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INEQUALITIES FOR A CLASS OF MEROMORPHIC FUNCTIONS WHOSE ZEROS ARE WITHIN OR OUTSIDE A GIVEN DISK[1]

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Abstract: In this paper, we consider a class of meromorphic functions $r(z)$ having an s-fold zero at the origin and establish some inequalities of Bernstein and Turán type for the modulus of the derivative of rational functions in the sup-norm on the disk in the complex plane. These results produce some sharper inequalities while taking into account the placement of zeros of the underlying rational function. Moreover, many inequalities for polynomials and polar derivatives follow as special cases. In particular, our results generalize as well as refine a result due Dewan et al. [6].

Keywords: Polynomial, Rational function, s-fold zeros, Bernstein inequality.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials

$$p(z) := \sum_{j=0}^{n} a_j z^j$$

of degree at most n and $p'(z)$ denote the derivative of $p(z)$. Let D_k^- denote the region inside $T_k := \{z : |z| = k\}$ and D_k^+ denote the region outside T_k. For $\alpha_j \in \mathbb{C}$, we write

$$w(z) := \prod_{j=1}^{n}(z - \alpha_j); \quad B(z) := \prod_{j=1}^{n}\left(\frac{1 - \overline{\alpha_j} z}{z - \alpha_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(\alpha_1, \alpha_2, ..., \alpha_n) := \left\{\frac{p(z)}{w(z)} : \ p \in \mathcal{P}_n\right\},$$

the set of rational functions with poles $\alpha_1, \alpha_2, ..., \alpha_n$, such that $\alpha_j \in D_1^+$ and with finite limit at infinity. A famous result due to Bernstein states that:

Theorem 1 [5]. *If $p \in \mathcal{P}_n$, then for any $z \in \mathbb{C}$*

$$\max_{z\in T_1} |p'(z)| \leq n \max_{z\in T_1} |p(z)|.$$

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If $p(z) \neq 0$ for $z \in D_1^-$ then it was conjectured by Erdös and latter proved by Lax [9] that

$$\max_{z\in T_1} |p'(z)| \leq \frac{n}{2} \max_{z\in T_1} |p(z)|,$$

where as, if $p(z) \neq 0$ for $z \in D_1^+$, then Turán [11] proved:

$$\max_{z\in T_1} |p'(z)| \geq \frac{n}{2} \max_{z\in T_1} |p(z)|.$$

Li, Mohapatra and Rodriguez [10] obtained Bernstein-type inequalities for rational functions $r \in \mathcal{R}_n$ with prescribed poles $\alpha_1, \alpha_2, ..., \alpha_n$ replacing z^n by Blashke product $B(z)$. Among other things they proved the following results for rational functions with prescribed poles.

Theorem 2. *If $r \in \mathcal{R}_n$ has n zeros all lie in $T_1 \cup D_1^+$, then for $z \in T_1$, we have*

$$|r'(z)| \leq \frac{1}{2}|B'(z)||r(z)|.$$

The result is sharp and equality holds for $r(z) = aB(z) + b$, with $|a| = |b| = 1$.

As a refinement of Theorem 2, Aziz and Shah [2] proved the following:

Theorem 3. *Let $r \in \mathcal{R}_n$ be such that all the zeros of $r(z)$ lie in $T_1 \cup D_1^+$. If $t_1, t_2, ..., t_n$ are the zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $B(z) - \lambda, \lambda \in T_1$, then for $z \in T_1$*

$$|r'(z)| \leq \frac{|B'(z)|}{2}\Big\{\big(\max_{1\leq j\leq n} |r(t_j)|\big)^2 + \big(\max_{1\leq j\leq n} |r(s_j)|\big)^2\Big\}^{1/2}. \tag{1.1}$$

In this paper we prove some results which infact strengthen certain known inequalities for rational functions with prescribed poles and inturn produce refinements of some known polynomial inequalities. We first prove the following generalization as well as a refinement of a result due to Wali and Shah [12].

2. Main results

Theorem 4. *Let*

$$r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$p(z) = z^s\Big(a_0 + \sum_{j=1}^{m-s} a_j z^j\Big)$$

is a polynomial of degree m, having all zeros in $T_k \cup D_k^+$, $k \geq 1$ except an s-fold zero at the origin. If $t_1, t_2, ..., t_n$ are the zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $B(z) - \lambda, \lambda \in T_1$, then for $z \in T_1$

$$|r'(z)| \leq \frac{|B'(z)|}{2}\Big\{\big(\max_{1\leq j\leq n} |r(t_j)|\big)^2 + \big(\max_{1\leq j\leq n} |r(s_j)|\big)^2$$
$$-4\Big[\Big(\frac{k}{1+k}\Big(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\Big) - \frac{sk}{1+k} - \frac{2m - n(1+k)}{2(1+k)}\Big]\frac{|r(z)|^2}{|B'(z)|}\Big\}^{1/2}.$$

If we take $k = 1$ and $m = n$, in Theorem 4, we get the following:

Corollary 1. *Let*

$$r(z)=\frac{p(z)}{w(z)}\in\mathcal{R}_n,$$

where

$$p(z)=z^s\Big(a_0+\sum_{j=1}^{n-s}a_jz^j\Big)$$

is a polynomial of degree n, having all zeros in $T_1\cup D_1^+$ except a zero of multiplicity s at origin. If $t_1,t_2,...,t_n$ are the zeros of $B(z)+\lambda$ and $s_1,s_2,...,s_n$ are the zeros of $B(z)-\lambda$, $\lambda\in T_1$, then for $z\in T_1$

$$|r'(z)|\leq\frac{|B'(z)|}{2}\bigg\{\big(\max_{1\leq j\leq n}|r(t_j)|\big)^2+\big(\max_{1\leq j\leq n}|r(s_j)|\big)^2-2\bigg[\bigg(\frac{|a_0|-|a_{n-s}|}{|a_0|+|a_{n-s}|}\bigg)\frac{|r(z)|^2}{|B'(z)|}-s\bigg]\bigg\}^{1/2}. \quad (2.1)$$

On comparing inequalities (1.1) and (2.1) and noting that $|a_0|\geq|a_{n-s}|$, it is easy to see that for $s=0$, Corollary 1 is an improvement of Theorem 3 which is a result due to Aziz and Shah [2].

Remark 1. For $s=0,k=1$ and $m=n$, Theorem 4 reduces to a result due to Wali and Shah [12, Theorem 1].

It is to be noted that in the paper of Wali and Shah [12] an advanced tool (Osserman's lemma) has been used for its proof. However, we here use a simple application of mathematical induction to prove a more general result from which the result of Wali and Shah follows as special case.

If we take $s=0,m=n$ in Theorem 4, we have the following:

Corollary 2. *Let*

$$r(z)=\frac{p(z)}{w(z)}\in\mathcal{R}_n,$$

where

$$p(z)=\Big(a_0+\sum_{j=1}^{n}a_jz^j\Big)$$

is a polynomial of degree n, having all zeros in $T_k\cup D_k^+$, $k\geq1$. If $t_1,t_2,...,t_n$ are the zeros of $B(z)+\lambda$ and $s_1,s_2,...,s_n$ are the zeros of $B(z)-\lambda$, $\lambda\in T_1$, then for $z\in T_1$

$$|r'(z)|\leq\frac{|B'(z)|}{2}\bigg\{\big(\max_{1\leq j\leq n}|r(t_j)|\big)^2+\big(\max_{1\leq j\leq n}|r(s_j)|\big)^2$$
$$-4\bigg[\frac{n(k-1)}{2(k+1)}+\frac{k}{k+1}\bigg(\frac{|a_0|-k^n|a_n|}{|a_0|+k^n|a_n|}\bigg)\frac{|r(z)|^2}{|B'(z)|}\bigg]\bigg\}^{1/2}.$$

If we consider that $r(z)$ has a pole of order n at $z=\alpha$, then

$$r(z)=\frac{p(z)}{(z-\alpha)^n},$$

where $p(z)$ is a polynomial of degree m. Therefore, we have

$$r'(z)=\bigg(\frac{p(z)}{(z-\alpha)^n}\bigg)'=-\frac{(n-m)p(z)+D_\alpha p(z)}{(z-\alpha)^{n+1}},$$

where for any $\alpha \in \mathbb{C}$, $D_\alpha p(z)$ denotes the polar derivative of the polynomial $p(z)$. Also

$$B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n = \frac{w^*(z)}{w(z)},$$

with $B(z) \to z^n$ as $\alpha \to \infty$, and

$$B'(z) = \frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^{n-1}.$$

Further for $z \in T_1$,

$$|B'(z)| = \frac{n(|\alpha|^2-1)}{|z-\alpha|^2}.$$

Using these observations with $m = n$ in Theorem 4 and letting $|\alpha| \to \infty$, we get the following:

Corollary 3. *Let $p \in \mathcal{P}_n$ be such that all the zeros of*

$$p(z) = z^s\Big(a_0 + \sum_{j=1}^{n-s} a_j z^j\Big)$$

lie in $T_k \cup D_k^+$ except an s-fold zero at the origin. If $t_1, t_2, ..., t_n$ are the zeros of $z^n + \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $z^n - \lambda, \lambda \in T_1$, then for $z \in T_1$

$$|p'(z)| \leq \frac{n}{2}\Bigg\{\big(\max_{1\leq j\leq n}|p(t_j)|\big)^2 + \big(\max_{1\leq j\leq n}|p(s_j)|\big)^2 \\ -4\left[\frac{k}{1+k}\left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|}\right) - \frac{sk}{1+k} - \frac{n(1-k)}{2(1+k)}\right]\frac{|p(z)|^2}{n}\Bigg\}^{1/2}.$$

By taking $k = 1$ in Corollary 3, we get the following:

Corollary 4. *Let $p \in \mathcal{P}_n$ be such that all the zeros of*

$$p(z) = z^s\Big(a_0 + \sum_{j=1}^{n-s} a_j z^j\Big)$$

lie in $T_1 \cup D_1^+$ except an s-fold zero at the origin. If $t_1, t_2, ..., t_n$ are the zeros of $z^n + \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $z^n - \lambda$, $\lambda \in T_1$, then for $z \in T_1$

$$|r'(z)| \leq \frac{|B'(z)|}{2}\Bigg\{\big(\max_{1\leq j\leq n}|p(t_j)|\big)^2 + \big(\max_{1\leq j\leq n}|p(s_j)|\big)^2 - 2\left[\left(\frac{|a_0| - |a_{n-s}|}{|a_0| + |a_{n-s}|}\right)\frac{|p(z)|^2}{n} - s\right]\Bigg\}^{1/2}.$$

Taking $s = 0$, and noting that $|a_0| \geq |a_{n-s}|$, it can easily be seen that Corollary 4 is an improvement of a result due to Aziz [1, Theorem 4].

We next prove the following:

Theorem 5. *Let $r \in \mathcal{R}_n$ be such that all zeros of $r(z)$ lie in $T_k \cup D_k^-$, $k \leq 1$ with an s-fold zero at the origin, then for some γ with $|\gamma| \leq 1$ and for any $z \in T_1$*

$$\left|zr'(z) + \frac{\gamma}{2}\Big(|B'(z)| + \frac{2ks + n(1-k)}{1+k}\Big)r(z)\right| \geq \left|\Big(1 + \frac{\gamma}{2}\Big)|B'(z)| + \frac{\gamma}{2}\Big(\frac{2ks + n(1-k)}{1+k}\Big)\right| \inf_{z\in T_k}|r(z)|.$$

By taking $s = 0,\ k = 1$, Theorem 5 reduces to the result due to Hans et al. [8, Theorem 1].

Again substituting for $r(z), r'(z)$ and $|B'(z)|$ the values as in Corollary 3 and letting $|\alpha| \to \infty$, we get the next property from Theorem 5.

Corollary 5. *Let $p \in \mathcal{P}_n$ be such that all the zeros of a polynomial $p(z)$ lie in $T_k \cup D_k^-$ except an s-fold zero at the origin, then for some γ with $|\gamma| \le 1$ and for any $z \in T_1$*

$$\left|zp'(z) + \frac{\gamma}{2}\left(n + \frac{2ks + n(1-k)}{1+k}\right)p(z)\right| \ge \left|(1 + \frac{\gamma}{2})n + \frac{\gamma}{2}\left(\frac{2ks + n(1-k)}{1+k}\right)\right| \min_{z \in T_k} |p(z)|. \tag{2.2}$$

Remark 2. For $s = 0, k = 1$, (2.2) reduces to a result due to Dewan and Hans [6, Theorem 1].

3. Lemmas

For the proof of these theorems we need the following lemmas.

Lemma 1. *If*

$$B(z) = \prod_{j=1}^{n} \frac{1 - \overline{\alpha_j} z}{z - \alpha_j}.$$

Then for $z \in T_1$

$$\text{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

The above lemma is due to Aziz and Zargar [3].

Lemma 2. *If $(x_j)_{j=1}^{\infty}$ be a sequence of real numbers such that $x_j \ge 1, j \in \mathbb{N}$. Then*

$$\sum_{j=1}^{n} \frac{1 - x_j}{1 + x_j} \le \frac{1 - \prod_{j=1}^{n} x_j}{1 + \prod_{j=1}^{n} x_j}$$

for all $n \in \mathbb{N}$.

The proof of Lemma 2 is a simple consequence of the principle of mathematical induction.

Lemma 3. *Suppose $r \in \mathcal{R}_n$ and if $t_1, t_2, ..., t_n$ are the zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $B(z) - \lambda, \lambda \in T_1$, then for $z \in T_1$*

$$|r'(z)|^2 + |r^{*'}(z)|^2 \le \frac{|B'(z)|^2}{2}\left\{\left(\max_{1 \le j \le n} |r(t_j)|\right)^2 + \left(\max_{1 \le j \le n} |r(s_j)|\right)^2\right\}.$$

The above lemma is due to Aziz and Shah [2].

Lemma 4. *Let $r \in \mathcal{R}_n$ be such that all zeros of $r(z)$ lie in $T_k \cup D_k^-, k \le 1$ with s-fold zeros at the origin, then for $z \in T_1$*

$$|zr'(z)| \ge \frac{1}{2}\left(|B'(z)| + \frac{1}{1+k}(2ks + n(1-k))\right)|r(z)|.$$

The above lemma follows from a result due to Akhter et al. [4].

Next lemma is due to Li, Mohapatra and Rodgriguez [10].

Lemma 5. *If A and B are two complex numbers, then*

(i) *if $|A| \ge |B|$ and $B \ne 0$, then $A \ne vB$ for some complex number v with $|v| < 1$;*

(ii) *conversely, if $A \ne vB$ for some complex number v, with $|v| < 1$, then $|A| \ge |B|$.*

4. Proofs of Theorems

Proof of Theorem 2. Since

$$r(z) = \frac{z^s h(z)}{w(z)},$$

where

$$h(z) = a_0 + \sum_{j=1}^{m-s} a_j z^j.$$

This implies

$$\frac{zr'(z)}{r(z)} = s + \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)}.$$

Equivalently, we get

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) = s + \mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) - \mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right).$$

Let $z_1, z_2, ..., z_{m-s}$ be the zeros of $h(z)$, such that $|z_j| \geq k > 1$. In particular for $z \in T_1$, we get by using Lemma 1.

$$\begin{aligned}
\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &= s + \mathrm{Re}\left(\sum_{j=1}^{m-s} \frac{z}{z - z_j}\right) - \mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) \\
&\leq s + \sum_{j=1}^{m-s} \frac{1}{1+|z_j|} - \mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) \\
&= s + \frac{m-s}{1+k} + \sum_{j=1}^{m-s}\left(\frac{1}{1+|z_j|} - \frac{1}{1+k}\right) - \left(\frac{n - |B'(z)|}{2}\right) \\
&= s + \frac{m-s}{1+k} + \frac{k}{1+k}\sum_{j=1}^{m-s} \frac{k - |z_j|}{k + |z_j|k} - \left(\frac{n - |B'(z)|}{2}\right) \\
&\leq s + \frac{m-s}{1+k} + \frac{k}{1+k}\sum_{j=1}^{m-s} \frac{k - |z_j|}{k + |z_j|} - \left(\frac{n - |B'(z)|}{2}\right).
\end{aligned}$$

Now using Lemma 2 with $|z_j|/k \geq 1$, we get

$$\begin{aligned}
\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &\leq s + \frac{m-s}{1+k} + \frac{k}{1+k}\left(\frac{1 - \prod_{j=1}^{m-s} |z_j|/k}{1 + \prod_{j=1}^{m-s} |z_j|/k}\right) - \left(\frac{n - |B'(z)|}{2}\right) \\
&= s + \frac{m-s}{1+k} + \frac{k}{1+k}\left(\frac{k^{m-s}|a_{m-s}| - |a_0|}{k^{m-s}|a_{m-s}| + |a_0|} - \left(\frac{n - |B'(z)|}{2}\right)\right) \qquad (4.3) \\
&= \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{1+k} + \frac{2sk}{1+k} - \frac{2k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right)\right\}.
\end{aligned}$$

Now

$$r^*(z) = B(z)\overline{r(1/\overline{z})},$$

therefore using the fact that

$$\frac{zB'(z)}{B(z)} = |B'(z)|,$$

(see also [10]) we get for any $z \in T_1$

$$|{r^*}'(z)| = \big||B'(z)|r(z) - zr'(z)\big|.$$

This implies for $z \in T_1$

$$\left|\frac{z{r^*}'(z)}{r(z)}\right|^2 = \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 = |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right). \tag{4.4}$$

Now using (4.3) in (4.4), we get

$$\left|\frac{z{r^*}'(z)}{r(z)}\right|^2 \geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2$$
$$-|B'(z)|\left(|B'(z)| + \frac{2m-n(1+k)}{1+k} + \frac{2sk}{1+k} - \frac{2k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right)\right).$$

This gives for $z \in T_1$

$$|{r^*}'(z)|^2 \geq |r'(z)|^2 + \left\{\frac{2k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right) - \frac{2sk}{1+k} - \frac{2m-n(1+k)}{1+k}\right\}|B'(z)||r(z)|^2.$$

This implies

$$2|r'(z)|^2 + \left\{\frac{2k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right) - \frac{2sk}{1+k} - \frac{2m-n(1+k)}{1+k}\right\}|B'(z)||r(z)|^2$$
$$\leq |r'(z)|^2 + |{r^*}'(z)|^2.$$

Using Lemma 3, we get

$$2|r'(z)|^2 + \left\{\frac{2k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right) - \frac{2sk}{1+k} - \frac{2m-n(1+k)}{1+k}\right\}|B'(z)||r(z)|^2$$
$$\leq \frac{|B'(z)|^2}{2}\Big\{\big(\max_{1\leq j\leq n}|r(t_j)|\big)^2 + \big(\max_{1\leq j\leq n}|r(s_j)|\big)^2\Big\}.$$

On simplification, it follows that

$$|r'(z)| \leq \frac{|B'(z)|}{2}\Bigg\{\big(\max_{1\leq j\leq n}|r(t_j)|\big)^2 + \big(\max_{1\leq j\leq n}|r(s_j)|\big)^2$$
$$-4\left[\left(\frac{k}{1+k}\left(\frac{|a_0| - k^{m-s}|a_{m-s}|}{|a_0| + k^{m-s}|a_{m-s}|}\right) - \frac{sk}{1+k} - \frac{2m-n(1+k)}{2(1+k)}\right]\frac{|r(z)|^2}{|B'(z)|}\Bigg\}^{1/2}.$$

This completely proves Theorem 2. □

Proof of Theorem 3. Suppose $r(z)$ has a zero on T_k, then

$$m = \inf_{z\in T_k}|r(z)| = 0$$

and the result holds trivially. We assume all the zeros of $r(z)$ lie in $D_k^-, k \leq 1$ with an s-fold zero at the origin. So that $m > 0$ and for $z \in D_k^-, |r(z)| \geq m$.

Since $|B(z)| \leq 1$ for $z \in T_1 \cup D_1^-$ (see [7, p. 40]), therefore $|B(z)| \leq 1$ for $z \in T_k,\ k \leq 1$. Hence it follows by Rouche's theorem that for some δ with $|\delta| < 1$,

$$F(z) = r(z) - \delta m B(z)$$

has all zeros in $D_k^-, k \leq 1$. Applying Lemma 4 to $F(z)$, we get for $z \in T_1$

$$|zF'(z)| \geq \frac{1}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}|F(z)|.$$

That is for $z \in T_1$

$$\Big|zr'(z) - \delta mzB'(z)\Big| \geq \frac{1}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}\big|r(z) - \delta mB(z)\big|.$$

Since $F(z) \neq 0$ in $T_k \cup D_k^+$, therefore for any complex number γ with $|\gamma| \leq 1$, we have from (i) of Lemma 5,

$$T(z) = zr'(z) - \delta mzB'(z) + \gamma\left\{\frac{2ks + n(1-k)}{2(1+k)} + \frac{|B'(z)|}{2}\right\}\big(r(z) - \delta mB(z)\big) \neq 0.$$

This gives for $z \in T_1$

$$T(z) = zr'(z) + \frac{\gamma}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}r(z)$$
$$-\delta m\left[zB'(z) + \frac{\gamma}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}B(z)\right] \neq 0.$$

Now using (ii) part of Lemma 5, we get for $|\delta| < 1$, $|\gamma| \leq 1$ and $k \leq 1$

$$\left|zr'(z) + \frac{\gamma}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}r(z)\right| \geq m\left|zB'(z) + \frac{\gamma B(z)}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}\right|.$$

Equivalently for $z \in T_1$, we have

$$\left|zr'(z) + \frac{\gamma}{2}\left\{\frac{2ks + n(1-k)}{1+k} + |B'(z)|\right\}r(z)\right|$$
$$\geq \left|\left(1 + \frac{\gamma}{2}\right)|B'(z)| + \frac{\gamma}{2}\left(\frac{2ks + n(1-k)}{1+k}\right)\right| \inf_{z \in T_k} |r(z)|.$$

This completely proves Theorem 3. □

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