the following statements hold*:

(i) $\overline{H} = \prod_{i=1}^{t} Sym_{n_i}$;

(ii) *for any* $K \leq G$ *such that* $H \leq K$*,* H *is pronormal in* K *if and only if* $\pi_i(H)$ *is pronormal in* $\pi_i(K)$ *for each* $i \in \{1, \dots, t\}$*; and*

(iii) *if* $t = 1$ *(here we put* $p_1 = p$ *and* $n_1 = n$ *for brevity), then the following statements hold*:

(1) *if* p *does not divide* n*, then* H *is pronormal in* G;

(2) *if* p *divides* n*, then* H *is pronormal in* G *if and only if* $V^- \leq H$;

(3) *if* $H \leq K < G$*, then* H *is pronormal in* K*.*

Remark 2. In is easy to see that for each positive integer n, $\mathbb{Z}_2 \wr Sym_n \in \mathbb{X}_2$ if Sym_n acts naturally on $\{1, \dots, n\}$. Thus, the requirement of oddness of each p_i can be omitted in parts (i) and (ii) of Theorem 2 by Lemma 10.

Consider an example of application the obtained results to solving General Problem for a given subgroup of odd index in the finite simple symplectic group $PSp_6(3)$.

Example 1. *Let* $\overline{G} = PSp_6(3)$ *and* $\overline{H} \leq_2 \overline{G}$*. Decide, is* $\overline{H}$ *pronormal in* $\overline{G}$*?*

Let $G = Sp_6(3)$, V be the natural module of G, and H be the complete preimage of $\overline{H}$ in G. It is easy to see that $H \leq_2 G$. Recall that by Lemma 3, H is pronormal in G if and only if $\overline{H}$ is pronormal in $\overline{G}$.

We can assume that H is a proper subgroup of G. Therefore, there exists a maximal subgroup M of G such that $H \leq M$. The index $|G : M|$ is odd.

Since $q = 3$ is prime, by Lemma 12, possibilities for M are the following:

Type (1) $M \cong Sp_2(3) \times Sp_4(3)$ is the stabilizer of a non-degenerate subspace of dimension 2 of V;

Type (2) $M \cong Sp_2(3) \wr Sym_3$ is the stabilizer of an orthogonal decomposition

$$V = \bigoplus_{i=1}^{3} V_i$$

into a sum of 3 pairwise isometric non-degenerate subspaces V_i of dimension 2.

Let $S \in Syl_2(G)$ such that $S \leq H$.

If there is a subgroup M of Type (1) such that $H \leq M$, then H is reducible on V. Moreover, H is pronormal in M by Lemma 17. By Lemma 11, we have $N_G(S) = N_M(S)$. Therefore, H is pronormal in G by Lemma 5.

If there is no a maximal subgroup M of Type (1) such that $H \leq M$, then by Lemma 12, H is irreducible on V and there is a maximal subgroup M of Type (2) such that $H \leq M \cong Sp_2(3) \wr Sym_3$, where Sym_3 acts naturally on the set $\{1, 2, 3\}$. By Lemma 11, we have $N_G(S) = N_M(S)$. Therefore, by Lemma 5, H is pronormal in G if and only if H is pronormal in M.

By [4], $Sp_2(3) \cong \mathbb{Z}_2.Alt_4$, therefore,

$$H \leq M \cong (\mathbb{Z}_2.Alt_4) \wr Sym_3.$$

Moreover, since H is irreducible on V and the dimension of V_i was chosen as maximal as possible (since there was the only choice $dim(V_i) = 2$), we conclude that H projects onto Sym_3.

By Lemma 3, H is pronormal in M if and only if $H/O_2(M)$ is pronormal in $M/O_2(M) \cong \mathbb{Z}_3 \wr Sym_3$. Now to decide is H pronormal in M, it is sufficient to apply Theorem 2 (really, in this case it is sufficient to apply Proposition 4 which is a part of Theorem 2). □

Thus, Problems 1 and 2 formulated in Section 1 are of interest. Moreover, the following problems naturally arise.

Problem 3. Find a criterion of pronormality of a given subgroup of odd index in the group $G = \prod_{i=1}^{t} \mathbb{Z}_{p_i} \wr Sym_{n_i}$, where each p_i is a prime and each Sym_{n_i} acts naturally on $\{1, \dots, n_i\}$.

Problem 4. Provide an effective algorithm which solves General Problem for an arbitrary subgroup of odd index in the group $G = \prod_{i=1}^{t} \mathbb{Z}_{p_i} \wr Sym_{n_i}$, where each p_i is a prime and each Sym_{n_i} acts naturally on $\{1, \dots, n_i\}$.

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