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AN $M^{[X]}/G/1$ QUEUE WITH OPTIONAL SERVICE AND WORKING BREAKDOWN

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Abstract: In this study, a batch arrival single service queue with two stages of service (second stage is optional) and working breakdown is investigated. When the system is in operation, it may breakdown at any time. During breakdown period, instead of terminating the service totally, it continues at a slower rate. We find the time-dependent probability generating functions in terms of their Laplace transforms and derive explicitly the corresponding steady state results. Furthermore, numerous measures indicating system performances, such as the average queue size and the average queue waiting time, has been obtained. Some of the numerical results and graphical representations were also presented.

Keywords: Non-Markovian queue, Second optional service, Working breakdown.

1. Introduction

Queueing theory refer to the study of people, their object and movement in line. It is working to create a well balanced system that serves customers faster and works efficiently without being too expensive. Queueing is widely performed to analyze and streamline staffing needs, scheduling, and inventory in order to enhance overall customer service. The system may have either a limited or an unlimited capacity for holding customers. The sources from which the customers come may be finite or infinite. Queueing models with a second optional service imply that all arriving consumers will receive the first essential service, while only few will request the second optional service.

Yang and Chen [17] examined $M/M/1$ queueing system which has optional service. The server is assumed to malfunction. Together they derived the condition at which the stability is obtained and have found the probability stationary distribution using Matrix geometric method. A queueing model $M^{[X]}/G/1$ queue with two phases of service was presented by Maragathasundari and Srinivasan [10]. In their study they have clearly analysed the steady state results and some performance measures. Finally, they demonstrated some good applications related to the model such as large scale industrial production lines which also include computer communication networks.

Second optional service with general service time distribution was studied by Al-Jararha and Madan [2]. They used the supplementary variable technique to study the model with respect to both the first essential and second optional service. They consider service time to follow general distribution. Madan [9] proposed the concept of a single server queue with a second optional service and furnished its real time applications.

Choudary and Paul [4] discussed an $M^{[X]}/G/1$ queueing system with a second optional service channel under N-policy. Only when minimum N customers are present in queue, server starts serving present customers in the queue which is stated as N-policy. They found the queue size distribution at random epoch and departure epoch.

Maragathasundari and Srinivasan [11] discussed a non-Markovian queueing model with multi-stage of service. The numerical results of this model have been presented in graphical form, and they have also discussed the practical large-scale industrial applications.

Thangaraj and Vanitha [16] investigated an $M/G/1$ queue with two-stage heterogeneous service and random breakdowns. They have modelled the queueing system that could unexpectedly fail, causing the server to stop operating until the system is fixed. Gupta et al. [6] studied the steady-state behaviour of the $M^{[X]}/G/1$ with server breakdown. Customers will arrive to the system in varied sizes of batches, but will be served one by one, according to this study. The repair process does not begin immediately after a breakdown, and there is a time delay for repairs to start. Choudary and Tadj [5] analysed an $M/G/1$ queue with two service phases that was subject to server failure and delayed repair. An $M^{[X]}/G/1$ queue with second optional service and server breakdown was explored by Singh and Kaur [15]. Their study has numerous applications in everyday life, including tremendous utility for system designers and managements.

Santhi [14] developed a single server retrial queue with a second optional service and working vacation, assuming that there is no available waiting space for an arriving client. They can abandon the service area and join an orbit consisting of a pool of blocked clients. Rajadurai et al. [13] looked into an $M/G/1$ feedback retrial queue that was subjected to server breakdown, repair and multiple working vacations. To investigate the system's impact they presented a cost optimization analysis. This model is a generalised version of a number of current queueing models.

Working breakdown is very common in many manufacturing industries and production process. In queueing models, it is the most used parameter. Kalidass and Kasturi [7] have studied a queueing model with working breakdown. According to their research, if a server crashes at a certain point but doesn't completely shut down, the server continues to run at a slower speed. Kim and Lee [8] have analysed an $M/G/1$ queueing system with disaster and working breakdown. This study presents an extension of the queueing system and results may provide a better decision making for many practical system. Yang et al. [18] presented a two-server queue with multiple vacations and working breakdowns. They have used Matrix-geometric method to obtain the steady state probabilities and performance measures.

Rajadurai [12] recently investigated a retrial queueing system with several features. One of his assumptions is that disaster causes all clients to exit the system, and at the same time the main server fails. After that, the main server is sent to the repair station, and the repair process begins immediately. Finally, cost optimization analysis and some numerical results are presented. Ammar et al. [3] analysed the preemptive priority retrial queueing system with disaster under working breakdown. This model has some good applications in computer processing systems. The inclusion of a preemptive priority retrial queueing system in the presence of working breakdown services is a unique feature of this study. The optimization analysis of the N-policy $M/G/1$ queue with working breakdown was discussed by Yen et al. [19]. They have illustrated the effectiveness of the two-stage optimization model in this study, as well as some numerical results have been shown. Ayyappan et al. [1] studied a single server queue which serves two classes of customers under non-preemptive priority services, working breakdown, Bernoulli vacation, admission and balking.

Our model is potentially applicable to cellular networks, as we know that in cellular network each cell has a base station that controls the call admissions and the quality of service of the network. If we want to model the base station properly and adequately, we should consider the possibility of many users (customers) accessing the internet on their mobiles at the same time. Thus the services provided by the base station controller is required. As any other electronic component, the server is also exposed to risks due to external shocks, and therefore subject to breakdowns. At the same time, the services to mobile users are very important. Hence, the service providers cannot afford full interruptions in their services leading to backup servers being relied upon to provide services at reduced rates whenever the main sever is under repair.

2. Mathematical model description

The following assumptions for this model are:

- Customers enter the system in batches of varying sizes according to compound Poisson process with rate λ, and they are served one by one under 'first come-first served' basis.
- Let $\lambda c_i (i = 1, 2, 3, ...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival.
- The first essential service is required by all arriving customers, and its distribution function and density function are $B_1(x)$ and $b_1(x)$ respectively.
- Let $\mu_1(x)dx$ be the conditional probability of completion of the first essential service during $(x, x + dx]$, given that the elapsed service time is x. Then

$$\mu_1(x) = \frac{b_1(x)}{1 - B_1(x)},$$

  and therefore $b_1(v) = \mu_1(v)e^{-\int_0^v \mu_1(x)dx}$.
- When a customer's first essential service is completely finished, the customer opts for the II-optional service with probability p and this optional service will immediately start. Otherwise, with probability $(1 - p)$ they may decide to exit the system, in which case a new customer (if any) is picked for their first essential service from the head of the queue.
- The second optional service time is also assumed to follow the general distribution, with distribution function density function as $B_2(x)$ and $b_2(x)$ respectively.
- Let $\mu_2(x)dx$ be the conditional probability of completion of the II-optional service during $(x, x + dx]$, given that the elapsed service time is x. Then

$$\mu_2(x) = \frac{b_2(x)}{1 - B_2(x)},$$

  and therefore $b_2(v) = \mu_2(v)e^{-\int_0^v \mu_2(x)dx}$.
- When servicing a customer at first stage or second stage, the system may get breakdown and the breakdown times are supposed to occur under Poisson process with parameter α.
- After breakdown, instead of stopping the service completely, the server will complete the current service at a slower rates $\beta_1(x)$ and $\beta_2(x)$ for first essential service and optional service respectively.
- The working breakdown service (both essential and optional) time is also assumed to follow the general distribution, with distribution function density function are $Q_i(x)$ and $q_i(x)$, $i = 1, 2$ respectively. Then

$$\beta_i(x) = \frac{q_i(x)}{1 - Q_i(x)}$$

  and therefore $q_i(v) = \beta_i(v)e^{-\int_0^v \beta_i(x)dx}, \quad i = 1, 2.$
- On completion of current service at a slower rate, the server is sent to repair. The repair time follows general distribution with the rate of $\eta(x)$.

- Meanwhile after the repair, when the server returns to the system and when there are no customers throughout the system, the server remains in the idle state and waits for the customers to arrive.
- Various stochastic process taking part in the system are considered to be independent of each other.

The structure of the system representation in Fig. 1.

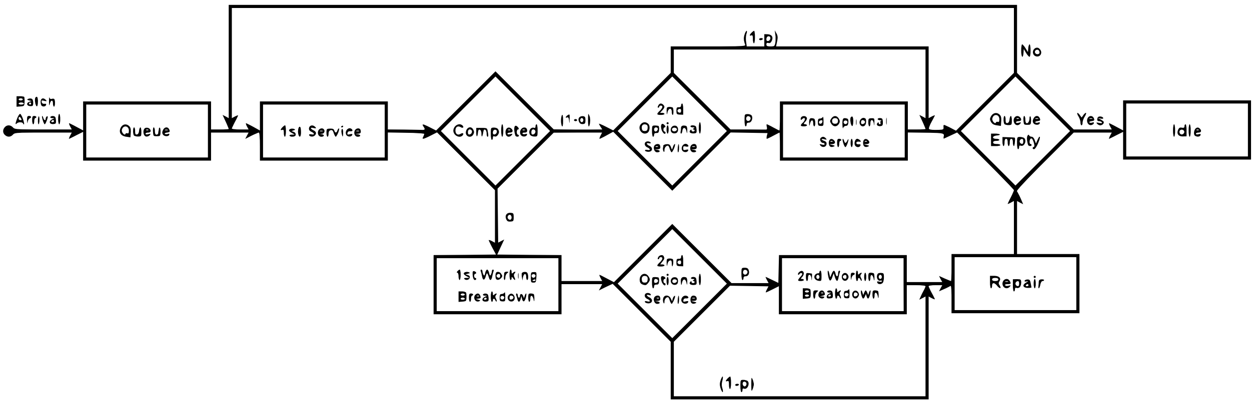

Figure 1. Diagrammatic representation of this model.

3. Notations and equations governing the system

Let $\sigma(t)$ denotes the server state: $N_s(t)$ denotes the number of customers in the service station, $N_q(t)$ denotes the number of customers in the queue.

Notations	Meaning
$P_n^{(1)}(x,t)$	$\sigma(t)$ = first essential service, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, with elapsed service duration x at time t
$P_n^{(1)}(t)=\int_{x=0}^{\infty} P_n^{(1)}(x,t)dx$	$\sigma(t)$ = first essential service, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, irrespective of the value of x
$P_n^{(2)}(x,t)$	$\sigma(t)$ = second optional service, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, with elapsed service duration x at time t
$P_n^{(2)}(t)=\int_{x=0}^{\infty} P_n^{(2)}(x,t)dx$	$\sigma(t)$ = second optional service, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, irrespective of the value of x
$Q_n^{(1)}(x,t)$	$\sigma(t)$ = first essential service at slower rate, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, with elapsed service duration x at time t
$Q_n^{(1)}(t)=\int_{x=0}^{\infty} Q_n^{(1)}(x,t)dx$	$\sigma(t)$ = first essential service at slower rate, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, irrespective of the value of x

$Q_n^{(2)}(x,t)$	$\sigma(t)=$ second optional service at slower rate, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, with elapsed service duration x at time t
$Q_n^{(2)}(t)=\int_{x=0}^{\infty}Q_n^{(2)}(x,t)dx$	$\sigma(t)=$ second optional service at slower rate, $N_s(t)=1$ and $N_q(t)=n(\geq 0)$, irrespective of the value of x
$R_n(x,t)$	$\sigma(t)=$ repair, $N_q(t)=n(\geq 0)$, with elapsed repair duration x at time t
$R_n(t)=\int_{x=0}^{\infty}R_n(x,t)dx$	$\sigma(t)=$ repair, $N_q(t)=n(\geq 0)$, irrespective of the value of x
$I(t)$	$\sigma(t)=$ Idle, $N_q(t)=0$ at time t

The Kolmogorov forward equations to govern the model are the following:

$$\begin{aligned}&\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)\\&=-(\lambda+\mu_1(x)+\alpha)P_n^{(1)}(x,t)+(1-\delta_{0n})\lambda\sum_{i=1}^{n}c_iP_{n-i}^{(1)}(x,t),\quad n=1,2,...,\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)\\&=-(\lambda+\mu_2(x)+\alpha)P_n^{(2)}(x,t)+(1-\delta_{0n})\lambda\sum_{i=1}^{n}c_iP_{n-i}^{(2)}(x,t),\quad n=1,2,...,\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\frac{\partial}{\partial x}Q_n^{(1)}(x,t)+\frac{\partial}{\partial t}Q_n^{(1)}(x,t)\\&=-(\lambda+\beta_1(x))Q_n^{(1)}(x,t)+(1-\delta_{0n})\lambda\sum_{i=1}^{n}c_iQ_{n-i}^{(1)}(x,t),\quad n=1,2,...,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\frac{\partial}{\partial x}Q_n^{(2)}(x,t)+\frac{\partial}{\partial t}Q_n^{(2)}(x,t)\\&=-(\lambda+\beta_2(x))Q_n^{(2)}(x,t)+(1-\delta_{0n})\lambda\sum_{i=1}^{n}c_iQ_{n-i}^{(2)}(x,t),\quad n=1,2,...,\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\frac{\partial}{\partial x}R_n(x,t)+\frac{\partial}{\partial t}R_n(x,t)\\&=-(\lambda+\eta(x))R_n(x,t)+(1-\delta_{0n})\lambda\sum_{i=1}^{n}c_iR_{n-i}(x,t),\quad n=1,2,...,\end{aligned}\tag{3.5}$$

$$\begin{aligned}\frac{d}{dt}I(t)=-\lambda I(t)+(1-p)\int_0^{\infty}P_0^{(1)}(x,t)\mu_1(x)dx+\int_0^{\infty}P_0^{(2)}(x,t)\mu_2(x)dx\\+\int_0^{\infty}R_0(x,t)\eta(x)dx,\quad n=1,2,\ldots.\end{aligned}\tag{3.6}$$

Equations (3.1) to (3.6) are to be solved subject to the following boundary conditions at $x=0$.

$$\begin{aligned}P_n^{(1)}(0,t)=\lambda c_{n+1}I(t)+(1-p)\int_0^{\infty}P_{n+1}^{(1)}(x,t)\mu_1(x)dx+\int_0^{\infty}P_{n+1}^{(2)}(x,t)\mu_1(x)dx\\+\int_0^{\infty}R_{n+1}\eta(x)(x,t)\gamma(x)dx,\end{aligned}\tag{3.7}$$

$$P_n^{(2)}(0,t) = p\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, \tag{3.8}$$

$$Q_n^{(1)}(0,t) = \alpha\int_0^\infty P_n^{(1)}(x,t)dx, \tag{3.9}$$

$$Q_n^{(2)}(0,t) = \alpha\int_0^\infty P_n^{(2)}(x,t)dx + p\int_0^\infty Q_n^{(1)}(x,t)\beta_1(x)dx, \tag{3.10}$$

$$R_n(0,t) = (1-p)\int_0^\infty Q_n^{(1)}(x,t)\beta_1(x)dx + \int_0^\infty Q_n^{(2)}(x,t)\beta_2(x)dx. \tag{3.11}$$

The initial conditions are

$$I(0) = 1, \quad P^{(1)}(0) = P^{(2)}(0) = Q^{(1)}(0) = Q^{(2)}(0) = R(0) = 0. \tag{3.12}$$

4. Generating functions of the queue length: the time-dependent solution

We define the probability generating functions,

$$A_q(x,z,t) = \sum_{n=0}^{\infty} z^n A_n(x,t), \quad C(z) = \sum_{n=1}^{\infty} z^n c_n(t).$$

Here $A = P^{(1)}, P^{(2)}, Q^{(1)}, Q^{(2)}, R$ which are convergent inside the circle given by $|z| \leq 1$. By taking Laplace transform from equations from (3.1) to (3.11) and solving those equations we get,

$$\bar{P}_q^{(1)}(x,z,s) = \bar{P}_q^{(1)}(0,z,s)e^{-(s+\lambda(1-C(z)+\alpha)x - \int_0^x \mu_1(t)dt}, \tag{4.1}$$

$$\bar{P}_q^{(2)}(x,z,s) = \bar{P}_q^{(2)}(0,z,s)e^{-(s+\lambda(1-C(z)+\alpha)x - \int_0^x \mu_2(t)dt}, \tag{4.2}$$

$$\bar{Q}_q^{(1)}(x,z,s) = \bar{Q}_q^{(1)}(0,z,s)e^{-(s+\lambda(1-C(z))x - \int_0^x \beta_1(t)dt}, \tag{4.3}$$

$$\bar{Q}_q^{(2)}(x,z,s) = \bar{Q}_q^{(2)}(0,z,s)e^{-(s+\lambda(1-C(z))x - \int_0^x \beta_2(t)dt}, \tag{4.4}$$

$$\bar{R}_q(x,z,s) = \bar{R}_q(0,z,s)e^{-(s+\lambda(1-C(z))x - \int_0^x \eta(t)dt}. \tag{4.5}$$

Again on integrating equations from (4.1) to (4.5) by parts with respect to x we get

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(f(z))}{[f(z)]}\right], \tag{4.6}$$

$$\bar{P}_q^{(2)}(z,s) = \bar{P}_q^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right], \tag{4.7}$$

$$\bar{Q}_q^{(1)}(z,s) = \bar{Q}_q^{(1)}(0,z,s)\left[\frac{1-\bar{Q}_1[g(z)]}{[g(z)]}\right], \tag{4.8}$$

$$\bar{Q}_q^{(2)}(z,s) = \bar{Q}_q^{(2)}(0,z,s)\left[\frac{1-\bar{Q}_2[g(z)]}{[g(z)]}\right], \tag{4.9}$$

$$\bar{R}_q(z,s) = \bar{R}_q(0,z,s)\left[\frac{1-\bar{R}[g(z)]}{[g(z)]}\right], \tag{4.10}$$

where

$$f(z) = s + \lambda(1 - C(z)) + \alpha, \quad g(z) = s + \lambda(1 - C(z)).$$

Now multiplying both sides of equation (4.1) by $\mu_1(x)$ and equation (4.2) by $\mu_2(x)$ and equation (4.3) by $\beta_1(x)$, equation (4.4) by $\beta_2(x)$ and equation (4.5) by $\eta(x)$ and then integrating over x we obtain

$$\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}_q^{(1)}(0,z,s)\bar{B}_1[f(z)], \tag{4.11}$$

$$\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}_q^{(2)}(0,z,s)\bar{B}_2[f(z)], \tag{4.12}$$

$$\int_0^\infty \bar{Q}_q^{(1)}(x,z,s)\beta_1(x)dx = \bar{Q}_q^{(1)}(0,z,s)\bar{Q}_1[g(z)], \tag{4.13}$$

$$\int_0^\infty \bar{Q}_q^{(2)}(x,z,s)\beta_2(x)dx = \bar{Q}_q^{(2)}(0,z,s)\bar{Q}_2[g(z)], \tag{4.14}$$

$$\int_0^\infty \bar{R}_q(x,z,s)\eta(x) = \bar{R}_q(0,z,s)\bar{R}[g(z)]. \tag{4.15}$$

Using equations (4.11) and (3.8), we get

$$\bar{P}_q^{(2)}(0,z,s) = p[\bar{P}_q^{(1)}(0,z,s)\bar{B}_1[f(z)]].$$

Using equations (4.6) and (3.9), we get

$$\bar{Q}_q^{(2)}(0,z,s) = \alpha\bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right].$$

Performing similar operation in equations (3.10) and (3.11), we obtain

$$\bar{Q}_q^{(1)}(0,z,s) = \alpha p\bar{P}_q^{(1)}(0,z,s)\left\{\bar{B}_1[f(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right] + \bar{Q}_1[g(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right]\right\},$$

$$\begin{aligned}\bar{R}_q(0,z,s) = \bar{P}_q^{(1)}(0,z,s)\Bigg\{&(1-p)\alpha\bar{Q}_1[g(z)]\left[\frac{1-\bar{B}_1}{[f(z)]}\right] + \alpha p\bar{B}_1[f(z)]\bar{Q}_2[g(z)]\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right]\\ &+\alpha p\bar{Q}_1[f(z)]\bar{Q}_2[g(z)]\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right]\Bigg\}.\end{aligned}$$

Using equations (4.11), (4.12) and (4.15), to solve $\bar{P}_q^{(1)}(0,z,s)$

$$\bar{P}_q^{(1)}(0,z,s) = \frac{1-[g(z)]\bar{I}(s))}{\left\{\begin{array}{c} z-(1-p)\bar{B}_1[f(z)]+p\bar{B}_2[f(z)]\bar{B}_2[g(z)]+(1-p)\alpha\bar{Q}_1[g(z)] \\ \left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right] + \alpha p\bar{B}_1[f(z)]\bar{Q}_2[g(z)]\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right] \\ +\alpha p\bar{Q}_1[g(z)]\bar{Q}_2[g(z)]\bar{R}[g(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right] \end{array}\right\}}.$$

We see that equations (4.6) to (4.10) become to be as follows

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right], \tag{4.16}$$

$$\bar{P}_q^{(2)}(z,s) = p\bar{P}_q^{(1)}(0,z,s)\bar{B}_1[f(z)]\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right], \tag{4.17}$$

$$\bar{Q}_q^{(1)}(z,s) = \alpha\bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right]\left[\frac{1-\bar{Q}_1[g(x)]}{[g(x)]}\right], \tag{4.18}$$

$$\begin{aligned}\bar{Q}_q^{(2)}(z,s) = {}&\alpha p\bar{P}_q^{(1)}(0,z,s)\bar{B}_1[f(z)]\left[\frac{1-\bar{B}_2[f(z)]}{f(z)}\right]\left[\frac{1-\bar{Q}_2[g(z)]}{[g(z)]}\right]\\ &+\alpha p\bar{P}_q^{(1)}(z,0)\bar{Q}_1[g(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right]\left[\frac{1-\bar{Q}_2[g(z)]}{[g(z)]}\right],\end{aligned} \tag{4.19}$$

$$\bar{R}_q(z,s) = \bar{P}_q(0,z,s)\left\{\alpha(1-p)\bar{Q}_1[g(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right]\left[\frac{1-\bar{R}[g(z)]}{[g(z)]}\right]\right.$$
$$+\alpha p\bar{P}_q(z,0)\bar{B}_1[f(z)]\bar{Q}_2[g(z)]\left[\frac{1-\bar{B}_2[f(z)]}{[f(z)]}\right]\left[\frac{1-\bar{R}[g(z)]}{[g(z)]}\right] \tag{4.20}$$
$$\left.+\alpha p\bar{P}_q(z,0)\bar{Q}_1[g(z)]\bar{Q}_2[g(z)]\left[\frac{1-\bar{B}_1[f(z)]}{[f(z)]}\right]\right\}\left[\frac{1-\bar{R}[g(z)]}{[g(z)]}\right].$$

5. The steady state results

For the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t).$$

Let $P(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (4.16) to (4.20) we obtain

$$P(z) = \frac{\begin{bmatrix} I[1-B_1(g(z))][f(z)] + pB_1(g(z))[1-B_2(g(z))][f(z)] \\ +\alpha[1-B_1(g(z))][1-Q_1(f(z))] \\ +\alpha p[1-B_2(g(z))][1-Q_2(f(z))]B_1[g(z)] \\ +\alpha pQ_1[f(z)][1-B_1(g(z))][1-Q_2(f(z))] \\ +\alpha(1-p)[Q_1(f(z))][1-B_1(g(z))][1-R(f(z))] \\ +\alpha p[B_1(f(z))][Q_2(f(z))][1-B_2(g(z))][1-R(f(z))] \\ +\alpha p[Q_1(f(z))][Q_2(f(z))][1-B_1(g(z))][1-R(f(z))] \end{bmatrix}}{\begin{bmatrix} z(g(z)) - (1-p)B_1[g(z)]g(z) + pB_1[g(z)]B_2[g(z)]g(z) \\ +\alpha(1-p)Q_1[f(z)]R[f(z)][1-B_1(g(z))] \\ +\alpha pB_1[g(z)]Q_2[f(z)]R[f(z)][1-B_2[g(z)]] \\ +\alpha pQ_1[f(z)]Q_2[f(z)]R[g(z)][1-B_1[g(z)]] \end{bmatrix}}. \tag{5.1}$$

We see that for $z=1$, $P(z)$ is indeterminate of the form $0/0$. Therefore, we apply L'Hopital's rule and after simplification we obtain,

$$Q_1(0)=1,\quad Q_2(0)=1,\quad R(0)=1,\quad -Q_1'(0)=E(Q_1),\quad -Q_2'(0)=E(Q_2),$$
$$-R'(0)=E(R),\quad Q_1''(0)=E(Q^2),\quad Q_2''(0)=E(Q^2),\quad R''(0)=E(R^2).$$

$$P(1) = \frac{\begin{bmatrix} -I\lambda[E(X)]\{1-B_1(\alpha)] + p[B_1(\alpha)][1-B_2(\alpha)] + \alpha[1-B_1(\alpha)] \\ [E(Q_1)] + \alpha[E(R)][1-B_1(\alpha)] + \alpha p[B_1(\alpha)E(Q_2)[1-B_2(\alpha)] \\ +B_1(\alpha) + B_1(\alpha)E(R)[1-B_1(\alpha)] + E(Q_2)[1-B_1(\alpha)]]\} \end{bmatrix}}{\begin{bmatrix} \alpha - \lambda[E(X)]\{1-B_1(\alpha) + P[B_1(\alpha)] + \alpha\big[E(Q_1)[1-B_1(\alpha)] + E(R) \\ [1-B_1(\alpha)]\big] + \alpha p\big[B_1(\alpha)E(Q_2)[1-B_2(\alpha)] \\ +B_1(\alpha)E(R)[1-B_2(\alpha)] - B_1(\alpha)B_2'(\alpha)\big]\} \end{bmatrix}}. \tag{5.2}$$

In order to determine I, we use the normalizing condition

$$P_q^{(1)}(1) + P_q^{(2)}(1) + Q_q^{(1)}(1) + Q_q^{(2)}(1) + R_q(1) + I = 1 \tag{5.3}$$

and we get

$$I = \frac{\begin{bmatrix} \alpha - \lambda[E(x)][1 - B_1(\alpha)] + \alpha[E(Q_1) + E(R)][1 - B_1(\alpha)] \\ +p[B_1(\alpha)] + \alpha p\big\{B_1(\alpha)E(Q_2)[1 - B_2(\alpha)] \\ +B_1(\alpha)E(R)[1 - B_2(\alpha)]\big\} - B_1(\alpha)B_2'(\alpha) \end{bmatrix}}{\begin{bmatrix} \alpha - \lambda[E(x)]\big\{[1 - B_1(\alpha)] + \alpha[E(Q_1) + E(R)][1 - B_1(\alpha)] + p[B_1(\alpha)] \\ +\alpha p\big\{B_1(\alpha)E(Q_2)[1 - B_2(\alpha)] + B_1(\alpha)E(R)[1 - B_2(\alpha)]\big\} \\ -pB_1(\alpha)B_2(\alpha) + \alpha p[E(Q_2)[1 - B_1(\alpha)] - B_1(\alpha)B_2'(\alpha)]\big\} \end{bmatrix}}. \tag{5.4}$$

Hence the utilization factor ρ of the system is given by

$$\rho = 1 - I, \tag{5.5}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (5.4) gives the probability that the server is idle.

6. Performance measures

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \lim_{z \to 1} \frac{d}{dt} P_q(z), \tag{6.1}$$

$$L_q = \lim_{z \to 1} \frac{d}{dt} \frac{N(z)}{D(z)}, \tag{6.2}$$

where

$$\begin{aligned} N(z) = I[1 - B_1(g(z))][f(z)] + pB_1(g(z))[1 - B_2(g(z))][f(z)] + \alpha[1 - B_1(g(z))][1 - Q_1(f(z))] \\ +\alpha p[1 - B_2(g(z))][1 - Q_2(f(z))]B_1[g(z)] + \alpha p Q_1[f(z)][1 - B_1(g(z))][1 - Q_2(f(z))] \\ +\alpha(1 - p)[Q_1(f(z))][1 - B_1(g(z))][1 - R(f(z))] \\ +\alpha p[B_1(f(z))][Q_2(f(z))][1 - B_2(g(z))][1 - R(f(z))] \\ +\alpha p[Q_1(f(z))][Q_2(f(z))][1 - B_1(g(z))][1 - R(f(z))], \end{aligned}$$

$$\begin{aligned} D(z) = z(g(z)) - (1 - p)B_1[g(z)]g(z) + pB_1[g(z)]B_2[g(z)]g(z) \\ +\alpha(1 - p)Q_1[f(z)]R[f(z)][1 - B_1(g(z))] + \alpha pB_1[g(z)]Q_2[f(z)]R[f(z)][1 - B_2[g(z)]] \\ +\alpha p Q_1[f(z)]Q_2[f(z)]R[g(z)][1 - B_1[g(z)]], \end{aligned}$$

therefore

$$L_q = \frac{[D'(1)N''(1) - N'(1)D''(1)]}{2[D'(1)]^2},$$

$$\begin{aligned} N'(1) = -I\lambda[E(X)]\big\{[1 - B_1(\alpha)] + p[B_1(\alpha)][1 - B_2(\alpha)] + \alpha[E(Q_1)][1 - B_1(\alpha)] \\ +\alpha[E(R)][1 - B_1(\alpha)] + \alpha p\big[B_1(\alpha) + B_1(\alpha)E(Q_2)[1 - B_2(\alpha)] \\ +B_1(\alpha)E(R)[1 - B_2(\alpha)] + E(Q_2)[1 - B_1(\alpha)]\big]\big\}, \end{aligned}$$

$$N''(1) = \lambda^2[E(X)]^2\big\{ - B_1'(\alpha) + p\big[B_1'(\alpha) - B_1'(\alpha)B_2(\alpha) - B_1(\alpha)B_2'(\alpha)\big]$$
$$-\alpha\big[B_1'(\alpha)E(Q_1) + B_1'(\alpha)E(R) + E(Q_1^2)[1 - B_2(\alpha)] - E(R^2)[1 - B_1(\alpha)]$$
$$+E(Q_1)E(R)[1 - B_1(\alpha)]\big] - \alpha p\big[B_1(\alpha)B_2'(\alpha)E(Q_2) + B_1(\alpha)B_2(\alpha)$$
$$-B_1'(\alpha)E(Q_2)[1 - B_1(\alpha)] + B_1'(\alpha)E(Q_2) + E(Q_1)E(Q_2)[1 - B_1(\alpha)]$$
$$+E(Q_1)E(R)[1 - B_1(\alpha)] - B_1(\alpha)E(Q_2)E(R) + B_1(\alpha)E(R^2)[1 - B_2(\alpha)]$$
$$+E(Q_2)E(R) + E(Q_2^2)\big]\big\} + \lambda E(X^2)\big\{[1 - B_1(\alpha)] - pB_1(\alpha)[1 - B_2(\alpha)]$$
$$-\alpha\big[E(Q_1)[1 - B_1(\alpha)] + E(R)[1 - B_1(\alpha)]\big] - \alpha p\big[E(R)B_1(\alpha)[1 - B_2(\alpha)]$$
$$+E(Q_2) - B_1(\alpha)E(R)[1 - B_2(\alpha)]\big]\big\},$$

$$D'(1) = \alpha - \lambda[E(X)]\big\{1 - B_1(\alpha) + P[B_1(\alpha)] + \alpha\big[E(Q_1)[1 - B_1(\alpha)] + E(R)[1 - B_1(\alpha)]\big]$$
$$+\alpha p\big[B_1(\alpha)E(Q_2)[1 - B_2(\alpha)] + B_1(\alpha)E(R)[1 - B_2(\alpha)] - B_1(\alpha)B_2'(\alpha)\big]\big\},$$

$$D''(1) = -2\lambda[E(X)] - \lambda^2[E(X)]^2\big\{(1 - p)[2B_1'(\alpha) + 2B_2'(\alpha)]$$
$$+p\big[\alpha B_1''(\alpha)B_2(\alpha) + \alpha B_1(\alpha)B_2''(\alpha) + 2B_1'(\alpha)B_2(\alpha) + 2B_1(\alpha)B_2'(\alpha) + 2\alpha B_1'(\alpha)B_2'(\alpha)\big]$$
$$+\alpha(1 - p)\big[E(Q_1^2)[1 - B_1(\alpha)] + E(R^2)[1 - B_1(\alpha)] + 2E(Q_1)B_1'(\alpha)$$
$$+2E(R)B_1'(\alpha) + 2E(Q_1)E(R)[1 - B_1(\alpha)]\big]$$
$$+\alpha p\big[B_1''(\alpha)[1 - B_2(\alpha)] - B_1(\alpha)B_2''(\alpha) + B_1(\alpha)E(Q_2^2)[1 - B_2(\alpha)]$$
$$+B_1(\alpha)E(R^2)[1 - B_2(\alpha)] + 2B_1(\alpha)B_2'(\alpha)E(Q_2) + 2B_1(\alpha)B_2'(\alpha)E(R)\big]$$
$$-2B_1'(\alpha)B_2'(\alpha) - 2B_1'(\alpha)E(Q_2)[1 - B_2(\alpha)] - 2B_1'(\alpha)E(R)[1 - B_2(\alpha)]$$
$$+2B_1(\alpha)E(Q_2)E(R)[1 - B_2(\alpha)]\big\} - \lambda[E(X^2)]\big\{z + (1 - p)[\alpha B_1'(\alpha) - B_1(\alpha)]$$
$$+p\big[\alpha B_1'(\alpha)B_2(\alpha) + \alpha B_1(\alpha)B_2'(\alpha) + B_1(\alpha)B_2(\alpha)\big] + \alpha(1 - p)\big[E(Q_1)[1 - B_1(\alpha)]$$
$$+E(R)[1 - B_1(\alpha)] + B_1(\alpha)\big] + \alpha p\big[B_1'(\alpha) - B_1'(\alpha)B_2(\alpha) - B_1(\alpha)B_2'(\alpha)\big]$$
$$-E(Q_2)B_1(\alpha)[1 - B_2(\alpha)] - E(R)B_1(\alpha)[1 - B_2(\alpha)]\big\}.$$

Let W_q denote the average waiting time of customers in the queue by Little's formula

$$W_q = \frac{L_q}{\lambda}.$$

Idle I has been found in (5.4) and substituting values of $N'(1)$, $N''(1)$, $D'(1)$ and $D''(1)$ in (6.2) we obtain L_q in closed form, further we define the average system size L by using Little's formula. Thus, we have

$$L = L_q + \rho,$$

where L_q has been found in equation (6.2) and ρ is obtained from equation (5.5) as

$$\rho = 1 - I.$$

7. Numerical results

This section presents numerical examples related to specific work. Various parameters specified for the system performance measures are illustrated using MATLAB. We consider service times and working breakdown times are exponentially distributed. Analytical results are validated with numerical results. The set of values which satisfy the stability condition, are taken for the table calculation.

For the Table 1, we choose the following arbitrary values

$$\lambda = 2, \quad \mu_1 = 3, \quad \mu_2 = 3, \quad \beta_1 = 2.6, \quad \beta_2 = 2.6, \quad \eta = 5, \quad p = 0.6.$$

It clearly shows that as long as the breakdown rate (α) increases, the idle time (I) decreases, the mean queue size (L_q) increases and the mean waiting time of the customers (W_q) also increases. Fig. 2 shows that the idle time I decreases for the increasing values of the breakdown rate (α).

Table 1. Effective of breakdown.

α	I	L_q	W_q
0.20	0.0055	0.2609	0.1304
0.25	0.0054	0.2868	0.1434
0.30	0.0048	0.3149	0.1575
0.35	0.0038	0.3461	0.1730
0.40	0.0024	0.3811	0.1905

Similarly, Fig. 3 and Fig. 4 show that both the average queue length (L_q) and the average waiting time of the customers in the queue (W_q) for the increasing values of the breakdown rate (α).

From the Table 2, we choose the following values

$$\lambda = 1.3, \quad \mu_2 = 0.3, \quad \beta_1 = 17, \quad \beta_2 = 0.36, \quad \eta = 0.95, \quad \alpha = 0.9, \quad p = 0.6.$$

For increasing service rate (μ_1), the idle time (I) increases, the mean queue size (L_q) decreases and the mean waiting time of the customers (W_q) also decreases.

Table 2. Effective of service rate.

μ_1	I	L_q	W_q
11	0.4916	9.6167	7.3975
12	0.4940	9.4750	7.2884
13	0.4960	9.3573	7.1979
14	0.4977	9.2580	7.1216
15	0.4992	9.1732	7.0563

Fig. 5 shows that the idle time (I) increases for the increasing values of the service rate (μ).

Similarly, Fig. 6 and Fig. 7 show that the average queue length (L_q) and the average waiting time in the queue (W_q) decrease for the increasing values of the service rate (μ).

8. Conclusion

We considered an $M^{[X]}/G/1$ queue with second optional service and working breakdown. Using the supplementary variable method, important performance measures are derived. Numerical illustrations are made to examine the validity of analytical results. Slower rate service instead of stopping service can reduce waiting time and queue length. It helps to avoid heavy loss in production and manufacturing industries.

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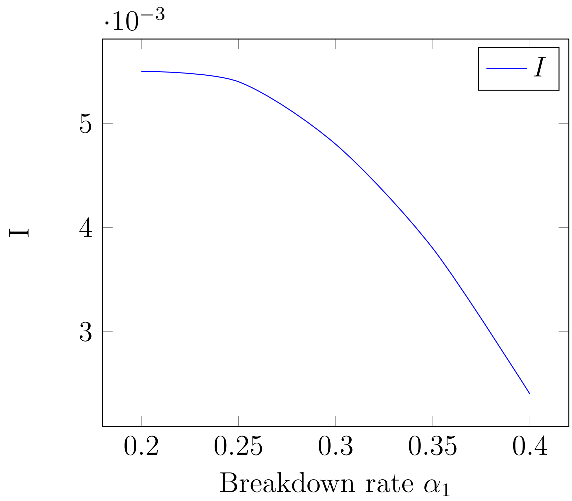

Figure 2. Breakdown rate vs Idle.

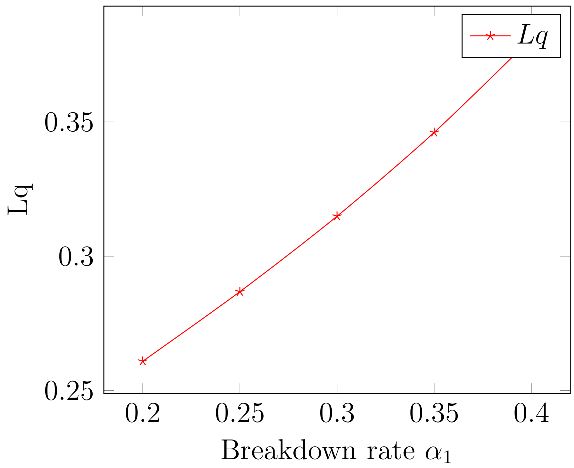

Figure 3. Breakdown rate vs Queue length.

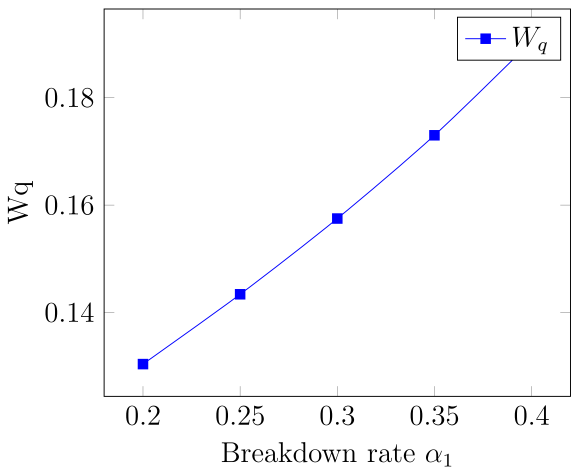

Figure 4. Breakdown rate vs Waiting time.

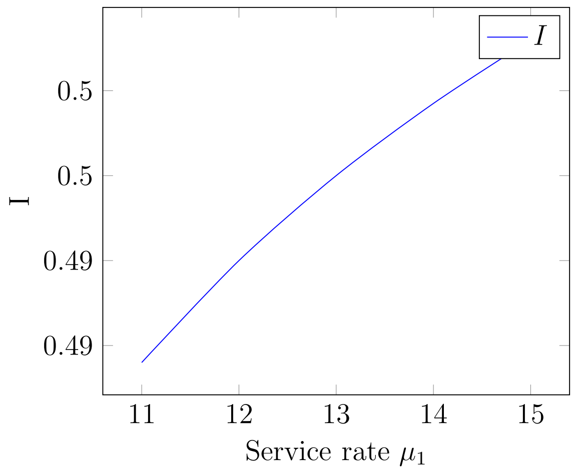

Figure 5. Service rate vs Idle.

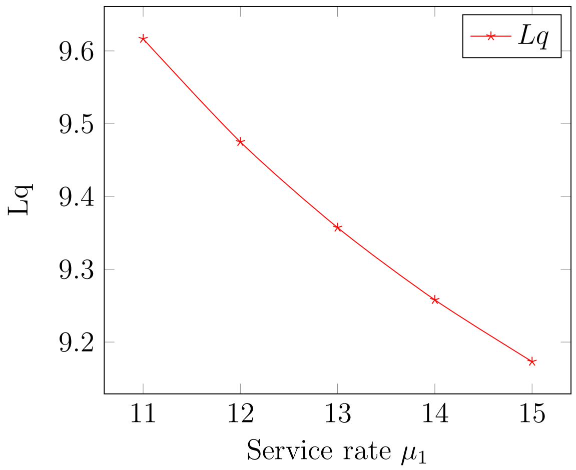

Figure 6. Service rate vs Queue length.

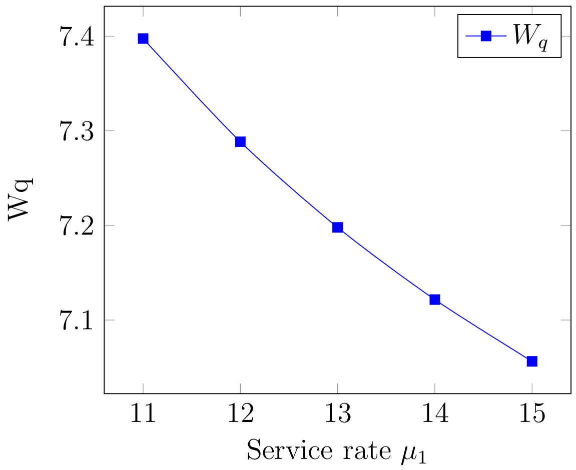

Figure 7. Service rate vs Waiting time.