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SOME TRIGONOMETRIC SIMILARITY MEASURES OF COMPLEX FUZZY SETS WITH APPLICATION

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Abstract: Similarity measures of fuzzy sets are applied to compare the closeness among fuzzy sets. These measures have numerous applications in pattern recognition, image processing, texture synthesis, medical diagnosis, etc. However, in many cases of pattern recognition, digital image processing, signal processing, and so forth, the similarity measures of the fuzzy sets are not appropriate due to the presence of dual information of an object, such as amplitude term and phase term. In these cases, similarity measures of complex fuzzy sets are the most suitable for measuring proximity between objects with two-dimensional information. In the present paper, we propose some trigonometric similarity measures of the complex fuzzy sets involving similarity measures based on the sine, tangent, cosine, and cotangent functions. Furthermore, in many situations in real life, the weight of an attribute plays an important role in making the right decisions using similarity measures. So in this paper, we also consider the weighted trigonometric similarity measures of the complex fuzzy sets, namely, the weighted similarity measures based on the sine, tangent, cosine, and cotangent functions. Some properties of the similarity measures and the weighted similarity measures are discussed. We also apply our proposed methods to the pattern recognition problem and compare them with existing methods to show the validity and effectiveness of our proposed methods.

Keywords: Complex fuzzy set, Similarity measures, Pattern recognition.

1. Introduction

The fuzzy set theory introduced by L.A. Zadeh [34] has been revealed to be a valuable apparatus for designating situations in which the data are imprecise or vague. Fuzzy sets (FSs) describe such cases by assigning a degree to which a particular object belongs to a set. It is a robust system where no precise inputs are required; as a result, it has been applied in numerous branches of science and engineering with great success. However, in real life, there is a lot of uncertain data that cannot be described by fuzzy sets due to the presence of dual information about the object, such as the amplitude term and the phase term. To describe such uncertain data, Ramot et al. [25] introduced the concept of complex fuzzy set (CFS), in which the membership function is characterized by a complex number in the polar form $r_A(x)e^{i\omega_A(x)}$ belonging to the unit circle of the complex plane, where $r_A(x)$ and $e^{i\omega_A(x)}$ denote the amplitude term and the phase term of an element $x \in A \subseteq X$, respectively. The amplitude term consistent with the membership degree gives the extent of belonging of an object to a CFS, and the phase term allied with the membership degree provides supplementary data associated with periodicity. The phase term is a unique parameter of the membership degree and is the crucial difference between a traditional FS and a CFS. Due to the presence of the phase term in a CFS, the uncertainty of an object can be described more accurately than by an FS. As a result, the concept of CFSs has been applied by a host of researchers in many areas of our real-life situations, such as image processing [19], signal processing [16, 25, 35], decision making [1, 3, 4, 21, 22], and so on by using different mathematical tools, such as distance measures, aggregation operations, entropy measures, and so forth.

On the other hand, the similarity measure is a core method to see how two objects are related together. The similarity measures between the fuzzy sets are significant topics in fuzzy mathematics which have obtained much attention for their wide applications in various fields, such as pattern recognition [15], decision making [27, 28], image processing [29], clustering [7, 10, 13], approximate reasoning [26, 33], and many other fields (see [8, 32]). The theoretical aspects of the similarity measures of the FSs are also studied by a host of researchers (see [2, 5, 6, 9, 11, 12, 17, 18, 23, 24, 30, 31, 36]).

However, if the data sets are related to a two-dimensional aspect, then the similarity measures of the fuzzy sets fail to compare the proximity among the data sets. To overcome these situations, Guo et al. [14] introduced the cosine similarity measure of complex fuzzy sets and applied it to measure the robustness of the complex fuzzy connectives and the complex fuzzy inference. Moreover, trigonometric similarity measures are important in solving numerous complicated problems in pattern recognition, medical diagnosis, signal processing, etc. But still now, as far as known from the literature, there are no trigonometric similarity measures for CFSs. So, in this paper, we introduce some trigonometric similarity measures for CFSs. Also, the weighted trigonometric similarity measures for CFSs are established. Finally, an application in the pattern recognition problem is illustrated by using our proposed similarity measures.

The paper is organized as follows. In Section 2, we describe some basic properties of CFSs. In Section 3, we introduce some trigonometric similarity measures involving similarity measures based on the sine, tangent, cosine, and cotangent functions. We also define weighted trigonometric similarity measures based on the sine, tangent, cosine, and cotangent functions. In Section 4, a practical example illustrates using these similarity measures and weighted similarity measures. The comparison studies with existing methods and advantages of our proposed methods are also described in Section 4. Finally, a concluding remark is given.

2. Preliminaries

In this section, we describe some basic concepts of CFSs from [25], which are essential to the rest of the paper.

2.1. Complex fuzzy set

A complex fuzzy set defined on a universal set X is characterized by a membership function $\mu_A(x)$ that assigns a complex-valued grade of membership in A to any element $x \in X$. By definition, all values of $\mu_A(x)$ lie within the unit circle in the complex plane and are expressed in the form $r_A(x) \cdot e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ are both real-valued, $r_A(x) \in [0,1]$, and $\omega_A(x) \in [0, 2\pi]$. A complex fuzzy set may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in X\} = \{(x, r_A(x) \cdot e^{i\omega_A(x)}) : x \in X\}.$$

For every two CFSs

$$A = \{(x, r_A(x) \cdot e^{i\omega_A(x)})\} \quad \text{and} \quad B = \{(x_j, r_B(x) \cdot e^{i\omega_B(x)})\},$$

$A \subseteq B$ if $r_A(x) \leq r_B(x)$ and $\omega_A(x) \leq \omega_B(x)$.

3. Some trigonometric similarity measures of complex fuzzy sets

In this section, we propose similarity measures based on the sine, tangent, cosine, and cotangent functions of CFSs. In many decision-making problems, sometimes we need the weight of attributes to describe precisely any situation. These weights of attributes play an important role in making decisions properly. As a result, we consider the weighted similarity measures based on the sine, tangent, cosine, and cotangent functions of CFSs. Some properties of these similarity measures and weighted similarities are also described.

3.1. Similarity measures based on the sine function

Let
$$A=\left\{(x_j, r_A(x_j)\cdot e^{i\omega_A(x_j)})\right\}, \quad B=\left\{(x_j, r_B(x_j)\cdot e^{i\omega_B(x_j)})\right\}$$
be two CFSs in the universe of discourse $X=\{x_1,x_2,\dots,x_n\}$, $x_j\in X$. Then the similarity measures based on the sine function between A and B can be defined as follows:
$$SSF^1(A,B)=1-\frac{1}{n}\sum_{j=i}^{n}\sin\left\{\frac{\pi}{2}\left[|r_A(x_j)-r_B(x_j)|\vee\frac{1}{2\pi}(|\omega_A(x_j)-\omega_B(x_j)|)\right]\right\},$$
$$SSF^2(A,B)=1-\frac{1}{n}\sum_{j=i}^{n}\sin\left\{\frac{\pi}{4}\left[|r_A(x_j)-r_B(x_j)|+\frac{1}{2\pi}(|\omega_A(x_j)-\omega_B(x_j)|)\right]\right\},$$
where the symbol $\vee$ denotes the maximum operator.

Proposition 1. *For two CFSs A and B in $X=\{x_i,x_2,\dots,x_n\}$, the similarity measures $SSF^k(A,B)$, $k=1,2$, have the following properties*:

(1) $0\le SSF^k(A,B)\le 1$;

(2) $SSF^k(A,B)=SSF^k(B,A)$;

(3) $SSF^k(A,B)=1$ *if and only if* $A=B$;

(4) *if C is a CFS in X and $A\subseteq B\subseteq C$, then $SSF^k(A,B)\ge SSF^k(A,C)$ and $SSF^k(B,C)\ge SSF^k(A,C)$.*

P r o o f. (1) It is known that the sine function monotonically increases in the interval $[0,\pi/2]$ and takes values from $[0,1]$. Therefore, we have $0\le SSF^k(A,B)\le 1$.

(2) This property is obvious.

(3) If $A=B$, then $r_A(x_j)=r_B(x_j)$ and $\omega_A(x_i)=\omega_B(x_i)$ for $j=1,2\dots,n$. Therefore, $SSF^k(A,B)=1$.

(4) If $A\subseteq B\subseteq C$, then $r_A(x_j)\le r_B(x_j)\le r_C(x_j)$ and $\omega_A(x_j)\le\omega_B(x_j)\le\omega_C(x_j)$ for $j=1,2,\dots,n$. Then we have
$$|r_A(x_j)-r_B(x_j)|\le|r_A(x_j)-r_C(x_j)|,$$
$$|r_B(x_j)-r_C(x_j)|\le|r_A(x_j)-r_C(x_j)|,$$
and
$$|\omega_A(x_j)-\omega_B(x_j)|\le|\omega_A(x_j)-\omega_C(x_j)|,$$
$$|\omega_B(x_j)-\omega_C(x_j)|\le|\omega_A(x_j)-\omega_C(x_j)|.$$
Hence,
$$SSF^2(A,B)=1-\frac{1}{n}\sum_{j=i}^{n}\sin\left\{\frac{\pi}{4}\left[|r_A(x_j)-r_B(x_j)|+\frac{1}{2\pi}(|\omega_A(x_j)-\omega_B(x_j)|)\right]\right\}$$
$$\ge 1-\frac{1}{n}\sum_{j=i}^{n}\sin\left\{\frac{\pi}{4}\left[|r_A(x_j)-r_C(x_j)|+\frac{1}{2\pi}(|\omega_A(x_j)-\omega_C(x_j)|)\right]\right\}=SSF^2(A,C).$$

Similarly, we can prove that $SSF^2(B,C) \geq SSF^2(A,C)$ as well as $SSF^1(A,B) \geq SSF^1(A,C)$ and $SSF^1(B,C) \geq SSF^1(A,C)$. Hence, $SSF^k(A,B) \geq SSF^k(A,C)$ and $SSF^k(B,C) \geq SSF^k(A,C)$. □

Taking a real-valued weight of x_j, we propose to consider the following weighted similarity measures based on the sine function between CFSs A and B:

$$WSSF^1(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \rho_j \sin\Big\{\frac{\pi}{2}\big[|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$
$$WSSF^2(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \rho_j \sin\Big\{\frac{\pi}{4}\big[|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_n)^T$ and ρ_j is the weight vector of x_j $(j = 1, 2, \ldots, n)$, $\rho_j \in [0,1]$, and $\sum_{j=1}^{n} \rho_j = 1$. If we take $\rho_j = 1/n$, $j = 1, 2, \ldots, n$, then $WSSF^k(A,B) = SSF^k(A,B)$, $k = 1, 2$.

Proposition 2. *For two CFSs A and B in $X = \{x_i, x_2, \ldots, x_n\}$, the weighted similarity measures based on the sine function $SSF^k(A,B)$, $k = 1, 2$, have the following properties*:

(1) $0 \leq WSSF^k(A,B) \leq 1$;

(2) $WSSF^k(A,B) = WSSF^k(B,A)$;

(3) $WSSF^k(A,B) = 1$ *if and only if* $A = B$;

(4) *if C is a CFS in X and $A \subseteq B \subseteq C$, then $WSSF^k(A,B) \geq WSSF^k(A,C)$ and $WSSF^k(B,C) \geq WSSF^k(A,C)$.*

P r o o f. Similarly to the previous proof methods, we can prove the above four properties. □

3.2. Similarity measures based on the tangent function

Let
$$A = \big\{(x_j, r_A(x_j) \cdot e^{i\omega_A(x_j)})\big\}, \quad B = \big\{(x_j, r_B(x_j) \cdot e^{i\omega_B(x_j)})\big\}$$
be two CFSs in $X = \{x_1, x_2, \ldots, x_n\}$, $x_j \in X$. Then the similarity measures based on the tangent function between A and B are defined as follows:

$$STF^1(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \tan\Big\{\frac{\pi}{4}\big[|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$
$$STF^2(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \tan\Big\{\frac{\pi}{8}\big[|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\}.$$

Proposition 3. *For two CFSs A and B in $X = \{x_i, x_2, \ldots, x_n\}$, the similarity measures $STF^k(A,B)$, $k = 1, 2$, have the following properties*:

(1) $0 \leq STF^k(A,B) \leq 1$;

(2) $STF^k(A,B) = STF^k(B,A)$;

(3) $STF^k(A,B) = 1$ *if and only if* $A = B$;

(4) *if C is a CFS in X and $A \subseteq B \subseteq C$, then $STF^k(A,B) \geq STF^k(A,C)$ and $STF^k(B,C) \geq STF^k(A,C)$.*

P r o o f. The proofs are similar to the proofs for the similarity measures based on the sine function. ☐

Taking a real valued weight of x_j, we propose to consider the following weighted similarity measures based on the tangent function between CFSs A and B:

$$WSTF^1(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \rho_j \sin\Big\{\frac{\pi}{2}\big[|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$
$$WSTF^2(A,B) = 1 - \frac{1}{n}\sum_{j=i}^{n} \rho_j \sin\Big\{\frac{\pi}{4}\big[|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_n,)^T$ and ρ_j is the weight vector of x_j $(j = 1, 2, \ldots, n)$, $\rho_j \in [0,1]$, $\sum_{j=1}^{n} \rho_j = 1$. If we take $\rho_j = 1/n$, $j = 1, 2, \ldots, n$, then $WSTF^k(A,B) = STF^k(A,B)$, $k = 1, 2$.

Proposition 4. *For two CFSs A and B in $X = \{x_i, x_2, \ldots, x_n\}$, the weighted similarity measures based on the tangent function $WTSF^k(A,B)$, $k = 1, 2$, have the following properties*:

(1) $0 \leq WSTF^k(A,B) \leq 1$;

(2) $WSTF^k(A,B) = WSTF^k(B,A)$;

(3) $WSTF^k(A,B) = 1$ *if and only if* $A = B$;

(4) *if C is a CFS in X and $A \subseteq B \subseteq C$, then $WSTF^k(A,B) \geq WSTF^k(A,C)$ and $WSTF^k(B,C) \geq WSTF^k(A,C)$.*

P r o o f. The proofs are similar to the proofs for the similarity measures based on the sine function. ☐

3.3. Similarity measures based on the cosine function

Let

$$A = \big\{(x_j, r_A(x_j) \cdot e^{i\omega_A(x_j)})\big\}, \quad B = \big\{(x_j, r_B(x_j) \cdot e^{i\omega_B(x_j)})\big\}$$

be two CFSs in $X = \{x_1, x_2, \ldots, x_n\}$, $x_j \in X$. We define two similarity measures between A and B based on the cosine function as follows:

$$SCF^1 = \frac{1}{n}\sum_{j=i}^{n} \cos\Big\{\frac{\pi}{2}\big[|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\},$$
$$SCF^2 = \frac{1}{n}\sum_{j=i}^{n} \cos\Big\{\frac{\pi}{4}\big[|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\big]\Big\}.$$

Proposition 5. *For two CFSs A and B in $X = \{x_i, x_2, \ldots, x_n\}$, the similarity measures $SCF^k(A,B)$, $k = 1, 2$, have the following properties*:

(1) $0 \leq SCF^k(A,B) \leq 1$;

(2) $SCF^k(A, B) = SCF^k(B, A)$;

(3) $SCF^k(A, B) = 1$ *if and only if* $A = B$;

(4) *if* C *is a CFS in* X *and* $A \subseteq B \subseteq C$, *then* $SCF^k(A, C) \leq SCF^k(A, B)$ *and* $SCF^k(A, C) \leq SCF^k(B, C)$.

P r o o f. The proofs of properties (1)–(3) are trivial. Let us prove property (4).

If $A \subseteq B \subseteq C$, then $r_A(x_j) \leq r_B(x_j) \leq r_C(x_j)$ and $\omega_A(x_j) \leq \omega_B(x_j) \leq \omega_C(x_j)$ for $j = 1, 2, \ldots, n$. Then, we have

$$|r_A(x_j) - r_B(x_j)| \leq |r_A(x_j) - r_C(x_j)|,$$
$$|r_B(x_j) - r_C(x_j)| \leq |r_A(x_j) - r_C(x_j)|,$$

and

$$|\omega_A(x_j) - \omega_B(x_j)| \leq |\omega_A(x_j) - \omega_C(x_j)|,$$
$$|\omega_B(x_j) - r_C(x_j)| \leq |\omega_A(x_j) - \omega_C(x_j)|.$$

Hence, $SCF^k(A, C) \leq SCF^k(A, B)$ and $SCF^k(A, C) \leq SCF^k(B, C)$ for $k = 1, 2$. □

Taking a real weight of x_j, we define the weighted similarity measures based on the cosine function as follows:

$$WSCF^1 = \sum_{j=i}^{n} \rho_j \cos\left\{\frac{\pi}{2}\left[|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\right]\right\},$$
$$WSCF^2 = \sum_{j=i}^{n} \rho_j \cos\left\{\frac{\pi}{4}\left[|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\right]\right\},$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_n,)^T$ and ρ_j is the weight vector of x_j $(j = 1, 2, \ldots, n)$,$\rho_j \in [0, 1]$, $\sum_{j=1}^{n} \rho_j = 1$. If we take $\rho_j = 1/n$, $j = 1, 2, \ldots, n$, then $WSCF^k(A, B) = SCF^k(A, B)$, $k = 1, 2$.

It is clear that the weighted similarity measures based on the cosine function between CFSs A and B also satisfy the following statement.

Proposition 6. *For two CFSs* A *and* B *in* $X = \{x_i, x_2, \ldots, x_n\}$, *the similarity measures* $WSCF^k(A, B)$, $k = 1, 2$, *have the following properties*:

(1) $0 \leq WSCF^k(A, B) \leq 1$;

(2) $WSCF^k(A, B) = WSCF^k(B, A)$;

(3) $WSCF^k(A, B) = 1$ *if and only if* $A = B$;

(4) *if* C *is a CFS in* X *and* $A \subseteq B \subseteq C$, *then* $WSCF^k(A, C) \leq WSCF^k(A, B)$ *and* $WSCF^k(A, C) \leq WSCF^k(B, C)$.

We can prove (1)–(4) using methods similar to the above proofs.

3.4. Similarity measures based on the cotangent function

For any two CFSs A and B, the similarity measure between A and B based on the cotangent function is defined as follows:

$$SCTF^1 = \frac{1}{n}\sum_{j=i}^{n} \cot\left[\frac{\pi}{4} + \frac{\pi}{4}(|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\right].$$

Taking a real weight of x_j, we define the weighted similarity measure based on the cotangent function between two CFSs A and B as follows:

$$WSCTF^1 = \sum_{j=i}^{n} \rho_j \cot\left[\frac{\pi}{4} + \frac{\pi}{4}(|r_A(x_j) - r_B(x_j)| \vee \frac{1}{2\pi}(|\omega_A(x_j) - \omega_B(x_j)|)\right],$$

where $\rho_j = (\rho_1, \rho_2, \ldots, \rho_n,)^T$ and ρ_j is the weight vector of x_j $(j = 1, 2, \ldots, n)$, $\rho_j \in [0, 1]$, $\sum_{j=1}^{n} \rho_j = 1$. If we take $\rho_j = 1/n$, $j = 1, 2, \ldots, n$, then $WSCTF^1(A, B) = SCTF^1(A, B)$.

4. Application of the proposed similarity measures in the pattern recognition problem

4.1. Application in pattern recognition

Pattern recognition is one of the most essential decision-making skills in problems of choice. It consists in finding an appropriate pattern from some unknown patterns. Pattern recognition with fuzzy data is becoming increasingly popular and important in research in medicine, engineering, computer science, psychology, and physiology, among others. However, in many cases of pattern recognition, such as digital images, speech, audio signals, voice, and language, among others, we face some problems due to the dual characteristics of an element. In this case, pattern recognition with complex fuzzy data is more suitable for describing such situations. In this section, we describe the problem of pattern recognition using our proposed similarity measures.

In general, we can formulate the pattern recognition problem in complex fuzzy sets as follows.

Problem Formulation. Let

$$B = \left\{(x_j, r_B(x_j) \cdot e^{i\omega_B(x_j)}) :\ x_j \in X,\ j = 1, 2, \ldots, n\right\}$$

be an ideal pattern characterized by a complex fuzzy set in $X = \{x_1, x_2, \ldots x_n\}$. Let $\{A_1, A_2, \ldots, A_m\}$ be some sample patterns characterized by complex fuzzy sets in $X = \{x_1, x_2, \ldots, x_n\}$ as follows:

$$A_k = \left\{(x_j, r_{A_k}(x_j) \cdot e^{i\omega_{A_k}(x_j)}) :\ x_j \in X,\ j = 1,\ 2 \ldots, n\right\}, \quad k = 1, 2, \ldots, m.$$

Aim. Determine which sample pattern is close to ideal.

Solution. The sample patterns A_k, $k = 1, 2 \ldots, m$, should be close to the ideal pattern B, which has the maximum similarity.

Now, we illustrate the performance of our proposed similarity measures with the help of a practical example. First, we describe our proposed unweighted similarity measures of an element of the universe of discourse. Second, we depict the weighted similarity measures.

Table 1. The representation of sample signals and an ideal signal by complex fuzzy sets.

	A_1	A_2	A_3	A_4	B
x_1	$0.5e^{i2\pi(0.3)}$	$0.8e^{i2\pi(0.2)}$	$0.6e^{i2\pi(0.4)}$	$0.7e^{i2\pi(0.6)}$	$0.8e^{i2\pi(0.9)}$
x_2	$0.4e^{i2\pi(0.5)}$	$0.7e^{i2\pi(0.4)}$	$0.7e^{i2\pi(0.6)}$	$0.1e^{i2\pi(0.2)}$	$0.6e^{i2\pi(0.5)}$
x_3	$0.2e^{i2\pi(0.1)}$	$0.1e^{i2\pi(0.4)}$	$0.5e^{i2\pi(0.6)}$	$0.9e^{i2\pi(0.2)}$	$0.6e^{i2\pi(0.3)}$

Table 2. The similarity measures between the sample signals and the ideal signal.

	(A_1, B)	(A_2, B)	(A_3, B)	(A_4, B)
$SSF^1(A_j, B)$	0.4316	0.4153	**0.5610**	0.4618
$SSF^2(A_j, B)$	0.5802	0.6225	**0.7685**	0.7169
$STF^1(A_j, B)$	0.6692	0.6316	**0.7558**	0.7020
$STF^2(A_j, B)$	0.7731	0.8056	**0.8827**	0.8180
$SCF^1(A_j, B)$	0.7828	0.7165	**0.8620**	0.8298
$SCF^2(A_j, B)$	0.8798	0.9105	**0.9644**	0.9562
$SCTF^1(A_j, B)$	0.5210	0.5035	**0.6277**	0.5473
$SCTF^2(A_j, B)$	0.6433	0.6763	**0.7142**	0.6549

Example 1. In this example, we practice our proposed similarity measures in audio signal processing. Suppose we have four sample audio signals $A = \{A_1, A_2, A_3, A_4\}$ and an ideal audio signal B that are characterized by complex fuzzy sets given in Table 1. Here $X = \{x_1, x_2, x_3\}$ represents different points on the signals and $r_{A_k}(x_j)$ $(j = 1, 2, 3,\ k = 1, 2, 3, 4)$ and $r_B(x_j)$ $(j = 1, 2, 3)$ represent the frequency of the sample signals and the ideal signal, respectively. The terms $\omega_{A_k}(x_j)$ $(j = 1, 2, 3,\ k = 1, 2, 3, 4)$ and $\omega_B(x_j)$ $(j = 1, 2, 3)$ represent the amplitude of the sample signals and the ideal signal, respectively. We aim to detect which sample signal is close to the ideal signal. To determine this, we use our proposed similarity measures. The results are given in Table 2. From the numerical results in Table 2, we observe that the signal A_3 is the closest to the signal B.

Again, we consider, the real valued weight vector of $x_j (j = 1, 2, 3)$ is $\rho = (0.30, 0.34, 0.36)^T$. Then by using table 1 and methods of weighted similarity measures of complex fuzzy sets, we obtain the results of weighted similarity measures between sample audio signals and ideal audio signal given in Table 3. From the numerical results in Table 3, we also observe that the signal A_3 is the closest to the signal B.

4.2. Comparison studies

As known from the literature, there is only a similarity measure, namely the cosine similarity measure [14], for CFSs, in which the range of similarity measurement value is from -1 to 1. However, the similarity measurement value in our proposed methods ranges from 0 to 1. So, for comparing the performance of our proposed methods, we consider some existing distance methods of CFSs proposed in [1, 20].

1. Applying the distance measure denoted by d_1 and using equation (4) from [1], we get the

Table 3. The weighted similarity measures between the sample signals and the ideal signal.

	(A_1, B)	(A_2, B)	(A_3, B)	(A_4, B)
$WSSF^1(A_j, B)$	0.4408	0.4251	**0.5714**	0.4601
$WSSF^2(A_j, B)$	0.5887	0.6268	**0.7745**	0.7159
$WSTF^1(A_j, B)$	0.6765	0.6404	**0.7626**	0.7008
$WSTF^2(A_j, B)$	0.7787	0.8079	**0.8858**	0.8144
$WSCF^1(A_j, B)$	0.7911	0.7262	**0.8688**	0.8286
$WSCF^2(A_j, B)$	0.8848	0.9124	**0.9662**	0.9559
$WSCTF^1(A_j, B)$	0.5286	0.5122	**0.6359**	0.5459
$WSCTF^2(A_j, B)$	0.6500	0.6797	**0.7206**	0.6534

following measurement value for each sample signal $\{A_1, A_2, \ldots, A_m\}$ compared with the ideal signal B:

$$d_1(A_1, B) = 0.85, \quad d_1(A_2, B) = 0.75, \quad d_1(A_3, B) = 0.65, \quad d_1(A_4, B) = 0.8.$$

The minimum value of $d_1(A_j, B)$ is considered the best alternative. Since the measurement value of A_3 is the minimum among all these values, we conclude that A_3 is the closest to B.

2. Utilizing the complex fuzzy weighted discrimination measures denoted by wd_1 as defined in [20], we get the following measurement values by using Table 1 and the real-valued weight vector $\rho = (0.30, 0.34, 0.36)^T$ of $x_j (j = 1, 2, 3)$:

$$wd_1(A_1, B) = 0.24, \quad wd_1(A_2, B) = 0.27, \quad wd_1(A_3, B) = 0.15, \quad wd_1(A_4, B) = 0.19.$$

In this case, the minimum value of $wd_1(A_j, B)$ is also considered the best alternative. Since the measurement value of A_3 is the minimum among all these values, we again conclude that A_3 is the closest to B.

4.3. Advantages of our proposed methods

From the study of existing literature and our proposed methods, we address the following advantages of our proposed methods to apply in different branches of science and engineering.

1. A CFS is an extension of FS considering two-dimensional information, such as the amplitude term and the phase term in a single element, whereas a real FS contains only the amplitude term in a single element. So the primary advantage of our proposed methods is capturing more information about an element when uncertainty arises in the case of decision-making, pattern recognition, image processing, signal processing, audio recognition, and others.

2. It is disclosed from our study that some trigonometric similarity measures under the CFSs are particular forms of trigonometric similarity measures of real FSs, so we can use these similarity measures to solve many problems where a one-dimensional term presents in a single element by taking the phase term zero.

5. Conclusion

In our present study, an endeavor has been taken to develop some trigonometric similarity measures under the CFSs environment. Different kinds of similarity measures have been defined in the FS environment where the membership degree of an element is a subset of real numbers. But in our proposed methods, the membership degree of an element is taken as a two-dimensional value which enables describing the uncertainty of an element more precisely. On the other hand, in the existing cosine similarity measure of CFS, which was introduced in [14], the range of similarity measurement values was taken from -1 to 1, but the similarity measurement value should range from 0 till 1. In our proposed methods, we also develop this limitation. In the future, we will extend our methods to an interval-valued complex fuzzy set, complex intuitionistic fuzzy set, complex Pythagorean fuzzy set, complex picture fuzzy set, and so forth. Also, other similarity measures for CFSs will be considered.

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