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ON TWO-SIDED UNIDIRECTIONAL MEAN VALUE INEQUALITY IN A FRÉCHET SMOOTH SPACE[1]

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Abstract: The paper is devoted to a new unidirectional mean value inequality for the Fréchet subdifferential of a continuous function. This mean value inequality finds an intermediate point and localizes its value both from above and from below; for this reason, the inequality is called two-sided. The inequality is considered for a continuous function defined on a Fréchet smooth space. This class of Banach spaces includes the case of a reflexive space and the case of a separable Asplund space. As some application of these inequalities, we give an upper estimate for the Fréchet subdifferential of the upper limit of continuous functions defined on a reflexive space.

Keywords: Smooth Banach space, Fréchet subdifferential, Unidirectional mean value inequality, Upper limit of continuous functions.

1. Introduction

Consider the following mean value inequalities.

Proposition 1 [12, Theorems 2.1 and 2.2]. *Let a scalar function f be defined and lower semicontinuous on a Fréchet smooth Banach space $\mathbb{X}$. Let points $\check{u}$ and $\check{v}$ in $\mathbb{X}$ be given. Then, for arbitrary numbers $\check{s} < f(\check{v}) - f(\check{u})$ and $\check{\varkappa} > 0$, there exist a point $z_- \in [\check{u};\check{v}] + \check{\varkappa}B$ and a Fréchet subgradient $\zeta_- \in \hat{\partial} f(z_-)$ such that*

$$\check{s} < \zeta_-(\check{v} - \check{u}) \quad \textit{and} \quad f(z_-) < f(\check{u}) + \max(0, \check{s}) + \check{\varkappa}. \tag{1.1}$$

Furthermore, if f is continuous, there are a point $z_+ \in [\check{u};\check{v}] + \check{\varkappa}B$ and a Fréchet subgradient $\zeta_+ \in \hat{\partial} f(z_+)$ such that

$$\check{s} < \zeta_+(\check{v} - \check{u}) \quad \textit{and} \quad f(z_+) > f(\check{v}) - \max(0, \check{s}) - \check{\varkappa}. \tag{1.2}$$

Note that inequalities (1.1) and (1.2) are similar. This suggests that, in the case of continuity of f, it is possible to get a common point $z_+ = z_-$ such that the value $f(z)$ is localized from both above and below. Proving the corresponding two-sided unidirectional mean inequality is the primary goal of this paper.

As part of the historical background, note that the existence of a pair (z_-, ζ_-) satisfying inequalities like (1.1) has been widely studied (see, for example, [13, Subsect. 3.4.8] and [14, Sect. 4.4]). Unlike different variants of Lagrange's mean value theorem for certain classes of Lipschitz continuous functions, they ensure an upper bound of $f(v) - f(u)$ through some subgradient ζ. These inequalities apply to any lower semicontinuous function. Furthermore, the corresponding to the

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Fréchet subdifferentials unidirectional mean value inequality is equivalent to the Asplund property of a Banach space [14, 17], and therefore is equivalent to several basic principles of variational analysis [1, 18], for example, to the inspired by [16] and [5] multidirectional mean value inequality [4, Subsection 3.6.1]; for more recent references, see [10] and [8]. However, the multidirectional mean value theorem as well as the previous unidirectional mean value inequality also localizes $f(\hat{z})$ on one side only.

The rest of the paper is organized as follows. In Section 3, we will prove the desired two-sided unidirectional mean value inequality for continuous functions. Then, applying this result, in Section 4, we will show an upper estimate for the Fréchet subdifferential of the upper limit of continuous functions. But first, we will recall several elementary definitions and notions.

2. Definitions and notation

We will use elementary notions from the set-valued and variational analyses [4, 13, 15].

For a nonempty set $\mathcal{X}$ of some real Banach space $\mathbb{X}$, denote by $\operatorname{cl}\mathcal{X}$ and $\operatorname{co}\mathcal{X}$ the closure and the convex hull of $\mathcal{X}$. For a point $x \in \mathcal{X}$, the contingent (Bouligand tangent) cone to $\mathcal{X}$ at x is the set $T(x;\mathcal{X})$ of all $v \in \mathbb{X}$ such that one finds a decreasing to 0 sequence of positive t_n and a converging to v sequence of $v_n \in \mathbb{X}$ such that $x + t_n v_n \in \mathcal{X}$ for all positive integers n. For a point $x \in \mathbb{X}$, we say that $\zeta \in \mathbb{X}^*$ is a Fréchet normal to $\mathcal{X}$ at x if one has $x \in \mathcal{X}$ and

$$\limsup_{n\to\infty} \frac{\zeta(z_n - x)}{\|z_n - x\|} = 0$$

for all converging to x sequences of $z_n \in \mathcal{X}$. Denote by $\hat{N}(x;\mathcal{X})$ the set of all Fréchet normals to $\mathcal{X}$ at x.

We call a Banach space $\mathbb{X}$ Fréchet smooth if this space has an equivalent norm that is C^1-smooth off the origin. Note that any reflexive Banach space and any separable Asplund space are Fréchet smooth [4, Theorem 6.1.6]. It is worth mentioning that each Fréchet smooth space has a C^1-smooth Lipschitz function with bounded nonempty support [3, Ex. 4.3.9].

Denote by B and B^* the unit closed balls in $\mathbb{X}$ and $\mathbb{X}^*$, respectively.

Given an extended-real-valued function $g : \mathbb{X} \to \mathbb{R} \cup \{-\infty, \infty\}$, define its lower semicontinuous envelope $\operatorname{lsc} g$ by the rule:

$$\operatorname{lsc} g(x) \stackrel{\triangle}{=} \liminf_{\varkappa\downarrow 0} \inf_{z\in x+\varkappa B} g(z) \text{ for all } x \in \mathbb{X}.$$

Note that this function is lower semicontinuous. In addition, a function g is lower semicontinuous iff its epigraph

$$\operatorname{epi} g \stackrel{\triangle}{=} \{(x, a) \in \mathbb{X} \times \mathbb{R} \mid a \geq g(x)\}$$

is closed. In the case of lower semicontinuous function g, define the Fréchet subdifferential of g at x as

$$\hat{\partial} g(x) \stackrel{\triangle}{=} \{\zeta \in \mathbb{X}^* \mid (\zeta, -1) \in \hat{N}(x, g(x); \operatorname{epi} g)\}$$

for a point $x \in \mathbb{X}$ with finite $g(x)$; let also $\hat{\partial} g(x) = \varnothing$ if $|g(x)| = \infty$.

3. Two-sided mean value inequality

Theorem 1. *Let $\mathbb{X}$ be a Fréchet smooth space. Let a continuous function $f : \mathbb{X} \to \mathbb{R}$ and some closed interval $[u; v]$ in $\mathbb{X}$ be given. Then, for a real number $s < f(v) - f(u)$ and positive ε, there exist some point $\hat{z} \in [u; v] + \varepsilon B$ and Fréchet subgradient $\hat{\zeta} \in \hat{\partial} f(\hat{z})$ such that*

$$s < \hat{\zeta}(v - u) \quad \textit{and} \quad |f(\hat{z}) - f(u)| \leq |s| + \varepsilon. \tag{3.1}$$

P r o o f. Without loss of generality, we can assume that $u = 0$ and $f(u) = 0$. Now, the initial inequality can be written as $s < f(v)$.

Case $s < 0$. Let s be negative. Choose a positive number $\varepsilon < \min(|s|, f(v) - s)$. Define $\bar{\varepsilon} = \varepsilon/4$ and $\bar{s} \triangleq s + 3\bar{\varepsilon}$. Since $\bar{s} < 0 = f(0) < |\bar{s}|$ and $\bar{s} < f(v)$, one finds a positive number $\hat{t} < 1$ such that

$$|\bar{s}| > f(\hat{t}v) > -\hat{t}|\bar{s}| > -|\bar{s}| = \bar{s} \tag{3.2}$$

because f is continuous on $[0; v]$. For the same reason, there is a positive $\varkappa < \bar{\varepsilon}$ such that

$$|f(z) - f(0)| < \bar{\varepsilon} \quad \text{for all} \quad z \in [0; v] \cap \varkappa B. \tag{3.3}$$

We claim that there exist some $\hat{z} \in [0; \hat{t}v] + \varkappa B$ and $\hat{\zeta} \in \hat{\partial} f(\hat{z})$ such that

$$s < -|\bar{s}| < \hat{\zeta} v \quad \text{and} \quad |f(\hat{z})| < |\bar{s}| + 2\bar{\varepsilon} < |s|. \tag{3.4}$$

To this end, consider the continuous map

$$[0; \hat{t}\,] \ni \tau \mapsto h(\tau) = f(\tau v) - \tau f(\hat{t}v)/\hat{t}.$$

Since $h(\hat{t}) = h(0) = 0$ holds, due to the intermediate value theorem, there exists positive $\hat{\tau} \leq \hat{t}$ that satisfies the equality $h(\hat{\tau}) = 0$ and at least one of the following conditions:

(i) $\hat{\tau} < \varkappa$; (ii) $h|_{[0,\hat{\tau}]}$ is nonpositive; (iii) $h|_{[0,\hat{\tau}]}$ is nonnegative.

Now, the relations $0 < \hat{\tau} \leq \hat{t} < 1$, $h(\hat{\tau}) = 0$, and (3.2) yield the inequality

$$-|\bar{s}| \leq -|\bar{s}|\hat{\tau} \overset{(3.2)}{<} \hat{\tau} f(\hat{t}v)/\hat{t} = f(\hat{\tau}v) - f(0). \tag{3.5}$$

Let us apply Proposition 1 to this inequality with

$$\check{u}_- \triangleq 0, \quad \check{v}_+ \triangleq \hat{\tau}v, \quad \check{s} \triangleq -|\bar{s}|\hat{\tau}, \quad \text{and} \quad \varkappa_n \triangleq \varkappa/n$$

for all positive integers n. This gives some $r_-, r_+ \in [0; \hat{\tau}]$, $z_-, z_+ \in \mathbb{X}$, $\zeta_- \in \hat{\partial} f(z_-)$, and $\zeta_+ \in \hat{\partial} f(z_+)$ such that

$$\begin{aligned} -|\bar{s}|\hat{\tau} < \hat{\tau}\zeta_- v, \quad \|r_- v - z_-\| < \varkappa_n, \quad f(z_-) - f(0) < \varkappa_n, \\ -|\bar{s}|\hat{\tau} < \hat{\tau}\zeta_+ v, \quad \|r_+ v - z_+\| < \varkappa_n, \quad f(z_+) - f(\hat{\tau}v) > -\varkappa_n. \end{aligned}$$

Next, taking into account the inequalities $\hat{\tau} > 0$ and $f(0) + \varkappa_n = \varkappa_n < \bar{\varepsilon}$, we have

$$-|\bar{s}| < \zeta_\pm v, \quad z_\pm = r_\pm v + \varkappa_n B \subset [0; v] + \varkappa B,$$

$$f(z_-) < f(0) + \varkappa_n < \bar{\varepsilon}, \quad \text{and} \quad f(z_+) > -\varkappa_n + f(\hat{\tau}v) \overset{(3.5)}{\geq} -\varkappa_n - |\bar{s}| > -\bar{\varepsilon} - |\bar{s}|. \tag{3.6}$$

Now, in the case of $\hat{\tau} < \varkappa$ (condition (i)) and in the case of nonpositive $h|_{[0;\hat{\tau}]}$ (condition (ii)), let us set $\hat{z}_n \triangleq z_+$, $\hat{r}_n \triangleq r_+$, and $\hat{\zeta}_n \triangleq \zeta_+$ for all positive integers n; and in the case of nonnegative $h|_{[0;\hat{\tau}]}$ (condition (iii)), set $\hat{z}_n \triangleq z_-$, $\hat{r}_n \triangleq r_-$, and $\hat{\zeta}_n \triangleq \zeta_-$ for all positive integers n. Then, in all these cases and for all positive integers n, we have proved the first inequality in (3.4). So, it is required to check only

$$|f(\hat{z}_n)| \leq |\bar{s}| + 2\bar{\varepsilon}$$

for at least one positive integer n.

Note that all $\hat{r}_n v$ lie in the compact set $[0; \hat{\tau} v]$. Passing to a subsequence, we can assume that this sequence converges. By $\|\hat{z}_n - \hat{r}_n v\| \to 0$, the both sequences of $\hat{z}_n$ and $\hat{r}_n v$ has the common limit. The sequences of $f(\hat{z}_n)$ and $f(\hat{r}_n v)$ are the same by the continuity of f; in particular, one finds a positive integer N such that

$$|f(\hat{z}_N) - f(\hat{r}_N v)| < \bar{\varepsilon}. \tag{3.7}$$

So, it is required to check only the inequality

$$|f(\hat{r}_N v)| < |\bar{s}| + \bar{\varepsilon}.$$

Now, in the case of nonnegative $h|_{[0;\hat{\tau}]}$ (condition (iii)), by the choice of $\hat{r}_N = r_-$ and $\hat{z}_N = z_-$, we obtain

$$0 \leq h(r_-) = f(r_- v) - r_- f(\hat{t} v)/\hat{t} \leq f(r_- v) + |f(\hat{t} v)|$$

and

$$\bar{\varepsilon} \overset{(3.6)}{>} f(\hat{z}_N) = f(z_-) \overset{(3.7)}{\geq} f(r_- v) - \bar{\varepsilon} \geq -|f(\hat{t} v)| - \bar{\varepsilon} \overset{(3.2)}{\geq} -|s| - \bar{\varepsilon}.$$

In the case $h|_{[0;\hat{\tau}]} \leq 0$ (condition (ii)), one has

$$0 \geq h(r_+) \geq f(r_+ v) - |f(\hat{t} v)|$$

and

$$-\bar{\varepsilon} - |\bar{s}| \overset{(3.6)}{<} f(\hat{z}_N) = f(z_+) \overset{(3.7)}{\leq} f(r_+ v) + \bar{\varepsilon} \leq |f(\hat{t} v)| + \bar{\varepsilon} \overset{(3.2)}{\leq} |\bar{s}| + \bar{\varepsilon}.$$

Finally, in the case $\hat{\tau} < \varkappa$ (condition (i)), (3.3) and (3.7) yield

$$|f(r_- v)| < 2\bar{\varepsilon} < |\bar{s}| + \bar{\varepsilon}.$$

Inequalities (3.4) have been proved.

Case $s \geq 0$. Assume that s is nonnegative. Recall that $s < f(v)$. Choose a positive ε such that $s + \varepsilon < f(v)$. Define

$$\bar{s} \overset{\triangle}{=} s + \varepsilon/2.$$

This entails

$$0 < \bar{s} < s + \varepsilon < f(v),$$

and one can choose positive $\bar{\varepsilon}$ such that

$$\bar{\varepsilon} + (1 + \bar{\varepsilon})^3 \bar{s} < s + \varepsilon.$$

Consider the map $\bar{f} : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ defined as

$$\bar{f}(x, r) \overset{\triangle}{=} f(x) - (1 + \bar{\varepsilon}) r \bar{s} \quad \text{for all} \quad x \in \mathbb{X}, \quad r \in \mathbb{R}.$$

Then, we have $\bar{f}(0, 0) = 0$,

$$\hat{\partial} \bar{f}(x, r) = \hat{\partial} f(x) \times \{-(1 + \bar{\varepsilon}) \bar{s}\},$$
$$\bar{f}(v, 1) = \bar{f}(v, 1) - \bar{f}(0, 0) = f(v) - (1 + \bar{\varepsilon}) \bar{s} > \bar{s} - (1 + \bar{\varepsilon}) \bar{s} = -\bar{\varepsilon} \bar{s}.$$

Since $-\bar{\varepsilon} \bar{s} < 0$, we can apply the first case of our theorem to the inequality

$$-\bar{\varepsilon} \bar{s} < \bar{f}(v, 1) - \bar{f}(0, 0).$$

Then, there exist some number $r \in [0;1]$, point

$$\bar{z} = (\hat{z}, \hat{r}) \in (rv, r) + \bar{\varepsilon}B,$$

and subgradient

$$\bar{\zeta} = (\hat{\zeta}, -(1+\bar{\varepsilon})\bar{s}) \in \hat{\partial}\bar{f}(\hat{z}, \hat{r})$$

that satisfy (3.4); i.e.,

$$-\bar{\varepsilon}\bar{s} < \hat{\zeta}v - (1+\bar{\varepsilon})\bar{s} \quad \text{and} \quad |\bar{f}(\hat{z}, \hat{r})| \leq \bar{\varepsilon}|\bar{s}| + \bar{\varepsilon}.$$

Now, the first inequality leads to $s < \bar{s} < \hat{\zeta}v$ by $s < \bar{s}$; on the other hand, the second inequality entails

$$|f(\hat{z})| = \left|\bar{f}(\hat{z}, \hat{r}) + (1+\bar{\varepsilon})\hat{r}\bar{s}\right| < \bar{\varepsilon}\bar{s} + \bar{\varepsilon} + (1+\bar{\varepsilon})|\hat{r}|\bar{s} < \bar{\varepsilon}\bar{s} + \bar{\varepsilon} + (1+\bar{\varepsilon})^2\bar{s} < (1+\bar{\varepsilon})^3\bar{s} + \bar{\varepsilon} < s + \varepsilon$$

by $|\hat{r}| < |r| + \bar{\varepsilon} \leq 1 + \bar{\varepsilon}$ and the choice of $\bar{\varepsilon}$.

The theorem is proved. □

Remark 1. As [12, Example 2.1] has shown, (1.2) can be violated if $f : \mathbb{R} \to \mathbb{R}$ is only lower semicontinuous. Therefore, the assumption of the continuity of f is essential in this theorem as well.

Remark 2. In the case of Lipschitz continuous function f, for its G-subdifferential, there exists a variant of unidirectional mean value inequality that guaranties the inclusion $z \in [u;v]$ instead of $z \in [u;v] + \varepsilon B$ (see [9, Theorem 4.70]). However, this is not true for a Fréchet subdifferential. Indeed, for the Lipschitz continuous function

$$\mathbb{R}^2 \ni (x,y) \mapsto f(x,y) \triangleq -|x|,$$

its Fréchet subdifferential is empty on the interval $[(0,0);(0,1)]$; in particular, no Fréchet subgradient ζ satisfies (3.1).

Remark 3. It may mistakenly seem that Theorem 1 does not essentially use the asymmetry of a Fréchet subdifferential and can be directly extended to the symmetric case. Indeed, Lebourg's mean value theorem [6, Theorem 2.4] for Clarke subdifferentials, the mean value theorem [2] for MP-subdifferentials, and the symmetric subdifferential mean value theorem [13, Theorem 3.47], [14, Theorem 4.11] give the corresponding gradient ζ of f at some $\hat{z} \in [u;v]$ that satisfies the symmetric bound

$$|f(v) - f(u)| = \left|\hat{\zeta}(v-u)\right|. \tag{3.8}$$

This bound is exactly the limit of bounds

$$s_+ < \hat{\zeta}(v-u) + \varepsilon \quad \text{and} \quad -s_- < (-\hat{\zeta})(v-u) + \varepsilon$$

as $s_+ \uparrow f(v) - f(u)$, $-s_- \uparrow (-f)(u) - (-f)(v)$, and $\varepsilon \downarrow 0$. Similarly, passing to the limit in $|f(\hat{z}) - f(u)| < |s| + \varepsilon$, we could hope for the eatimate

$$|f(\hat{z}) - f(u)| \leq \left|\hat{\zeta}(v-u)\right| \tag{3.9}$$

together with (3.8). However, in the case

$$f(x) \triangleq x(x-2) \quad \text{and} \quad [u,v] \triangleq [0,2],$$

inequalities (3.8) and (3.9) should give $f'(\hat{z}) = \zeta = 0$ and $|f(\hat{z})| \leq 0$; i.e., $1 = \hat{z} \in \{0,2\}$. This contradiction negates the hope of adding two-side estimate (3.9) to (3.8).

4. Subdifferentials of the upper limit of continuous functions

Let a family of continuous functions $f_\theta : \mathbb{X} \to \mathbb{R} \cup \{-\infty, \infty\}$, $\theta \in [0; \infty)$ be given. Define a function $f_{\sup} : \mathbb{X} \to \mathbb{R} \cup \{-\infty, \infty\}$ by the following rule:

$$f_{\sup}(x) \stackrel{\triangle}{=} \limsup_{\theta \uparrow \infty} f_\theta(x) \quad \text{for all} \quad x \in \mathbb{X}. \tag{4.1}$$

For every positive δ, denote by $Z_\delta(\check{x})$ the set of all $\zeta \in \mathbb{X}^*$ for which there exists a pair $(\theta, x) \in [0; \infty) \times \mathbb{X}$ such that $\zeta \in \hat{\partial} f_\theta(x)$,

$$\theta > 1/\delta, \quad x \in \check{x} + \delta B, \quad \text{and} \quad |f_\theta(x) - (\operatorname{lsc} f_{\sup})(\check{x})| < \delta. \tag{4.2}$$

The following estimate of the subdifferential of the upper limit function is the enlargement of [11, Lemma 6] on reflexive spaces as well as the refinement of [12, Theorem 6.1(a)] in the case of continuous functions; its proof is similar to that of [12, Theorem 6.1(a)].

Proposition 2. *Assume that $\mathbb{X}$ is a reflexive space, a family of scalar functions f_θ, $\theta \in [0; +\infty)$, continuous on $\mathbb{X}$ is given, and $f_{\sup}$ is defined by (4.1). For all $\check{x} \in \mathbb{X}$ and $\xi \in \hat{\partial} \operatorname{lsc} f_{\sup}(\check{x})$, for every positive δ, there exist some $N \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_N \in [0; 1]$, and $\zeta_1, \zeta_2, \ldots, \zeta_N \in Z_\delta\big(\check{x}, \operatorname{lsc} f_{\sup}(\check{x})\big)$ such that $\alpha_1 + \cdots + \alpha_N = 1$ and*

$$\xi \in \sum_{k=1}^{N} \alpha_k \zeta_k + \delta B^*. \tag{4.3}$$

P r o o f. **The special case: $\check{x} = 0$ is a local minimum of $\operatorname{lsc} f_{\sup}$.** Assume that $\check{x} \stackrel{\triangle}{=} 0$ and $\xi \stackrel{\triangle}{=} 0$; furthermore, assume that

$$(\operatorname{lsc} \limsup_{\theta \uparrow \infty} f_\theta)(0) = \inf_{x \in \delta_0 B} \limsup_{\theta \uparrow \infty} f_\theta(x) = 0$$

for some positive δ_0. Then $0 \in \hat{\partial} f_{\sup}(0) = \hat{\partial} \operatorname{lsc} f_{\sup}(0)$.

Note that $f_{\sup}(z) = \inf_{T>0} E(T, z)$ for all $z \in \mathbb{X}$; here $E : [0; +\infty) \times \mathbb{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined as

$$E(T, x) \stackrel{\triangle}{=} \sup_{\theta \geq T} f_\theta(x) \quad \text{for all} \quad T > 0, \quad x \in \mathbb{X}.$$

Fix a vector $v \in B$ and a positive number $\delta < \min(\delta_0, 1/3)$. Define $t = \delta^2$. Since 0 is a local minimum of $\operatorname{lsc} f$, there exists a point $\check{z} \in tB$ such that

$$0 \leq \operatorname{lsc} f_{\sup}(\check{z}) \leq f_{\sup}(\check{z}) < \delta^4.$$

Then,

$$\|\check{z} + tv\| < 2\delta^2 < \delta < \delta_0 \quad \text{and} \quad f_{\sup}(\check{z} + tv) \geq \operatorname{lsc} f_{\sup}(\check{z} + tv) \geq 0.$$

So,

$$f_{\sup}(\check{z} + tv) - f_{\sup}(\check{z}) > -\delta^4 = -\delta^2 t.$$

Further, we can find positive numbers $\bar{T} \geq 1/\delta$ and $\hat{\theta} > \bar{T}$ such that

$$\delta^2 t > E(\bar{T}, \check{z}) - f_{\sup}(\check{z}) \quad \text{and} \quad \delta^2 t + f_{\hat{\theta}}(\check{z} + tv) > E(\bar{T}, \check{z} + tv). \tag{4.4}$$

By definition of E, we also have

$$0 \leq E(\bar{T}, \check{z} + tv) - f_{\sup}(\check{z} + tv) \quad \text{and} \quad f_{\hat{\theta}}(\check{z}) \leq E(\bar{T}, \check{z}). \tag{4.5}$$

Subtracting the sum of inequalities (4.5) from the sum of inequalities (4.4), we have

$$2\delta^2 t + f_{\hat\theta}(\check z + tv) - f_{\hat\theta}(\check z) > f_{\sup}(\check z + tv) - f_{\sup}(\check z).$$

From $f_{\sup}(\check z + tv) - f_{\sup}(\check z) > -\delta^2 t$ and $\delta < 1/3$, it follows that

$$f_{\hat\theta}(\check z + tv) - f_{\hat\theta}(\check z) > -\delta t.$$

Now, Theorem 1 for $f = f_{\hat\theta}$ with $u = \check z$, $v = \check z + tv$, $s = -\delta t$, and $\varepsilon = \delta(\delta - t)$ gives a number $r \in [0, t]$, a point $\hat z \in \mathbb{X}$, and a subgradient $\hat\zeta \in \hat\partial f_{\hat\theta}(\hat z)$ such that

$$-\delta t < t\hat\zeta v, \quad \|\hat z - \check z\| < \|\hat z - rv\| + r \leq t + \delta(\delta - t) < 2\delta^2, \quad \text{and} \quad |f_{\hat\theta}(\hat z) - f_{\hat\theta}(\check z)| < \delta t + \delta(\delta - t) = \delta^2.$$

Then, by the choice of $\check z$, we obtain

$$\|\hat z\| \leq \|\check z\| + 2\delta^2 < 3\delta^2 < \delta \quad \text{and} \quad |f_{\hat\theta}(\hat z)| \leq |f_{\hat\theta}(\check z) - f_{\hat\theta}(u)| + \delta^2 \leq 2\delta^2 < \delta.$$

So, we show (4.2) for $(\check x, \check y) = (0, 0)$, $(\theta, x) = (\hat\theta, \hat z)$, therefore we obtain $\hat\zeta \in Z_\delta(0, 0)$. Hence, for each $v \in B$, we have found $\hat\zeta \in Z_\delta(0, 0)$ such that $\hat\zeta v > -\delta$. This entails

$$-\delta < \inf_{v\in B} \sup_{\zeta \in Z_\delta(0,0)} \zeta v \leq \inf_{v\in B} \sup_{\zeta\in \mathrm{cl\,co}\, Z_\delta(0,0)} \zeta v.$$

The set B is an weak compact subset of $\mathbb{X}^{**} = \mathbb{X}$ and, together with $\mathrm{cl\,co}\, Z_\delta(0, 0)$, is convex. In addition, the map $(\zeta, v) \mapsto \zeta v$ is continuous and linear in $(\zeta, v) \in \mathbb{X}^* \times \mathbb{X}^{**}$. Therefore, the nonsymmetrical Minimax Theorem [4, Theorem 3.6.14] ensures

$$-\delta < \inf_{v\in B} \sup_{\zeta\in \mathrm{cl\,co}\, Z_\delta(0,0)} \zeta v = \sup_{\zeta\in \mathrm{cl\,co}\, Z_\delta(0,0)} \inf_{v\in B} \zeta v.$$

Since there exists $\zeta \in \mathrm{cl\,co}\, Z_\delta$ such that $\delta > -\zeta v$ for all $v \in B$, we obtain $\|\zeta\| \leq \delta$. Therefore, (4.3) holds in the special case. The special case of this lemma is proved.

The general case. Let a point $\check x \in \mathbb{X}$ and a subgradient $\xi \in \hat\partial\, \mathrm{lsc}\, f_{\sup}(\check x)$ be given. Define $\check y = \mathrm{lsc}\, f_{\sup}(\check x)$. Choose a positive number $\delta < 1/3$.

Since $\mathbb{X}$ is a Fréchet smooth space, by [7, Theorem 4.6 (i)], there exist a C^1-smooth function g and a positive number $\delta_1 < \delta^2$ such that

$$\xi = g'(\check x), \quad \mathrm{lsc}\, f_{\sup}(\check x) = g(\check x), \quad \text{and} \quad \mathrm{lsc}\, f_{\sup}(\check x + tv) - \mathrm{lsc}\, f_{\sup}(\check x) \geq g(\check x + tv) - g(\check x)$$

if $x \in \check x + \delta_1 B$. Further, decreasing δ_1 if necessary, we can also ensure $\xi \in g'(x) + \delta^2 B^*$ and $g(x) \in g(\check x) + \delta^2 B$ for all $x \in \check x + \delta_1 B$. So,

$$\mathrm{lsc}(f_{\sup} - g)(0) \leq (f_{\sup} - g)(x) \quad \text{for all} \quad x \in \delta_1 B.$$

Using the special case for the maps

$$\mathbb{X} \ni x \mapsto \bar f_\theta(x) = f_\theta(x - \check x) - g(x - \check x),$$

and a positive number δ^2, we find $\zeta \in \mathrm{cl\,co}\, \bar Z_{\delta^2}(\check x, \check y) \cap \delta^2 B^*$. Then, in the account of the fuzzy sum rule, one finds a positive integer N, points $x_1, \dots, x_N \in \mathbb{X}$, subgradients $\bar\zeta_1 \in \hat\partial f_{\theta_1}(\bar x_1) - g'(\bar x_1), \dots, \zeta_N \in \hat\partial \bar f_{\theta_N}(\bar x_N) - g'(\bar x_N)$, and convex coefficients $\alpha_i \in [0, 1]$ such that $\alpha_1 + \ldots + \alpha_N = 1$,

$$\bar x_i \in \check x + 2\delta^2 B, \quad \theta_i \geq \delta^{-2}, \quad |\bar f_{\theta_i}(\bar x_i) - g(\bar x_i) - \mathrm{lsc}\, f_{\sup}(\check x) + g(\check x)| \leq 2\delta^2$$

for all i and
$$\sum_{k=1}^{N}\alpha_k\zeta_k \in 2\delta^2 B^*.$$
Define
$$\bar\zeta_i' \triangleq \bar\zeta_i + g'(\bar x_1) \in \hat\partial f_{\theta_1}(\bar x_1).$$
By the choice of a positive number δ_1 and a smooth function g, we obtain
$$|\bar f_{\theta_i}(\bar x_i) - \operatorname{lsc} f_{\sup}(\check x)| \le |g(\bar x_i) - g(\check x)| + 2\delta^2 < \delta,$$
$$\Big\|\sum_{k=1}^{N}\alpha_k\zeta_k' - \sum_{k=1}^{N}\alpha_k\bar\zeta_k\Big\| \le \max_{i\in[1:N]}\big\|g'(\bar x_i) - \xi\big\| < 2\delta^2 < \delta, \quad \text{and} \quad \xi \in \sum_{k=1}^{N}\alpha_k\zeta_k' + \delta B^*.$$
So, the proposition is proved. □

Remark 4. If $\mathbb{X} \triangleq \mathbb{R}^d$, by the famous Carathéodory theorem [15, Theorem 2.29], any finite convex sum of a (co)vectors can be represented by some finite convex sum of no more than $d+1$ of them. So, we can assume that $N \le d+1$.

Remark 5. If every f_θ is C^1-smooth, we conclude that every $\hat\partial f_\gamma(x)$ is a singleton; therefore, $\zeta_i = f'_{\gamma_i}(x_i)$ for all i.

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