

# FIXED POINT THEOREM FOR MULTIVALUED NON-SELF MAPPINGS SATISFYING JS-CONTRACTION WITH AN APPLICATION

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**Abstract:** In this paper, we present some fixed point results for multivalued non-self mappings. We generalize the fixed point theorem due to Altun and Minak [2] by using Jleli and Sameti [9]  $\vartheta$ -contraction. To validate the results proved here, we provide an appropriate application of our main result.

**Keywords:** JS-contraction mapping, Multivalued mapping, Metric space, Non-self mapping, Fixed point.

## 1. Introduction and preliminaries

In 1922, in Banach's PhD thesis a remarkable fixed point theorem well known as the Banach contraction principal was initiated. It's simplicity, usefulness and application made it a supreme tool in finding the existence and uniqueness of solution in numerous branches of mathematical analysis and applied sciences. Following the Banach contraction principal, some authors, Nadler [13], Assad and Kirk [4], Itoh [8] and several others have extended and generalized this theorem in several ways. In fact, Nadler [13] introduced the concept of using Hausdorff metric on multi-valued contraction of self mappings in the study of fixed points. Assad and Kirk [4] proved the Banach contraction mapping theorem for multi-valued contraction of non-self mappings and Itoh [8] generalized the theorems due to Assad and Kirk, and many other researchers have made significant contributions in this area (see [3, 7, 11]). In 2013, Alghamdi et al. [1] proved fixed point results for multivalued nonself almost contractions on convex metric spaces. Recently, Altun and Minak [2] introduced a new approach to Assad and Kirk fixed point theorem and a new real generalization of it, by using  $\vartheta$ -contractiveness of a multivalued mapping. Jleli and Samet [9] introduced  $\vartheta$ -contraction and established a new fixed point theorem for such mappings in the setting of generalized metric spaces. Following the notion  $\vartheta$ , Hussain et al. [6] supposed that  $\Theta$  is the set of all functions  $\vartheta : [0, \infty) \rightarrow [1, \infty)$  satisfying the following conditions:

- ( $\vartheta_1$ )  $\vartheta$  is nondecreasing and  $\vartheta(t) = 1$  if and only if  $t = 0$ ;
- ( $\vartheta_2$ ) for each sequence  $\{t_n\} \subseteq (0, \infty)$ ,  $\lim_{n \rightarrow \infty} \vartheta(t_n) = 1$  if and only if the limit of  $\lim_{n \rightarrow \infty} t_n = 0$ ;
- ( $\vartheta_3$ ) there exists  $r \in (0, 1)$  and  $l \in (0, \infty]$  such that  $\lim_{t \rightarrow 0^+} \frac{\vartheta(t) - 1}{t^r} = l$ ;
- ( $\vartheta_4$ )  $\vartheta(a + b) \leq \vartheta(a)\vartheta(b)$  for all  $a, b > 0$ .

Throughout this paper we shall denote by  $\Theta$  the set of all functions  $\vartheta$  satisfying ( $\vartheta_1$ ) – ( $\vartheta_4$ ).

Next, we present some definitions and preliminaries that are required to prove the main result of this paper.

Since we are dealing with multivalued mapping it is important to state a brief description of the Hausdorff metric. The Hausdorff metric measures the distance between subsets of a metric space. One among many interesting properties of this metric space, which will be our focus in this paper is that the Hausdorff induced metric space is complete if our original metric space is complete. Now, we define the Hausdorff metric as follows:

**Definition 1.** [5] Let  $(M, \varrho)$  be a metric space. Denote by  $CB(M)$  the collection of non-empty closed bounded subsets of  $M$ . For  $A, B \in CB(M)$  and  $u \in M$ , define

$$\rho(u, A) = \inf_{a \in A} \varrho(u, a)$$

and

$$\mathcal{H}(A, B) = \max \left\{ \sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A) \right\}.$$

It is seen that  $\mathcal{H}$  is a metric on  $CB(M)$ .  $\mathcal{H}$  is called the Hausdorff metric induced by  $\varrho$ . The completion of  $(M, \varrho)$  implies that  $(CB(M), \mathcal{H})$  is a complete metric space.

Before proceeding further, it is important to know that we will need an extra condition, call it  $(\vartheta_5)$ , a very useful part of our tool to help us to prove our main results in multivalued mapping and  $\Theta$  satisfies  $(\vartheta_5)$ .

$$(\vartheta_5) \vartheta(\inf A) = \inf \vartheta(A) \text{ for all } A \subset (0, \infty) \text{ with } \inf A > 0.$$

The following definition is important for future work in this paper.

**Definition 2.** [10].

- (i) A sequence  $\{u_n\}$  in a metric space  $(M, \varrho)$  is said to converge or to be convergent if there is  $u \in M$  such that

$$\lim_{n \rightarrow \infty} \varrho(u_n, u) = 0.$$

- (ii) A sequence  $\{u_n\}$  in a metric space  $(M, \varrho)$  is said to be Cauchy sequence if for every  $\epsilon > 0$  there is a number  $N = N(\epsilon)$  such that

$$\varrho(u_n, u_m) < \epsilon$$

for every  $m, n > N$ .

- (iii) A metric space  $(M, \varrho)$  is said to be complete if every Cauchy sequence in  $M$  converges to an element of  $M$ .

The following is the description of a metrically convex metric space and some of its properties are stated.

The following definition is due to Assad and Kirk [4].

**Definition 3.** [4] A metric space  $(M, \varrho)$  is said to be metrically convex if for any  $u, v \in M$  with  $u \neq v$ , there exists a point  $z \in M, (u \neq z \neq v)$  such that

$$\varrho(u, v) = \varrho(u, z) + \varrho(z, v).$$

The following result is taken from Assad [4] where  $\partial K$  denotes the boundary of  $K$ .

**Lemma 1.** [4] *If  $K$  is a closed subset of the complete and convex metric space  $M$  and if  $u \in K$ ,  $v \notin K$ , then there exists a point  $z \in \partial K$ , such that*

$$\varrho(u, v) = \varrho(u, z) + \varrho(z, v),$$

where  $\partial K$  denotes the boundary of  $K$ .

Assad and Kirk [4] proved the following fixed point theorem.

**Theorem 1.** [4] *Let  $(M, \varrho)$  be a complete and metrically convex metric space,  $K$  be a non-empty closed subset of  $M$ ,  $T : K \rightarrow CB(M)$  be a mapping such that, for all  $u, v \in K$ ,*

$$\rho(Tu, Tv) \leq k\varrho(u, v),$$

for some  $k \in (0, 1)$ . If  $Tu \subseteq K$  for each  $u \in \partial K$ , then  $T$  has a fixed point in  $K$ .

In 2014, Jleli and Sameti [9] gave a new generalization of Banach contraction mapping theorem in the setting of Banciari metric spaces as follows:

**Theorem 2.** [9] *Let  $(M, \varrho)$  be a complete generalized metric space and  $T : M \rightarrow M$  be a mapping. Suppose that there exist  $\vartheta \in \Theta$  and  $k \in (0, 1)$  such that for  $u, v \in M$ ,*

$$\varrho(Tu, Tv) \neq 0 \implies \vartheta(\varrho(Tu, Tv)) \leq [\vartheta(\varrho(u, v))]^k.$$

Then  $T$  has a unique fixed point.

Recently, Altun and Minak [2] obtained a new approach to Theorem 2 and a new generalization of it, by using  $\vartheta$ -contraction as follows:

**Theorem 3.** [2] *Let  $(M, \varrho)$  be a complete and metrically convex metric space,  $K$  be a nonempty closed subset of  $M$ ,  $T : K \rightarrow CB(M)$  be a mapping such that for all  $u, v \in K$  with  $\mathcal{H}(Tu, Tv) > 0$ ,*

$$\vartheta(\mathcal{H}(Tu, Tv)) \leq [\vartheta(\varrho(u, v))]^k$$

for some  $k \in (0, 1)$  and  $\vartheta \in \Theta$ . If  $Tu \subseteq K$  for each  $u \in \partial K$ , then  $T$  has a fixed point in  $K$ .

Suppose we want to use a different  $\vartheta$ -contraction in the above theorem. Hussain et al. [6] introduced a new concept that we can apply in the proof of the above theorem and obtain a new result.

**Definition 4.** [6] *Let  $(M, \varrho)$  be a metric space and let  $T : M \rightarrow M$  be a mapping.  $T$  is said to be a JS-contraction whenever there is a function  $\vartheta \in \Theta$  and positive real numbers  $\tau_1, \tau_2, \tau_3, \tau_4$  with  $0 \leq \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$  such that*

$$\vartheta(\varrho(Tu, Tv)) \leq [\vartheta(\varrho(u, v))]^{\tau_1} [\vartheta(\varrho(u, Tu))]^{\tau_2} [\vartheta(\varrho(v, Tv))]^{\tau_3} [\vartheta(\varrho(u, Tv) + \varrho(v, Tu))]^{\tau_4}$$

for all  $u, v \in M$ .

## 2. JS-contraction fixed point theorem

We give now a definition of a generalized multivalued JS-contraction mapping.

**Definition 5.** Let  $(M, \varrho)$  be a metric space and  $K$  be a nonempty closed subset of  $M$ . Let  $T$  be a mapping of  $K$  into  $CB(M)$ . Then  $T$  is said to be a generalized JS-contraction mapping whenever there is a function  $\vartheta \in \Theta$  and nonnegative real numbers  $\tau_1, \tau_2, \tau_3, \tau_4$  with

$$0 \leq \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$$

such that

$$\vartheta(\mathcal{H}(Tu, Tv)) \leq [\vartheta(\varrho(u, v))]^{\tau_1} [\vartheta(\varrho(u, Tu))]^{\tau_2} [\vartheta(\varrho(v, Tv))]^{\tau_3} [\vartheta(\varrho(u, Tv) + \varrho(v, Tu))]^{\tau_4}. \quad (2.1)$$

for all  $u, v \in K$ .

We now present an extended version of Theorem 3.

**Theorem 4.** Let  $(M, \varrho)$  be a complete and metrically convex metric space,  $K$  be a nonempty closed subset of  $M$ . Let  $T : K \rightarrow CB(M)$  be a generalized multivalued JS-contraction mapping. If for any  $u \in \partial K$ ,  $Tu \subseteq K$  and

$$\frac{(1 + \tau_1 + \tau_2 + \tau_4)(\tau_1 + \tau_2 + \tau_4)}{(1 - \tau_3 - \tau_4)^2} < 1,$$

then there is  $z \in K$  such that  $z \in T(z)$ .

**P r o o f.** We construct two sequences  $\{u_n\}$  and  $\{v_n\}$  in  $K$  in the following way: let  $u_0 \in K$  and  $v_1 \in Tu_0$ . If  $v_1 \in K$ , let  $u_1 = v_1$ . If  $v_1 \notin K$ , then from Lemma 1, there exists  $u_1 \in \partial K$  such that

$$\varrho(u_0, u_1) + \varrho(u_1, v_1) = \varrho(u_0, v_1).$$

Thus,  $u_1 \in K$ . Now, we claim that  $\varrho(v_1, Tu_1) \geq 0$ . Suppose  $\varrho(v_1, Tu_1) = 0$ . If  $v_1 \in K$ , then  $u_1$  is a fixed point of  $T$ , which is a contradiction. If  $v_1 \notin K$ , then  $u_1 \in \partial K$  and so  $Tu_1 \subseteq K$ . Therefore,  $v_1 \notin Tu_1$ , which is a contradiction. Thus,  $\varrho(v_1, Tu_1) \geq 0$ . Now, since  $\varrho(v_1, Tu_1) \leq \mathcal{H}(Tu_0, Tu_1)$ , then we have

$$\begin{aligned} \vartheta(\varrho(v_1, Tu_1)) &\leq \vartheta(\mathcal{H}(Tu_0, Tu_1)) \\ &\leq [\vartheta(\varrho(u_0, u_1))]^{\tau_1} [\vartheta(\varrho(u_0, Tu_0))]^{\tau_2} [\vartheta(\varrho(u_1, Tu_1))]^{\tau_3} \\ &\quad \times [\vartheta(\varrho(u_0, Tu_1) + \varrho(u_1, Tu_0))]^{\tau_4}. \end{aligned} \quad (2.2)$$

On the other hand, from  $\vartheta_5$  we get

$$\vartheta(\varrho(v_1, Tu_1)) = \vartheta(\inf\{\varrho(v_1, m) : m \in Tu_1\}) = \inf\{\vartheta(\varrho(v_1, m)) : m \in Tu_1\}$$

and so from condition (2.2) we get

$$\begin{aligned} \inf\{\vartheta(\varrho(v_1, m)) : m \in Tu_1\} &\leq [\vartheta(\varrho(u_0, u_1))]^{\tau_1} [\vartheta(\varrho(u_0, Tu_0))]^{\tau_2} [\vartheta(\varrho(u_1, Tu_1))]^{\tau_3} \\ &\quad \times [\vartheta(\varrho(u_0, Tu_1) + \varrho(u_1, Tu_0))]^{\tau_4}. \end{aligned}$$

Thus, there exists  $v_2 \in Tu_1$  such that

$$\vartheta(\varrho(v_1, v_2)) \leq [\vartheta(\varrho(u_0, u_1))]^{\tau_1} [\vartheta(\varrho(u_0, Tu_0))]^{\tau_2} [\vartheta(\varrho(u_1, Tu_1))]^{\tau_3} [\vartheta(\varrho(u_0, Tu_1) + \varrho(u_1, Tu_0))]^{\tau_4},$$

where

$$0 \leq \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4 < 1.$$

If  $v_2 \in K$  let  $u_2 = v_2$ . If  $v_2 \notin K$ , select a point  $u_2 \in \partial K$  such that

$$\varrho(u_1, u_2) + \varrho(u_2, v_2) = \varrho(u_1, v_2).$$

Thus,  $u_2 \in K$ . We can show that  $\varrho(v_2, Tu_2) > 0$ . As above, we can find a point  $v_3 \in Tu_2$  such that

$$\vartheta(\varrho(v_2, v_3)) \leq [\vartheta(\varrho(u_1, u_2))]^{\gamma_1} [\vartheta(\varrho(u_1, Tu_1))]^{\gamma_2} [\vartheta(\varrho(u_2, Tu_2))]^{\gamma_3} [\vartheta(\varrho(u_1, Tu_2) + \varrho(u_2, Tu_1))]^{\gamma_4}.$$

Continuing the arguments, two sequences  $\{u_n\}$  and  $\{v_n\}$  are obtained such that for  $n \in \mathbb{N}$  we have

- (i)  $v_{n+1} \in Tu_n$ ,
- (ii)  $\vartheta(\varrho(v_n, v_{n+1})) \leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, Tu_{n-1}))]^{\gamma_2} [\vartheta(\varrho(u_n, Tu_n))]^{\gamma_3} \times [\vartheta(\varrho(u_{n-1}, Tu_n) + \varrho(u_n, Tu_{n-1}))]^{\gamma_4}$ ,

where  $v_{n+1} = u_{n+1}$  if  $v_{n+1} \in K$  or

$$\varrho(u_n, u_{n+1}) + \varrho(u_{n+1}, v_{n+1}) = \varrho(u_n, v_{n+1}) \quad (2.3)$$

if  $v_{n+1} \notin K$  and  $u_{n+1} \in \partial K$ .

Now, we consider sets

$$P = \{u_\xi \in \{u_n\} : u_\xi = v_\xi, \xi \in \mathbb{N}\}, \quad Q = \{u_\xi \in \{u_n\} : u_\xi \neq v_\xi, \xi \in \mathbb{N}\}.$$

Observe that if  $u_\xi \in Q$  for some  $\xi$ , then  $u_{\xi+1} \in P$ . Here, the intention is to estimate the distance  $\varrho(u_n, u_{n+1})$  for  $n \geq 2$ . Note that  $\varrho(u_n, u_{n+1}) > 0$ , otherwise,  $T$  has a fixed point. For this, three cases have to be considered:

**Case 1.** If  $u_n \in P$  and  $u_{n+1} \in P$ , then, we get

$$\begin{aligned} \vartheta(\varrho(u_n, u_{n+1})) &= \vartheta(\varrho(v_n, v_{n+1})) \\ &\leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, Tu_{n-1}))]^{\gamma_2} [\vartheta(\varrho(u_n, Tu_n))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, Tu_n) + \varrho(u_n, Tu_{n-1}))]^{\gamma_4} \\ &= [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, u_{n+1}) + \varrho(u_n, u_n))]^{\gamma_4} \\ &= [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1 + \gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} [\vartheta(\varrho(u_{n-1}, u_n) + \varrho(u_n, u_{n+1}))]^{\gamma_4} \\ &\leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1 + \gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_4} [\varrho(u_n, u_{n+1})]^{\gamma_4} \\ &= [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1 + \gamma_2 + \gamma_4} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3 + \gamma_4}. \end{aligned}$$

It follows that

$$\vartheta(\varrho(u_n, u_{n+1})) \leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_3 - \gamma_4}}.$$

**Case 2.** If  $u_n \in P$  and  $u_{n+1} \in Q$ , then, from condition (2.3), we get

$$\vartheta(\varrho(u_n, u_{n+1})) \leq \vartheta(\varrho(u_n, u_{n+1}) + \varrho(u_{n+1}, v_{n+1})) = \vartheta(\varrho(v_n, v_{n+1})) \leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_3 - \gamma_4}}.$$

**Case 3.** If  $u_n \in Q$  and  $u_{n+1} \in P$ , then, since

$$\begin{aligned} \vartheta(\varrho(v_n, v_{n+1})) &\leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, Tu_{n-1}))]^{\gamma_2} [\vartheta(\varrho(u_n, Tu_n))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, Tu_n) + \varrho(u_n, Tu_{n-1}))]^{\gamma_4}, \end{aligned}$$

$\Rightarrow$

$$\varrho(v_n, v_{n+1}) < (\varrho(u_{n-1}, u_n))(\varrho(u_{n-1}, Tu_{n-1}))(\varrho(u_n, Tu_n))(\varrho(u_{n-1}, Tu_n) + \varrho(u_n, Tu_{n-1})).$$

In our case, if we simplify we get

$$\begin{aligned} \vartheta(\varrho(v_n, v_{n+1})) &\leq [\vartheta(\varrho(u_{n-1}, u_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, u_{n+1}) + \varrho(u_n, v_n))]^{\gamma_4} \\ &\leq [\vartheta(\varrho(u_{n-1}, u_n) + \varrho(u_n, v_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_2} \\ &\quad \times [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} [\vartheta(\varrho(u_{n-1}, u_n) + \varrho(u_n, u_{n+1}) + \varrho(u_n, v_n))]^{\gamma_4} \\ &\leq [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_1} [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, v_n) + \varrho(u_n, u_{n+1}))]^{\gamma_4} \\ &\leq [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_1 + \gamma_2} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3} \\ &\quad \times [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_4} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_4} \\ &= [\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_1 + \gamma_2 + \gamma_4} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3 + \gamma_4}. \end{aligned}$$

It follows that

$$\begin{aligned} \vartheta(\varrho(u_n, u_{n+1})) &\leq \vartheta(\varrho(u_n, v_n) + \varrho(v_n, u_{n+1})) \\ &< \vartheta(\varrho(u_{n-1}, u_n) + \varrho(u_n, v_n) + \varrho(v_n, v_{n+1})) \\ &= \vartheta(\varrho(u_{n-1}, v_n) + \varrho(v_n, v_{n+1})) \\ &\leq \vartheta(\varrho(u_{n-1}, v_n))\vartheta(\varrho(v_n, v_{n+1})) \\ &\leq \vartheta(\varrho(u_{n-1}, v_n))[\vartheta(\varrho(u_{n-1}, v_n))]^{\gamma_1 + \gamma_2 + \gamma_4} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3 + \gamma_4} \\ &= [\vartheta(\varrho(u_{n-1}, v_n))]^{1 + \gamma_1 + \gamma_2 + \gamma_4} [\vartheta(\varrho(u_n, u_{n+1}))]^{\gamma_3 + \gamma_4}. \end{aligned}$$

Hence

$$[\vartheta(\varrho(u_n, u_{n+1}))]^{1 - \gamma_3 - \gamma_4} \leq [\vartheta(\varrho(u_{n-1}, v_n))]^{1 + \gamma_1 + \gamma_2 + \gamma_4}$$

by Case 2, since  $u_n \in Q$  implies  $u_{n-1} \in P$  we have

$$\vartheta(\varrho(u_{n-1}, v_n)) \leq [\vartheta(\varrho(u_{n-2}, u_{n-1}))]^{\frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_3 - \gamma_4}}.$$

Therefore,

$$\vartheta(\varrho(u_n, u_{n+1})) \leq [\vartheta(\varrho(u_{n-2}, u_{n-1}))]^{\frac{(1 + \gamma_1 + \gamma_2 + \gamma_4)(\gamma_1 + \gamma_2 + \gamma_4)}{(1 - \gamma_3 - \gamma_4)^2}}.$$

The case that  $u_n \in Q$  and  $u_{n+1} \in Q$  does not occur.

Since

$$\frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_3 - \gamma_4} \leq \frac{(1 + \gamma_1 + \gamma_2 + \gamma_4)(\gamma_1 + \gamma_2 + \gamma_4)}{(1 - \gamma_3 - \gamma_4)^2},$$

for  $n \geq 2$  we have

$$\vartheta(\varrho(u_n, u_{n+1})) \leq \begin{cases} [\vartheta(\varrho(u_{n-1}, u_n))]^\gamma, \\ [\vartheta(\varrho(u_{n-2}, u_{n-1}))]^\gamma. \end{cases}$$

Now we claim that

$$\vartheta(\varrho(u_n, u_{n+1})) \leq \delta \left( \gamma^{\frac{n-1}{2}} \right) \tag{2.4}$$

for all  $n \in \mathbb{N}$ , where

$$\delta = \max\{\vartheta(\varrho(u_0, u_1)), \vartheta(\varrho(u_1, u_2))\}.$$

Using (2.4) we obtain

$$\lim_{n \rightarrow \infty} \vartheta(\varrho(u_n, u_{n+1})) = 1.$$

From  $(\vartheta_2)$ ,  $\lim_{n \rightarrow \infty} \varrho(u_n, u_{n+1}) = 0$  and so from  $(\vartheta_3)$  there exists  $r \in (0, 1)$  and  $l \in (0, \infty]$  such that

$$\lim_{n \rightarrow \infty} \frac{\vartheta(\varrho(u_n, u_{n+1})) - 1}{[\varrho(u_n, u_{n+1})]^r} = l.$$

Suppose that  $l < \infty$ . In this case, let  $\Psi = l/2 > 0$ . Recall from the definition of the limit, there exists  $n_0 \in \mathbb{N}$  such that, for all  $n \geq n_0$ ,

$$\left| \frac{\vartheta(\varrho(u_n, u_{n+1})) - 1}{[\varrho(u_n, u_{n+1})]^r} - l \right| \leq \Psi.$$

This implies that, for all  $n \geq n_0$ ,

$$\frac{\vartheta(\varrho(u_n, u_{n+1})) - 1}{[\varrho(u_n, u_{n+1})]^r} \geq l - \Psi = \Psi.$$

Then, for all  $n \geq n_0$ ,

$$n[\varrho(u_n, u_{n+1})]^r \leq \Phi n[\vartheta(\varrho(u_n, u_{n+1})) - 1],$$

where  $\Phi = 1/\Psi$ . Thus, in all cases, there exist  $\Phi > 0$  and  $n_0 \in \mathbb{N}$  such that, for all  $n \geq n_0$ ,

$$n[\varrho(u_n, u_{n+1})]^r \leq \Phi n[\vartheta(\varrho(u_n, u_{n+1})) - 1].$$

Using (2.4), we obtain, for all  $n \geq n_0$ ,

$$n[\varrho(u_n, u_{n+1})]^r \leq \Phi n[\delta^{(\gamma \frac{n-1}{2})} - 1].$$

Letting  $n \rightarrow \infty$  in the above inequality, we obtain

$$\lim_{n \rightarrow \infty} n[\varrho(u_n, u_{n+1})]^r = 0.$$

Thus, there exists  $n_1 \in \mathbb{N}$  such that  $n[\varrho(u_n, u_{n+1})]^r \leq 1$  for all  $n \geq n_1$ . So, we have, for all  $n \geq n_0$

$$\varrho(u_n, u_{n+1}) \leq \frac{1}{n^{1/r}}. \quad (2.5)$$

In order to demonstrate that  $\{x_n\}$  is a Cauchy sequence consider  $m, n \in \mathbb{N}$  such that  $m > n \geq n_1$ . Now, applying the metric triangle inequality and from condition (2.5), we have

$$\begin{aligned} \varrho(u_n, u_m) &\leq \varrho(u_n, u_{n+1}) + \varrho(u_{n+1}, u_{n+2}) + \cdots + \varrho(u_{m-1}, u_m) \\ &= \sum_{\xi=n}^{m-1} \varrho(u_\xi, u_{\xi+1}) \leq \sum_{\xi=n}^{\infty} \varrho(u_\xi, u_{\xi+1}) \leq \sum_{\xi=n}^{\infty} \frac{1}{\xi^{1/r}}. \end{aligned}$$

Since the series  $\sum_{\xi=1}^{\infty} \frac{1}{\xi^{1/r}}$  converges, then passing to limit with  $n, m \rightarrow \infty$ , we get  $\varrho(u_n, u_m) \rightarrow 0$ .

It is an obvious implication that the sequence  $\{u_n\}$  is a Cauchy sequence in  $K$ . Since  $K$  is closed, the sequence  $\{u_n\}$  converges to some point  $z \in K$ . By our choice of  $\{u_n\}$ , there exists a subsequence  $\{u_{n_k}\}$  of  $\{u_n\}$  such that  $\{u_{n_k}\} \in P$  that is,  $\{u_{n_k}\} = \{v_{n_k}\}$ ,  $k \in \mathbb{N}$ . Note that

$\{u_{n_k}\} \in \{u_{n_{k-1}}\}$  for  $k \in \mathbb{N}$  and  $\{u_{n_k}\} \rightarrow z$  as  $k \rightarrow \infty$ . Also note that from condition (2.1) and  $(\vartheta_1)$  we get

$$\mathcal{H}(Tu, Tv) \leq (\varrho(u, v))(\varrho(u, Tu))(\varrho(v, Tv))(\varrho(v, Tu) + \varrho(u, Tv))$$

for all  $u, v \in K$  and so, we have

$$\begin{aligned} \varrho(u_{n_k}, Tz) &\leq \mathcal{H}(Tu_{n_{k-1}}, Tz) \\ &\leq \varrho(u_{n_{k-1}}, z)\varrho(u_{n_{k-1}}, Tu_{n_{k-1}})\varrho(z, Tz)\{\varrho(z, Tu_{n_{k-1}}) + \varrho(u_{n_{k-1}}, Tz)\} \end{aligned}$$

which on letting  $k \rightarrow \infty$  implies that  $\varrho(z, Tz) = 0$ , which is a contradiction. Therefore,  $T$  has a fixed point  $z \in K$ .  $\square$

*Remark 1.* If  $\tau_2 = \tau_3 = \tau_4 = 0$  and  $\tau_1 = \tau$  in Theorem 4 we obtain Theorem 3 of Altun [2].

For particular function  $\vartheta$  selections, some significant results are obtained. First, by setting  $\vartheta(\mu) = e^{\sqrt{\mu}}$  in Theorem 4, the following corollary is obtained:

**Corollary 1.** *Let  $K$  be a nonempty closed subset of a complete and metrically convex metric space  $M$ . Let  $T : K \rightarrow CB(M)$  be a mapping such that the following condition holds:*

$$\sqrt{\mathcal{H}(Tu, Tv)} \leq \tau_1 \sqrt{\varrho(u, v)} + \tau_2 \sqrt{\varrho(u, Tu)} + \tau_3 \sqrt{\varrho(v, Tv)} + \tau_4 \sqrt{\varrho(u, Tv) + \varrho(v, Tu)}$$

for all  $u, v \in K$ ,  $\vartheta \in \Theta$  and  $\tau_1, \tau_2, \tau_3, \tau_4 \geq 0$  with  $0 \leq \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$ . Then  $T$  has a unique fixed point.

And, by putting  $\vartheta(\mu) = e^{\sqrt[3]{\mu}}$  in Theorem 4, the following corollary is obtained:

**Corollary 2.** *Let  $K$  be a nonempty closed subset of a complete and metrically convex metric space  $M$ . Let  $T : K \rightarrow CB(M)$  be a mapping such that the following condition holds:*

$$\sqrt[3]{\mathcal{H}(Tu, Tv)} \leq \tau_1 \sqrt[3]{\varrho(u, v)} + \tau_2 \sqrt[3]{\varrho(u, Tu)} + \tau_3 \sqrt[3]{\varrho(v, Tv)} + \tau_4 \sqrt[3]{\varrho(u, Tv) + \varrho(v, Tu)}$$

for all  $u, v \in K$ ,  $\vartheta \in \Theta$  and  $\tau_1, \tau_2, \tau_3, \tau_4 \geq 0$  with  $0 \leq \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$ . Then  $T$  has a unique fixed point.

### 3. Application to nonlinear integral equations

Nonlinear integral equations can be solved using a variety of numerical approaches. The integral equation is usually transformed into a system of nonlinear algebraic equations. Solving these systems is difficult, or the solution may be impossible to find. Therefore, in this section, we describe how the fixed point approach may be used to solve Volterra–Hammerstein integral equations. This approach does not result in a system of nonlinear algebraic equations.

Now, consider the nonlinear integral equation below:

$$u(t) = g(t) + \int_a^b k(t, \tau) \mathcal{H}(\tau, u(\tau)) d\tau, \quad (3.1)$$

where  $t, \tau \in [a, b]$ ,  $a, b \in \mathbb{R}$ ,  $u \in C[a, b]$ ,  $g : [a, b] \rightarrow \mathbb{R}$ ,  $\mathcal{H} \in C[a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $k \in C^2[a, b]^2$  such that  $k(t, \tau) > 0$  are given functions.

Maleknejad [12] established some conditions which ensure the uniqueness of the solution and how the fixed point method approximates this solution.

Referring from Maleknejad [12] we are going to establish the following fixed point theorem.



**Theorem 5.** Let  $M = C[a, b]$  be a metric space endowed with the metric

$$\varrho(u, v) = \sup_{t \in [a, b]} |u(t) - v(t)|.$$

Define the mapping  $T : K \rightarrow CB(M)$  by

$$T(u)(t) = g(t) + \int_a^b k(t, \tau) \mathcal{H}(\tau, u(\tau)) d\tau.$$

Let  $u, v \in K$  and  $t \in [a, b]$ . Assume that  $g \in C[a, b]$ ,  $k \in C^2[a, b]^2$ , i.e. there exists a constant  $M > 0$  where

$$\left( \int_a^b k^2(t, \tau) d\tau \right)^{\frac{1}{2}} \leq M < \infty,$$

and  $\mathcal{H} : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there is  $\vartheta \in \Theta$  so that  $\vartheta(\sup f(t)) = \sup \vartheta(f(t))$  for arbitrary function  $f$  with

$$\vartheta \left( \int_a^b |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))| d\tau \right) \leq \int_a^b \vartheta(|\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))|) d\tau,$$

there is  $\tau_i \in (0, 1)$  where  $i = 1, 2, 3, 4$  such that

$$\begin{aligned} \vartheta(|\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))|) &\leq \left\{ [\vartheta(|u(t) - v(t)|)]^{\tau_1} [\vartheta(|u(t) - \int_a^b k(t, \tau) \mathcal{H}(\tau, u(\tau)) d\tau|)]^{\tau_2} [\vartheta(|v(t) \right. \\ &\quad \left. - \int_a^b k(t, \tau) \mathcal{H}(\tau, v(\tau)) d\tau|)]^{\tau_3} [\vartheta(|u(t) - \int_a^b k(t, \tau) \mathcal{H}(\tau, v(\tau)) d\tau| + |v(t) \right. \\ &\quad \left. - \int_a^b k(t, \tau) \mathcal{H}(\tau, u(\tau)) d\tau|)]^{\tau_4} \right\} / M(b-a). \end{aligned}$$

Then equation (3.1) has a unique solution.

*P r o o f.* We begin our proof by deriving the following relation where Cauchy–Schwartz inequality is used:

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left| \int_a^b k(t, \tau) (\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))) d\tau \right| \\ &\leq \int_a^b |k(t, \tau)| |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))| d\tau \\ &\leq \left( \int_a^b k^2(t, \tau) d\tau \right)^{1/2} \left( \int_a^b |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))|^2 d\tau \right)^{1/2} \\ &\leq M \left( \int_a^b |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))|^2 d\tau \right)^{1/2} \\ &\leq M \int_a^b |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))| d\tau. \end{aligned}$$

Now, we have

$$\begin{aligned} \vartheta(|Tu(t) - Tv(t)|) &= \vartheta \left( M \int_a^b |\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))| d\tau \right) \\ &\leq M \int_a^b \vartheta(|\mathcal{H}(\tau, u(\tau)) - \mathcal{H}(\tau, v(\tau))|) d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ M[\vartheta(|u(t) - v(t)|)]^{\tau_1} [\vartheta(|u(t) - \int_a^b k(t, \tau)\mathcal{H}(\tau, u(\tau))d\tau|)]^{\tau_2} \right. \\
&\quad \left. [\vartheta(|v(t) - \int_a^b k(t, \tau)\mathcal{H}(\tau, v(\tau))d\tau|)]^{\tau_3} [\vartheta(|u(t) - \int_a^b k(t, \tau)\mathcal{H}(\tau, v(\tau))d\tau| \right. \\
&\quad \left. + |v(t) - \int_a^b k(t, \tau)\mathcal{H}(\tau, u(\tau))d\tau|)]^{\tau_4} \right\} / M(b - a) \\
&\leq \frac{1}{b - a} \int_a^b [\vartheta(\varrho(u, v))]^{\tau_1} [\vartheta(\varrho(u, Tu))]^{\tau_2} [\vartheta(\varrho(v, Tv))]^{\tau_3} [\vartheta(\varrho(u, Tv) + \varrho(v, Tu))]^{\tau_4} \\
&= [\vartheta(\varrho(u, v))]^{\tau_1} [\vartheta(\varrho(u, Tu))]^{\tau_2} [\vartheta(\varrho(v, Tv))]^{\tau_3} [\vartheta(\varrho(u, Tv) + \varrho(v, Tu))]^{\tau_4}.
\end{aligned}$$

Thus, all the conditions of Theorem 4 are satisfied. Hence the integral equation (3.1) has a solution.  $\square$

#### 4. Conclusion

The main contribution of this study is Definition 5 and Theorem 4. This theorem is proved for multivalued non-self mappings in complete and metrically convex space. This theorem generalizes the fixed point theorem due to Altun and Minak [2] by using  $\vartheta$ -contraction due to Jleli and Sameti [9]. To validate the results proved here, we provide an appropriate application of our main result.

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