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ON $A^{\mathcal{I}^{\mathcal{K}}}$–SUMMABILITY

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Abstract: In this paper, we introduce and investigate the concept of $A^{\mathcal{I}^{\mathcal{K}}}$-summability as an extension of $A^{\mathcal{I}^*}$-summability which was recently (2021) introduced by O.H.H. Edely, where $A=(a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix and $\mathcal{I}$ and $\mathcal{K}$ represent two non-trivial admissible ideals in $\mathbb{N}$. We study some of its fundamental properties as well as a few inclusion relationships with some other known summability methods. We prove that $A^{\mathcal{K}}$-summability always implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability whereas $A^{\mathcal{I}}$-summability not necessarily implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability. Finally, we give a condition namely $AP(\mathcal{I},\mathcal{K})$ (which is a natural generalization of the condition AP) under which $A^{\mathcal{I}}$-summability implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability.

Keywords: Ideal, Filter, $\mathcal{I}$-convergence, $\mathcal{I}^{\mathcal{K}}$-convergence, $A^{\mathcal{I}}$-summa-bility, $A^{\mathcal{I}^{\mathcal{K}}}$-summability.

1. Introduction

In 2000, Kostrkyo and Salat [12] introduced the notion of ideal convergence. They studied several fundamental properties of $\mathcal{I}$ and $\mathcal{I}^*$-convergence and showed that their idea was the extended version of so many known convergence methods. Based on $\mathcal{I}$-convergence several generalizations were made by researchers and several analytical and topological properties have been investigated (see [1, 9, 11, 15–19, 21, 22] where many more references can be found) and this area becomes one of the most focused areas of research.

In 2011, M. Macaj and M. Sleziak [13] generalized the idea of $\mathcal{I}^*$-convergence to $\mathcal{I}^{\mathcal{K}}$-convergence by involving two ideals $\mathcal{I}$ and $\mathcal{K}$. In the case of $\mathcal{I}^{\mathcal{K}}$-convergence, the convergence along the large set is taken with regard to another ideal $\mathcal{K}$ instead of considering ordinary convergence. So from that point of view the concept of $\mathcal{I}^{\mathcal{K}}$-convergence being an extension of $\mathcal{I}^*$-convergence shows a strong analogy for further investigation. Recent developments in the direction of $\mathcal{I}^{\mathcal{K}}$-convergence from topological aspects can be found from the works of Das et al. [4, 5], Banerjee and Paul [2, 3] and many others.

If $x=(x_k)$ be a real-valued sequence and $A=(a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix, then Ax is the sequence having n^{th} term $A_n(x)=\sum_{k=1}^{\infty}a_{nk}x_k$. A sequence $x=(x_k)$ is said to be A-summable to L, if $\lim\limits_{n\to\infty}A_n(x)=L$. A matrix $A=(a_{nk})_{n,k=1}^{\infty}$ is said to be regular if it maps a convergent sequence into a convergent sequence keeping the same limit i.e., $A\in(c,c)_{reg}$ if $A\in(c,c)$ and $\lim\limits_{n\to\infty}A_n(x)=\lim\limits_{k\to\infty}x_k$. Here c, (c,c), and $(c,c)_{reg}$ denote the collection of all real-valued convergent sequences, collection of all matrices which maps an element of c to an element of c, and the collection of all regular matrices which maps an element of c to an element of c, respectively. The necessary and sufficient Silverman–Toeplitz conditions for an infinite matrix $A=(a_{nk})_{n,k=1}^{\infty}$ to be regular are as follows:

(i) $\sup\limits_{n}\sum_{k=1}^{\infty}|a_{nk}|<\infty$;

(ii) For any $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} a_{nk} = 0$;

(iii) $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

In 2008, Edely and Mursaleen [7] generalized the notion of A-summability to statistical A-summability by using the concept of natural density. Recently, Edely [6] further extended the notion of statistical A-summability to $A^{\mathcal{I}}$-summability, where $\mathcal{I}$ represents an ideal in $\mathbb{N}$. In this paper we intend to introduce the notion of $A^{\mathcal{I}^{\mathcal{K}}}$-summability which is a natural generalization of $A^{\mathcal{I}^*}$-summability. For more details regarding summability theory, one may refer to [8, 10, 14, 20].

Throughout the paper, we will use (y_n) to denote the image $(A_n(x))$ of the sequence $x = (x_k)$ under the transformation of the non-negative regular infinite matrix A.

2. Definitions and preliminaries

Definition 1. *A collection $\mathcal{I}$ containing subsets of a nonempty set X is called an ideal in X if and only if* (i) $\emptyset \in \mathcal{I}$, (ii) $P, Q \in \mathcal{I}$ *implies* $P \cup Q \in \mathcal{I}$ *(Additive), and* (iii) $P \in \mathcal{I}, Q \subset P$ *implies* $Q \in \mathcal{I}$ *(Hereditary).*

If for any $x \in X$ $\{\{x\}\} \subset \mathcal{I}$ then it is said that $\mathcal{I}$ satisfies the admissibility property or simply is called admissible. Also $\mathcal{I}$ is called non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Some standard examples of ideal are given below:

(i) The set $\mathcal{I}_f$ consisting of all subsets of $\mathbb{N}$ having finite cardinality is an admissible ideal in $\mathbb{N}$.

(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an ideal in $\mathbb{N}$ which is also admissible.

(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is an ideal in $\mathbb{N}$ which also has the so called admissibility property.

(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$, where $D_p \subset \mathbb{N}$ for any $p \in \mathbb{N}$ and for $i \neq j$, $D_i \cap D_j = \emptyset$. Then, the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [9] and [11].

Definition 2. *A collection $\mathcal{F}$ containing subsets of a nonempty set X is called a filter in X if and only if (i) $\emptyset \notin \mathcal{F}$ (ii) $M, N \in \mathcal{F}$ implies $M \cap N \in \mathcal{F}$ and (iii) $M \in \mathcal{F}, N \supset M$ implies $N \in \mathcal{F}$.*

If $\mathcal{I}$ is a proper non-trivial ideal in X, then the collection $\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists\, P \in \mathcal{I}\, such\, that\, M = X \setminus P\}$ forms a filter in X. It is known as the filter associated with the ideal $\mathcal{I}$.

Definition 3 [12]. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ which satisfies the admissibility property. A real-valued sequence $x = (x_k)$ is called $\mathcal{I}$-convergent to l if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ is contained in $\mathcal{I}$. The number l is called the $\mathcal{I}$-limit of the sequence $x = (x_k)$. Symbolically, $\mathcal{I} - \lim x = l$.*

Definition 4 [12]. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ which satisfies the admissibility property. A sequence $x = (x_k)$ is called $\mathcal{I}^*$-convergent to l, if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, for which $\lim\limits_{k} x_{m_k} = l$ holds.*

Definition 5 [13]. *Let $\mathcal{I}, \mathcal{K}$ denote two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is called $\mathcal{I}^{\mathcal{K}}$-convergent to l if the associated filter $\mathcal{F}(\mathcal{I})$ contains a set M such that the sequence $y = (y_k)$ defined by*

$$y_k = \begin{cases} x_k, & k \in M, \\ l, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to l.

If we consider $\mathcal{K} = \mathcal{I}_f$ then $\mathcal{I}^{\mathcal{K}}$-convergence concept coincides with $\mathcal{I}^*$-convergence [12].

Definition 6 [13]. *Let $\mathcal{K}$ be an ideal in $\mathbb{N}$. Then, $P \subset_{\mathcal{K}} Q$ denotes the property $P \setminus Q \in \mathcal{K}$. Also $P \subset_{\mathcal{K}} Q$ and $Q \subset_{\mathcal{K}} P$ together implies $P \sim_{\mathcal{K}} Q$. Thus $P \sim_{\mathcal{K}} Q$ if and only if $P \triangle Q \in \mathcal{K}$. A set P is said to be $\mathcal{K}$-pseudointersection of a system $\{P_i : i \in \mathbb{N}\}$ if for every $i \in \mathbb{N}$ $P \subset_{\mathcal{K}} P_i$ holds.*

Definition 7 [13]. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$. Then $\mathcal{I}$ is said to have the additive property with respect to $\mathcal{K}$ or the condition $AP(\mathcal{I}, \mathcal{K})$ holds if every sequence $(F_n)_{n \in \mathbb{N}}$ of sets from $\mathcal{F}(\mathcal{I})$ has $\mathcal{K}$-pseudointersection in $\mathcal{F}(\mathcal{I})$.*

Definition 8 [6]. *A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-summable to a real number L, if the transformed sequence $(A_n(x))$ is $\mathcal{I}$-convergent to L. Symbolically, it is written as $A^{\mathcal{I}} - \lim x_k = L$.*

Definition 9 [6]. *A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}^*}$-summable to a real number L, if there exists a set $M = \{m_1 < m_2 < ... < m_i < ...\} \in \mathcal{F}(\mathcal{I})$ such that*

$$\lim_{i \to \infty} \sum_k a_{m_i k} x_k = \lim_{i \to \infty} y_{m_i} = L.$$

3. Main results

Throughout the section, for a sequence $x = (x_k)$ we will use $y = (y_n)$ to denote the transformed sequence $(A_n(x))$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$.

Definition 10. *Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix and suppose $\mathcal{I}, \mathcal{K}$ be two admissible ideals in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}^{\mathcal{K}}}$-summable to $L \in \mathbb{R}$, if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $z = (z_k)$ defined by*

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L, where the sequence $y = (y_n)$ is defined as

$$y_n = A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

In this case we write, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.

Example 1. Consider the decomposition of $\mathbb{N}$ given by

$$\mathbb{N} = \bigcup_{i=1}^{\infty} D_i, \quad D_i = \{2^{i-1}(2s-1) : \; s = 1, 2, 3, ...\}.$$

Let $\mathcal{I}$ denotes the ideal consisting of all subsets of $\mathbb{N}$ which intersects finitely many of D_i's. Consider the sequence $x = (x_k)$ defined by $x_k = 1/i$ if $k \in D_i$ and the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as

$$a_{nk} = \begin{cases} 1, & k = n+2, \\ 0, & otherwise. \end{cases}$$

Then, the sequence is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to 0 for $\mathcal{K} = \mathcal{I}$.

Justification: Clearly,

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{i}, \quad n+2 \in D_i.$$

Let $M = \mathbb{N} \setminus D_1$. Then, $M \in \mathcal{F}(\mathcal{I})$ and it is easy to verify that the sequence $z = (z_k)$ defined by

$$z_k = \begin{cases} y_k, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $\mathcal{I}$-convergent to 0. Hence, $A^{\mathcal{I}^{\mathcal{I}}} - \lim x_k = 0$.

Theorem 1. *Let* $A^{\mathcal{I}^*} - \lim x_k = L$ *then* $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.

P r o o f. Let $A^{\mathcal{I}^*} - \lim x_k = L$. Then, there exists a set

$$M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$$

such that $\lim_i y_{m_i} = L$. This implies that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is usual convergent to L. Now by Theorem 2.1 of [11], we can say that for any ideal $\mathcal{K}$, the sequence $z = (z_k)$ is $\mathcal{K}$-convergent to L. Hence, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. □

Theorem 2. *Let* $A^{\mathcal{K}} - \lim x_k = L$ *then* $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$.

P r o o f. Since $A^{\mathcal{K}} - \lim x = L$, so for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \in \mathcal{K}. \tag{3.1}$$

Choose $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. Consider the sequence $z = (z_k)$ defined by $z_k = y_k,\ k \in M$. Then, using (3.1), we get for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - L| \geq \varepsilon\} \in \mathcal{K}$$

i.e. $z = (z_k)$ is $\mathcal{K}$-convergent to L. Hence $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. □

Remark 1. Converse of the above theorem is not necessarily true.

Example 2. Consider the ideals

$$\mathcal{I}_c = \{B \subseteq \mathbb{N} : \sum_{b \in B} b^{-1} < \infty\}, \quad \mathcal{I}_d = \{B \subseteq \mathbb{N} : d(B) = 0\}$$

and the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ defined by

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & \textit{otherwise.} \end{cases}$$

Let $x = (x_k)$ be the sequence defined as

$$x_k = \begin{cases} 1, & k \text{ is prime}, \\ 0, & k \text{ is not prime}. \end{cases}$$

Then, there exists set M of all non prime numbers $\in \mathcal{F}(\mathcal{I}_{\mathrm{d}})$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $\mathcal{I}_c$-converegnt to 0. Hence, $A^{\mathcal{I}_d{}^{\mathcal{I}_c}} - \lim x_k = 0$. But we claim that $A^{\mathcal{I}_c} - \lim x_k \neq 0$. Because if $A^{\mathcal{I}_c} - \lim x_k = 0$, then for any particular ε with $0 < \varepsilon < 1$, we have the set

$$\{k \in \mathbb{N} : |y_k - 0| \geq \varepsilon\} = \text{set of all prime numbers} \in \mathcal{I}_c,$$

it is a contradiction.

The next theorem gives the condition under which $A^{\mathcal{I}^{\mathcal{K}}}$-summability implies $A^{\mathcal{K}}$-summability.

Theorem 3. *Let $\mathcal{I}$ and $\mathcal{K}$ be two admissible ideals in $\mathbb{N}$. If $\mathcal{I} \subseteq \mathcal{K}$ then $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ implies $A^{\mathcal{K}} - \lim x_k = L$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{K}$. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ gives the assurance of the existence of a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L and subsequently, we have

$$\forall \varepsilon > 0, \quad \{k \in M : |y_k - L| \geq \varepsilon\} \in \mathcal{K}. \tag{3.2}$$

Now as the inclusion

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq \{k \in M : |y_k - L| \geq \varepsilon\} \cup (\mathbb{N} \setminus M)$$

holds and by our assumption, $\mathbb{N} \setminus M \in \mathcal{I} \subseteq \mathcal{K}$, from (3.2) we have

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \in \mathcal{K}.$$

Hence, $A^{\mathcal{K}} - \lim x_k = L$. □

Theorem 4. *If every subsequence of $x = (x_k)$ is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L, then x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L.*

P r o o f. If possible let us assume the contrary. Then, for every $M \in \mathcal{F}(\mathcal{I})$, the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is not $\mathcal{K}$-convergent to L. In other words, for any $M \in \mathcal{F}(\mathcal{I})$, there exists an $\varepsilon_M > 0$ such that

$$B = M \cap \{k \in \mathbb{N} : |y_k - L| \geq \varepsilon_M\} \notin \mathcal{K}.$$

Since $\mathcal{K}$ is admissible, so B is infinite. Let $B = \{b_1 < b_2 < ... < b_k < ...\}$. Construct a subsequence $w = (w_k)$ defined as $w_k = y_{b_k}$ for $k \in \mathbb{N}$. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim w_k \neq L$, we get a contradiction to the hypothesis. □

Theorem 5. *Let $x = (x_k)$ be a sequence such that $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. Then, every subsequence of x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L if and only if both $\mathcal{I}$ and $\mathcal{K}$ does not contain infinite sets.*

P r o o f. There are two possible cases.

Case I. Let $\mathcal{K}$ contain an infinite set. Suppose C be an infinite set and $C \in \mathcal{K}$. Then, $\mathbb{N} \setminus C \in \mathcal{F}(\mathcal{K})$ and $\mathbb{N} \setminus C$ is also infinite. Let $\varepsilon > 0$ be arbitrary. Choose $L_1 \in \mathbb{R}$ such that $L_1 \neq L$. Consider the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$, defined as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise, \end{cases}$$

and the sequence $x = (x_k)$ such that

$$x_k = \begin{cases} L_1, & k \in C, \\ L, & k \in \mathbb{N} \setminus C. \end{cases}$$

Then,

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq C \in \mathcal{K}.$$

This means that x is $A^{\mathcal{K}}$-summable to L. Therefore by Theorem 2, x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L. But clearly the subsequence $(x_k)_{k \in C}$ of x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L_1 and not to L.

Case II. Let $\mathcal{K}$ does not contain an infinite set. Then $\mathcal{K} = \mathcal{I}_f$ and $A^{\mathcal{I}^{\mathcal{K}}}$-summability concept coincides with $A^{\mathcal{I}^*}$-summability.

Subcase I: if $\mathcal{I}$ contains an infinite set. Let B be any infinite set such that $B \in \mathcal{I}$. Then, $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$ and $\mathbb{N} \setminus B$ is also infinite. Define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} \xi, & k \in B, \\ L, & k \in \mathbb{N} \setminus B, \end{cases}$$

where $\xi(\neq L) \in \mathbb{R}$. Clearly x is $A^{\mathcal{I}^*}$-summable to L for the infinite matrix considered in Case I. But clearly the subsequence $(x_k)_{k \in B}$ of x is not $A^{\mathcal{I}^*}$-summable to L.

Subcase II: if $\mathcal{I}$ does not contain an infinite set. In this subcase, we have $\mathcal{I} = \mathcal{K} = \mathcal{I}_f$ and therefore $A^{\mathcal{I}^{\mathcal{K}}}$-summability concept coincides with ordinary summability ([10]) so any subsequence of x is ordinary summable to L. □

Remark 2. If a sequence is $A^{\mathcal{I}^{\mathcal{K}}}$-summable then it may not be $A^{\mathcal{I}}$-summable.

Example 3. Let us consider the ideal $\mathcal{I}$ which is defined in Example 1 and the ideal

$$\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}.$$

Let $M = \{k \in \mathbb{N} : k = 2^p$ for some non-negative integer p$\}$. Then, for the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ defined as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise, \end{cases}$$

the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k \in M, \\ 0, & k \notin M \end{cases}$$

is $A^{\mathcal{I}^{\mathcal{I}_c}}$-summable to 0 but x is not $A^{\mathcal{I}}$-summable to 0.

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Let $x = (x_k)$ be any real-valued sequence. Then, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$ implies $A^{\mathcal{I}} - \lim x_k = L$ if and only if $\mathcal{K} \subseteq \mathcal{I}$.*

P r o o f. Let $\mathcal{K} \subseteq \mathcal{I}$ and suppose $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. Then, the result follows directly from the following inclusion

$$\{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \subseteq \{k \in M : |y_k - L| \geq \varepsilon\} \cup (\mathbb{N} \setminus M).$$

For the converse part, we assume the contrary. Then, there exists a set say $C \in \mathcal{K} \setminus \mathcal{I}$. Let L_1 and L_2 be two real numbers such that $L_1 \neq L_2$. Define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} L_1, & k \in C, \\ L_2, & k \in \mathbb{N} \setminus C \end{cases}$$

and the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise. \end{cases}$$

Then, for any $\varepsilon > 0$ we have,

$$\{k \in \mathbb{N} : |y_k - L_2| \geq \varepsilon\} \subseteq C \in \mathcal{K}$$

which means that x is $A^{\mathcal{K}}$-summable to L_2. Therefore by Theorem 2, x is $A^{\mathcal{I}^{\mathcal{K}}}$-summable to L_2. By hypothesis x is $A^{\mathcal{I}}$-summable to L_2. Therefore for $\varepsilon = |L_1 - L_2|$,

$$\{k \in \mathbb{N} : |y_k - L_2| \geq |L_1 - L_2|\} = C \in \mathcal{I},$$

it is a contradiction. Hence we must have $\mathcal{K} \subseteq \mathcal{I}$. □

Remark 3. If a sequence is $A^{\mathcal{I}}$-summable then it may not be $A^{\mathcal{I}^{\mathcal{K}}}$-summable. Consider the ideal $\mathcal{I}$ and the sequence $x = (x_k)$ defined in Example 1. Then, proceeding as Example 1 of [6], we can prove that $A^{\mathcal{I}^{\mathcal{I}_f}} - \lim x_k \neq 0$ although $A^{\mathcal{I}} - \lim x_k = 0$.

Theorem 7. *Let $\mathcal{I}$ and $\mathcal{K}$ be two admissible ideals of $\mathbb{N}$ such that the condition $AP(\mathcal{I}, \mathcal{K})$ holds. Then, for a sequence $x = (x_k)$, $A^{\mathcal{I}}$-summability implies $A^{\mathcal{I}^{\mathcal{K}}}$-summability to the same limit.*

P r o o f. Let $A^{\mathcal{I}} - \lim x_k = L$. Choose a sequence of rationales $(\varepsilon_i)_{i\in\mathbb{N}}$. Then, for every i,

$$M_i = \{k \in \mathbb{N} : |y_k - L| < \varepsilon_i\} \in \mathcal{F}(\mathcal{I}).$$

Thus by Definition 7, there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that for any $i \in \mathbb{N}$, $M \setminus M_i \in \mathcal{K}$. Consider the sequence $z = (z_k)_{k\in\mathbb{N}}$ defined by

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M. \end{cases}$$

To complete the proof, it is sufficient to show that the sequence $z = (z_k)$ is $\mathcal{K}$-convergent to L. Now,

$$\begin{aligned} \{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} &= \{k \in M : |z_k - L| < \varepsilon_i\} \cup \{k \in \mathbb{N} \setminus M : |z_k - L| < \varepsilon_i\} \\ &= (\mathbb{N} \setminus M) \cup \{k \in M : |z_k - L| < \varepsilon_i\} \\ &= (\mathbb{N} \setminus M) \cup (M_i \cap M) \\ &= \mathbb{N} \setminus (M \setminus M_i). \end{aligned}$$

Now as $M \setminus M_i \in \mathcal{K}$, so $\mathbb{N} \setminus (M \setminus M_i) \in \mathcal{F}(\mathcal{K})$ and consequently we have

$$\{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} \in \mathcal{F}(\mathcal{K})$$

i.e. $\mathcal{K} - \lim z_k = L$. Hence, $A^{\mathcal{I}^{\mathcal{K}}} - \lim x_k = L$. This completes the proof. □

Theorem 8. *Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be admissible ideals in $\mathbb{N}$ satisfying $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then,*

(i) $A^{\mathcal{I}_1^{\mathcal{K}}} - \lim x_k = L$ *implies* $A^{\mathcal{I}_2^{\mathcal{K}}} - \lim x_k = L$;

(ii) $A^{\mathcal{I}^{\mathcal{K}_1}} - \lim x_k = L$ *implies* $A^{\mathcal{I}^{\mathcal{K}_2}} - \lim x_k = L$.

P r o o f. (i) Suppose $A^{\mathcal{I}_1^{\mathcal{K}}} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(\mathcal{I}_1)$ such that the sequence $z = (z_k)$ defined as

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to L. Now since $M \in \mathcal{F}(\mathcal{I}_1)$, we have $\mathbb{N} \setminus M \in \mathcal{I}_1$ and therefore by hypothesis $\mathbb{N} \setminus M \in \mathcal{I}_2$, which again implies $M \in \mathcal{F}(\mathcal{I}_2)$. Hence we must have that $A^{\mathcal{I}_2^{\mathcal{K}}} - \lim x_k = L$.

(ii) Suppose $A^{\mathcal{I}^{\mathcal{K}_1}} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(\mathcal{I}_1)$ such that the sequence $z = (z_k)$ defined as,

$$z_k = \begin{cases} y_k, & k \in M, \\ l, & k \notin M \end{cases}$$

satisfies the following property $\forall \varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in \mathcal{K}_1.$$

Now by hypothesis the inclusion $\mathcal{K}_1 \subseteq \mathcal{K}_2$ holds, so we must have for $\forall \varepsilon > 0$,

$$\{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in \mathcal{K}_2.$$

Hence $A^{\mathcal{I}^{\mathcal{K}_2}} - \lim x_k = L$. □

4. Conclusion

Summability plays an important role in mathematics, particularly in mathematical analysis. In this paper, we introduce and investigate a few properties of $A^{\mathcal{I}^{\mathcal{K}}}$-summability. We generate a few examples and counterexamples in order to study some inclusion relationships with some known methods of summability. But the main focus was to link $A^{\mathcal{I}}$ and $A^{\mathcal{I}^*}$-summability with $A^{\mathcal{I}^{\mathcal{K}}}$-summability. We prove that the condition $AP(\mathcal{I},\mathcal{K})$ plays a crucial role in this regard. In the future, this idea can be utilized by the researchers to develop some other forms of summability.

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