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OUTPUT CONTROLLABILITY OF DELAYED CONTROL SYSTEMS IN A LONG TIME HORIZON[1]

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Abstract: In this paper, we consider the output controllability of finite-dimensional control systems governed by a distributed delayed control. For systems with ordinary controls, this problem was investigated earlier. Nevertheless, in many practical and technical problems the control acts with some delay. We give the necessary and sufficient condition for the output controllability. The main goal of our control is to govern the output of the system to some position on a subspace in a given instant, and then keep this output fixed for the remaining times. This property is called the long-time output controllability. For this, sufficient conditions are given. The introduced notions are applied for the investigation of averaged controllability of systems with delayed controls. The general approach for that is to approximate the system by the ordinary one. Some examples are considered.

Keywords: Output and averaged controllability, Delayed control, Approximation.

1. Introduction

In this paper, we deal with the output controllability of finite-dimensional control systems governed by a distributed delayed control. For systems with ordinary controls, this problem is investigated in [2]. It is known that in many practical and technical problems the controlling actions take place with some delays. The main goal of our control is to govern the output of the system to some position on a subspace in a given time $T > 0$, and then keep this output fixed for the remaining times $t > T$.

Consider a linear autonomous system with delayed controls and observation:

$$\dot{x}(t) = Ax(t) + \int_{-h}^{0} dB(s)u(t+s), \quad t \geq 0, \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \tag{1.1}$$

$$y(t) = Cx(t), \quad C \in \mathbb{R}^{p\times n}, \tag{1.2}$$

where elements of the matrix function $B(s)$ belong to $BV[-h, 0]$ (the space of functions of bounded variation) and they are left continuous on $(-h, 0]$, $B(s) = 0$ for $\forall s > 0$, and $B(s) = B(-h)$ for $\forall s \leq -h$. Since the matrix $B(s)$ generates a Borelian measure, any bounded Borelian m-vector function $u(t)$ can be used as a control. The notion of *output controllability* is as follows.

Definition 1. *We say that the system* (1.1) *is* C-output controllable*, if for every $x_0 \in \mathbb{R}^n$ and every $\bar{y} \in \operatorname{im} C = \{y \mid y = Cx,\, x \in \mathbb{R}^n\}$ there exist an instant $T > 0$ and a bounded Borelian control u on $[-h, T]$ such that the solution $x(t)$ of* (1.1) *with initial condition $x(0) = x_0$ satisfies $y(T) = Cx(T) = \bar{y}$.*

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Let us recall that the ordinary (state) controllability definition of the system (1.1) follows from Definition 1 when $C = I_n$. The symbol $I_n \in \mathbb{R}^{n\times n}$ means the unity matrix.

In this paper, we are interested in the conditions for having *long-time output controllability.* This controllability notion means that the output of the system enters the subspace and then remains on it for later times. This is defined as follows.

Definition 2. *Given $\bar y \in \operatorname{im} C$, the system (1.1) is said to be* C-long-time output controllable *(briefly C-LTOC on $\bar y$), if for every x_0 there exist a time $T > 0$ and a control u such that the solution of (1.1), with initial condition $x(0) = x_0$ satisfies $y(t) = \bar y$ for every $t \in [T, \infty)$.*

It is obvious that state controllability implies C-output controllability for any matrix C. But in order to save the property $y(t) = \bar y$ for every $t \geq T$ we need extra conditions on a delayed control. In this paper we only assume the C-output controllability of the system and give a criterion for this. Our main attention is directed to conditions of C-LTOC (or simply LTOC) for systems of the form (1.1), (1.2) and their applications.

The notions of output controllability and C-LTOC can be applied to averaged controllability property of finite-dimensional, parameter dependent systems with delayed controls. The averaged controllability has been introduced in the paper [4]. More precisely, let us consider d realizations of control systems,

$$\dot x_i(t) = A_i x(t) + \int_{-h}^{0} dB_i(s)u(t+s), \quad t \geq 0, \quad x_i(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad i \in 1:d, \tag{1.3}$$

and d parameters $p_i > 0$, $\sum_{i=1}^{d} p_i = 1$. Here the matrices $B_i(s)$ have the same properties as $B(s)$ in (1.1).

Definition 3. *We say that the flock of systems (1.3) is* controllable in average *for the weights $p_i > 0$ if for all initial states $x_{10}, \dots, x_{d0}$ and every $\bar y \in \mathbb{R}^n$ there exist an instant $T > 0$ and a bounded Borelian control u on $[-h, T]$ such that the solutions of (1.3) satisfy the equality $\sum_{i=1}^{d} p_i x_i(T) = \bar y$.*

Let us use the Matlab notation for matrices and vectors. We can see that the averaged controllability notion is exactly the C-output controllability of (1.1)–(1.2) with matrices:

$$A = \operatorname{diag}[A_1, \dots, A_d], \quad B(s) = [B_1(s); \dots; B_d(s)], \quad C = [p_1 I_n, \dots, p_d I_n],$$

where $x = [x_1; \dots; x_d] \in \mathbb{R}^{nd}$. The flock of systems (1.3) is called *simultaneously controllable* if corresponding system (1.1) is state controllable. Of course, the simultaneous controllability of (1.3) implies the averaged controllability. We can also define the notion of *long-time averaged controllability* (briefly LTAC on $\bar y$). We say that systems (1.3) are LTAC on $\bar y$ for the weights $p_i > 0$ if for every initial states $x_{10}, \dots, x_{d0}$ there exist an instant $T > 0$ and an admissible control u such that the corresponding mean value is the following

$$\sum_{i=1}^{d} p_i x_i(t) = \bar y$$

for every $t \in [T, \infty)$.

In this paper, we obtain conditions of C-output controllability and C-LTOC for general systems (1.1)–(1.2) and apply them for the LTAC property of (1.3). Besides, we get the algorithm for constructing of necessary control in special cases. Some examples are considered.

2. Output controllability of the system

First, note that the initial condition x_0 does not play any role in Definition 1. System (1.1), (1.2) is C-output controllable iff for every $\bar{y} \in \operatorname{im} C$ there exist an instant $T > 0$ and an admissible control u on $[-h, T]$ such that

$$C \int_0^T e^{A(T-\theta)} \int_{-h}^0 dB(s)u(\theta + s)d\theta = \bar{y}.$$

Setting $\alpha = \theta + s$ we have

$$\int_{-h}^0 dB(s)u(\theta + s) = \int_{\theta-h}^{\theta} dB(\alpha - \theta)u(\alpha),$$

and by Fubini's theorem we get the equivalent equality

$$\int_{-h}^T \mathcal{B}(T, \alpha)u(\alpha)d\alpha = \bar{y}, \quad \text{where} \quad \mathcal{B}(T, \alpha) = C \int_{\alpha \vee 0}^{(\alpha+h)\wedge T} e^{A(T-\theta)} dB(\alpha - \theta). \tag{2.1}$$

The $n \times m$-matrix function $\mathcal{B}(T, \alpha)$ is of bounded variation with respect to α and, therefore, belongs to the space $L_2^{p\times m}[-h, T]$ (the space of square integrable matrices or vectors). We can prove the following lemma.

Lemma 1. *System (1.1), (1.2) is C-output controllable iff there is a segment $[a, b]$, $-h \le a < b \le T$, such that*

$$\operatorname{rank} \left(\int_a^b \mathcal{B}(T, \alpha)\mathcal{B}'(T, \alpha)d\alpha \right) = \operatorname{rank} C. \tag{2.2}$$

P r o o f. Let condition (2.2) be satisfied. Since

$$\operatorname{im} \int_a^b \mathcal{B}(T, \alpha)\mathcal{B}'(T, \alpha)d\alpha \subset \operatorname{im} C,$$

we obtain the equality of subspaces in this inclusion. For every $\bar{y} \in \operatorname{im} C$ there exists a vector $v \in \mathbb{R}^p$ such that

$$\int_a^b \mathcal{B}(T, \alpha)\mathcal{B}'(T, \alpha)d\alpha v = \bar{y}.$$

Then $u(\alpha) = \mathcal{B}'(T, \alpha)v$ is a bounded Borelian control on $[a, b]$. We can take $u(\alpha) = 0$, $\alpha \notin [a, b]$, and satisfy (2.1) for any $T \ge b$. On the contrary, let condition (2.1) be valid, but there is a vector $\bar{y} \in \operatorname{im} C$ such that

$$\bar{y} \notin \operatorname{im} \int_{-h}^T \mathcal{B}(T, \alpha)\mathcal{B}'(T, \alpha)d\alpha.$$

Then we have a contradiction with (2.1) as

$$L_2^m[-h, T] = \{u(\alpha) : u(\alpha) = \mathcal{B}'(T, \alpha)v,\ v \in \mathbb{R}^p\} \bigoplus \left\{u(\alpha) : \int_{-h}^T \mathcal{B}(T, \alpha)u(\alpha)d\alpha = 0\right\}.$$

Therefore, there are no functions $u \in L_2^m[-h, T]$ satisfying (2.1). □

Corollary 1. *Condition (2.2) holds iff the equality*

$$l'\mathcal{B}(T,\alpha)=0 \quad \textit{a.e. on } [a,b] \quad \textit{implies that} \quad l\in\ker C'. \tag{2.3}$$

P r o o f. Condition (2.2) is equivalent to the equality

$$\ker C'=\ker\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha,$$

or, in other words, the implication (2.3) is fulfilled. □

Corollary 2. *The function $\mathcal{B}(T,\alpha)$ from (2.1) can be expressed in the form*

$$\mathcal{B}(T,\alpha)=Ce^{A(T-\alpha)}\boldsymbol{b}(T,\alpha), \quad \textit{where} \quad \boldsymbol{b}(T,\alpha)=\int_{(\alpha-T)\vee(-h)}^{\alpha\wedge 0} e^{As}dB(s). \tag{2.4}$$

If $T>h$ and $a=0$, $b=T-h$, then

$$\boldsymbol{b}(T,\alpha)=\int_{-h}^{0} e^{As}dB(s)=\text{const}$$

on $[a,b]$. Hence, the implication (2.3) is equivalent to the rank condition

$$\operatorname{rank} C\left[\int_{-h}^{0} e^{A(s+h)}dB(s), A\int_{-h}^{0} e^{A(s+h)}dB(s),\dots,A^{n-1}\int_{-h}^{0} e^{A(s+h)}dB(s)\right]=\operatorname{rank} C. \tag{2.5}$$

P r o o f. Setting $\alpha-\theta=s$ in (2.1) we get (2.4). As $\mathbf{b}(T,\alpha)=$const on segment $[a,b]$, the relation $l'\mathcal{B}(T,\alpha)=0$ can be differentiated with respect to α many times. So, we come to the equivalence of implication (2.3) and rank condition (2.5) by the theorem of Cayley–Hamilton [5, Theorem 7.2.4]. □

Let us discuss the Lemma 1 and its Corollaries. If condition (2.2) does not hold for some segment $[a,b]$, it can be hold for grater ones. The rank condition for C-output controllability is possible if the matrix function $B(s)$ is piecewise-constant as in the case of lumped delays. The simplest case of lumped delays is given by

$$B(s)=-B_0\chi_{(-\infty,0]}(s)-B_1\chi_{(-\infty,-1]}(s), \tag{2.6}$$

where the indicator function $\chi_{(a,b]}(s)=1$ if $s\in(a,b]$, and $\chi_{(a,b]}(s)=0$, elsewhere. Let $T>1$. We can divide the segment $[-1,T]=(T-1,T]\cup[0,T-1]\cup[-1,0)$ into three parts. On the first semi-interval the implication

$$l'Ce^{A(T-\alpha)}B_0\equiv 0\Rightarrow l\in\ker C'$$

is equivalent to the condition

$$\operatorname{rank} C[B_0,AB_0,\dots,A^{n-1}B_0]=\operatorname{rank} C \tag{2.7}$$

by the theorem of Cayley–Hamilton. On $[0,T-1]$ we have

$$\mathbf{b}(T,\alpha)=B_0+e^{-A}B_1,$$

and we are in the conditions of Corollary 2. On the remaining semi-interval the implication

$$l'Ce^{A(T-\alpha-1)}B_1\equiv 0\Rightarrow l\in\ker C'$$

is equivalent to condition

$$\operatorname{rank} Ce^{A(T-1)}[B_1, AB_1, \ldots, A^{n-1}B_1] = \operatorname{rank} C. \tag{2.8}$$

If $0 \le T < 1$, we have only two segments $[0, T]$ and $[-1, T-1]$. Therefore, we get two conditions in (2.7) and (2.8), where the matrix of $e^{A(T-1)}$ is absent. Conditions (2.5), (2.7) do not depend on T, but condition in (2.8) nevertheless depends. It distinguishes case of delayed controls from the ordinary one.

Remark 1. Of course, we can also consider partly different Definitions 1–3 with null initial controls, i.e. when $u(t) = 0$ for $t < 0$. Then the integral in (2.1) is considered on $[0, T]$, the parameter $a \ge 0$ in Lemma 1, and condition (2.8) is not necessary. In addition, we may demand that $u(t) = 0$ when $t \in [T-h, T]$, $T > h$. Then we have $0 \le a < b \le T - h$ in Lemma 1.

3. The property of LTOC

Suppose further that system (1.1)–(1.2) is C-output controllable at some instant $T > h$ and $u(t) = 0$ if $t \notin [0, T-h]$ as in the Remark 1. Then it is easily seen that the C-output controllability on $[0, T]$ is equivalent to C-output controllability on $[a, T+a]$ for all $a \ge 0$ when $u(t) = 0$ if $t \notin [a, T+a-h]$. Therefore, in order to get $y(t) \equiv \bar{y}$, $\forall t \ge T$, we need to obtain the conditions for the property

$$Cx_0 = Cx(t) \quad \forall t \ge 0. \tag{3.1}$$

By derivation with respect to t in (3.1), we have:

$$CAx(t) + C\int_{-h}^{0} dB(s)u(t+s) = 0 \quad \forall t \ge 0. \tag{3.2}$$

Introduce the subspace

$$\mathcal{U} = \left\{ x \in \mathbb{R}^n : \exists \text{ an admissible function } u(\cdot) \ s.t. \ x = \int_{-h}^{0} dB(s)u(s) \right\}. \tag{3.3}$$

To satisfy (3.2), one needs to have the inclusion $CAx(t) \in C\mathcal{U}$ for $\forall t \ge 0$. Since $\mathbb{R}^p = C\mathcal{U} \oplus C\mathcal{U}^{\perp}$, we can take an orthonormal basis $\{h_1, \ldots, h_q\}$ in $C\mathcal{U}^{\perp}$, where $q = p - \dim(C\mathcal{U})$, and the corresponding matrix $H = [h_1, \ldots, h_q] \in \mathbb{R}^{p\times q}$. As a result, we get a projector $P_0 = HH' \in \mathbb{R}^{p\times p}$ on the subspace $C\mathcal{U}^{\perp}$.

Consequently, condition (3.2) is $L_0x(t) = 0 \ \forall t \ge 0$, where $L_0 = P_0CA$. Therefore, if we introduce the matrix $C_1 = [C; L_0] \in \mathbb{R}^{2p\times n}$, then $C_1x_0 = C_1x(t) \ \ \forall t \ge 0$, as in (3.1).

We can iterate the process similar to ordinary case with no delays as in [2] to define $C_2 = [C, L_1]$, $L_1 = P_1C_1A$, and so on. After k steps we get

$$C_{k+1} = [C; L_k], \quad L_k = P_kC_kA \in \mathbb{R}^{(k+1)p\times n}, \tag{3.4}$$

where $P_k \in \mathbb{R}^{(k+1)p\times(k+1)p}$ is the orthogonal projector on $C_k\mathcal{U}^{\perp}$. The process stops when $\ker C_{k+1} = \ker C_k$. The condition (3.1) can be fulfilled iff $L_kx_0 = 0$. To be more exact, the following assertion holds.

Lemma 2. *We have* $\ker C_{k+1} \subset \ker C_k \subset \mathbb{R}^n$ *and* $\ker L_{k+1} \subset \ker L_k \subset \mathbb{R}^n$ *for every* $k \in \mathbb{N}\cup\{0\}$. *There exists a number* $K \in 0 : n$ *such that* $\ker C_{K+1} = \ker C_K$. *Here* $C_0 = C$. *For every* $i \in \mathbb{N}$ *we have* $\ker C_{K+i} = \ker C_K$.

P r o o f. We will argue by induction. As $\ker C_1 = \ker C_0 \cap \ker L_0$, we trivially obtain $\ker C_1 \subset \ker C_0$. Suppose that $\ker C_k \subset \ker C_{k-1}$ for some $k \in \mathbb{N}$. Then we notice that

$$\begin{aligned} L_k x = 0 \;\Leftrightarrow\; \exists\, u \in \mathcal{U} \quad s.t. \quad C_k(Ax-u)=0 \\ \Rightarrow\; C_{k-1}(Ax-u)=0 \;\Leftrightarrow\; L_{k-1}x=0. \end{aligned}$$

This means that $\ker L_k \subset \ker L_{k-1}$. Therefore, $\ker C_{k+1} \subset \ker C_k$. It is obvious that there exists a number $K \in 0:n$ such that $\ker C_{K+1} = \ker C_K \subset \mathbb{R}^n$. It follows that $\ker L_{K+1} = \ker L_K \subset \mathbb{R}^n$. Indeed,

$$\begin{aligned} L_K x = 0 \;\Leftrightarrow\; \exists\, u \in \mathcal{U} \quad s.t. \quad C_K(Ax-u)=0 \\ \Rightarrow\; C_{K+1}(Ax-u)=0 \;\Leftrightarrow\; L_{K+1}x=0. \end{aligned}$$

Hence, by induction, we obtain the final assertion. □

Note also that $\operatorname{im} L_k \subset \operatorname{im} C_k$ for every $k \in \mathbb{N} \cup \{0\}$. This is equivalent to the inclusion $\ker A'C_k'P_k \supset \ker C_k'$. Indeed, if $C_k'z = 0$, then $z \perp C_k\mathcal{U} \Rightarrow\; z \in C_k\mathcal{U}^{\perp} \Rightarrow\; P_k z = z$.

The problem of control with delays to ensure equality (3.1) is more difficult than for ordinary controls. Let us prove the lemma.

Lemma 3. *Let C_k, $k \in \mathbb{N} \cup \{0\}$, be the sequence defined by* (3.4)*, and let $K \in 0:n$ such that $\ker C_{K+1} = \ker C_K$. Then there exists a function $v(t) \in \mathcal{U}$ such that* (3.1) *holds where*

$$\dot{x}(t) = Ax(t) + v(t), \quad x(0) = x_0, \tag{3.5}$$

if and only if $L_K x_0 = 0$ with L_K defined by (3.4).

P r o o f. It follows from (3.1) that $C_K x_0 = C_K x(t)$ and $L_K x_0 = 0$. On the contrary, assume that $L_K x_0 = 0$. We need

$$C_K x_0 = C_K x(t) \quad \forall t \geq 0.$$

After derivation we get

$$C_K \dot{x}(t) = C_K(Ax(t) + v(t)), \quad v(t) \in \mathcal{U}. \tag{3.6}$$

If we find $v(t)$ with $C_K \dot{x}(t) = 0$, then the lemma is proved. We can write

$$C_K \dot{x}(t) = L_K x(t) + (I_{(K+1)p} - P_K)C_K A x(t) + C_K v(t).$$

Here $I_{(K+1)p} - P_K$ is a projector on $C_K\mathcal{U}$. Hence, there exists a continuous closed-loop control $v(x)$ such that

$$(I_{(K+1)p} - P_K)C_K A x + C_K v(x) = 0.$$

Relation (3.6) under such a control reduces to

$$C_K \dot{x}(t) = L_K x(t).$$

Let us write the orthogonal expansion for $x(t)$:

$$x(t) - x_0 = x_0(t) + x_1(t), \tag{3.7}$$

where $x_0(t) \in \ker C_K$ and $x_1(t) \in \operatorname{im} C_K'$. Then

$$C_k x(t) = C_K(x_0 + x_1(t))$$

and

$$L_K x(t) = L_K x_1(t)$$

as

$$\ker C_K = \ker C_{K+1} = \ker C_0 \cap \ker L_K.$$

Thus, we get that

$$C_K \dot{x}_1(t) = L_K x_1(t).$$

The matrix C_K is invertible on the subspace $\operatorname{im} C'_K$ and $x_1(0) = 0$ from (3.7). Therefore, $x_1(t) = 0$ $\forall t \geq 0$. The lemma is proved. □

It follows from Lemma 3 that conditions

$$L_K x_0 = 0 \quad \text{and} \quad \left(I_{(K+1)p} - P_K\right) C_K A x + C_K v(x) = 0$$

are necessary and sufficient for the solution $x(t)$ of equation (3.5) to satisfy (3.1). They define the function $\bar{v}(t) = v(x(t))$, but for our purposes we need a function $u(t)$ such that

$$\int_{(-h)\vee(-t)}^{0} dB(s)u(t+s) = \bar{v}(t), \quad t \geq 0. \tag{3.8}$$

This is an integral equation. It can have no solutions. Therefore, in next sections we consider the approximation scheme to exclude equations like (3.8). Now, we formulate the general result.

Theorem 1. *Let be given $\bar{y} \in \operatorname{im} C$. For every $x_0 \in \mathbb{R}^n$ there exists an admissible control for the system (1.1) such that the solution satisfies $Cx(t) = \bar{y}$ $\forall t \geq T$ if and only if*

$$[\bar{y}; 0] \in \operatorname{im} C_{K+1}$$

and system (1.1), (1.2) is C_{K+1}-output controllable in the sense of Remark 1, i.e. the condition like (2.2) holds for some $0 \leq a < b \leq T - h$ with

$$\mathcal{B}(T, \alpha) = C_{K+1} e^{A(T-\alpha)} \boldsymbol{b}(T, \alpha)$$

and $\operatorname{rank} C_{K+1}$*. Here C_k is the sequence defined by (3.4) and the number K is defined by Lemma 2. Besides, equation (3.8) has to be resolved for the function $\bar{v}(t)$ defined in Lemma 3.*

P r o o f. According to the Lemma 3 the control exists iff the system is transferred to the state $x(T)$ such that $C_{K+1}x(T) = [\bar{y}; 0]$. This is possible for every $\bar{y} \in \operatorname{im} C$ and every $x_0 \in \mathbb{R}^n$ iff the system is C_{K+1}-output controllable. After that we solve the problem as in Lemma 3 which does not depend of initial instant T. □

We do not give any sufficient conditions for the existence of a solution of integral equation (3.8). This is considered in some special cases. For example, in simplest case (2.6) we have the difference equation

$$B_0 u(t) + B_1 u(t-1) = \bar{v}(t), \quad u(t) = 0 \quad \text{if} \quad t < 0$$

which can be resolved step-by-step on segments $[i-1, i]$:

$$B_0 u_i(t) = \bar{v}(t) - B_1 u_{i-1}(t-1), \quad t \in [i-1, i], \quad i \in \mathbb{N}, \tag{3.9}$$

where $u_0(t) = 0$.

Example 1. Consider the flock of two systems of the form:

$$\dot{x}_1^1 = -x_1^2 + \sum_{i,j=1}^{2} b_{1ij} u_i(t-j+1), \quad \dot{x}_1^2 = x_1^1; \quad \text{first system},$$
$$\dot{x}_2^1 = x_2^1 + 2x_2^2 + \sum_{i,j=1}^{2} b_{2ij} u_i(t-j+1), \quad \dot{x}_2^2 = x_2^2; \quad \text{second system},$$

where we have the case with $p_1 = p_2 = 1/2$ and $C = [I_2, I_2]/2$. Condition (3.1) reduces to the requirement: $Cx_0 = 0$ implies $Cx(t) = 0$ if $t \geq 0$. Here $x(t) \in \mathbb{R}^4$ is the composed vector. Below we study the example in detail for various coefficients b_{lij}.

4. The system in the infinite-dimensional space

Let us now rewrite the system (1.1)–(1.2) in the infinite-dimensional space following [1]. We can write

$$\int_{-h}^{0} dB(s)u(s) = B_0 u(0) + \mathbb{B}u, \quad \text{where} \quad B_0 = -B(0), \quad \mathbb{B}u = \int_{[-h,0)} dB(s)u(s). \tag{4.1}$$

Formula (4.1) is true for continuous vector-functions u, but we want to use functions $\{u \in \mathcal{H} = L_2^m[-h, 0]\}$. In this case we consider the operator $\mathbb{B}$ as unbounded with dense domain $\mathbf{b}(\mathbb{B}) = W = W_{1,2}^m[-h, 0]$ (the Sobolev space). If $u \in W$, the function $\phi(t, s) = u(t + s)$ satisfies the equation in partial derivatives:

$$\dot{\phi}(t, s) = D\phi(t, s), \quad \phi(0, s) = u_0(s), \quad \phi(t, 0) = u(t), \tag{4.2}$$

with the operator $D = d/ds$. Equation (4.2) is considered in $\mathcal{H}$ with unbounded D. The left-shift C_0-semigroup S_t on $\mathcal{H}$ is defined by

$$(S_t u)(s) = \begin{cases} u(t+s), & s \in [-h, -t] \\ 0, & s \in (-t, 0] \end{cases} \quad \text{if} \quad t \leq h, \quad \text{and} \quad (S_t u)(s) = 0 \quad \text{if} \quad t > h.$$

The infinitesimal generator for S_t is D with dense domain

$$\mathbf{b}(D) = W^0 = \{u \in W : u(0) = 0\} \subset \mathcal{H}.$$

As shown in [1, Lemma 1.1], the solution

$$\phi(t, s) = \begin{cases} u_0(t+s), & s \in [-h, -t] \\ u(t+s), & s \in (-t, 0] \end{cases} \quad \text{if} \quad t \leq h, \quad \text{and} \quad \phi(t, s) = u(t+s) \quad \text{if} \quad t > h,$$

of equation (4.2), $\phi(t, \cdot) \in \mathcal{H}$, can be represented by

$$\phi(t) = S_t \phi_0 + \int_0^t S_{t-r} \Delta u(r) dr, \tag{4.3}$$

where the operator $\Delta \in \mathcal{L}(\mathbb{R}^m, W^*)$ (the space of linear operators) is given by the relation $(\Delta u, w) = u' w(0)$ for all $w \in W$. So, in spite of the fact that equality (4.3) is considered in $W^* \supset \mathcal{H} \supset W$ and the integration is also fulfilled in W^*, we have $\phi(t, \cdot) \in \mathcal{H}$ for every $u \in L_{2,loc}^m[0, \infty)$.

Introducing the operators $\mathbf{A} = [A, \mathbb{B}; 0, D]$, $\mathbf{B} = [B_0; \Delta]$, and C_0-semigroup $\mathbb{T}$ by

$$\mathbb{T}_{t-r} z = \left[e^{A(t-r)} x + \int_r^t e^{A(t-\alpha)} \mathbb{B} S_{\alpha-r} \phi d\alpha; S_{\alpha-r} \phi \right],$$

where $z = [x;\phi] \in Z = \mathbb{R}^n \times \mathcal{H}$, we can write the mild solution

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-r}\mathbf{B}u(r)dr, \quad \text{for the equation} \tag{4.4}$$

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}u(r), \quad z(0) = z_0. \tag{4.5}$$

Here the operator $\mathbf{A}$ is unbounded on Z with domains $\mathbf{b}(\mathbf{A}) = \mathbb{R}^n \times W^0$. For the operator B we have $\mathbf{B} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n \times W^*)$.

Equation (4.5) has no delays in control, but a recurrent procedure like in (3.4) is, unfortunately, impossible for infinite-dimensional system (4.5) to find a C-LTOC control. Therefore, we pass to finite-dimensional approximations of the obtained system.

5. Finite-dimensional approximation

We use the averaging approximation of the delayed system following [3]. For every positive integer N, we define the finite-dimensional linear subspace $\mathcal{H}^N$ of $\mathcal{H}$ by

$$\mathcal{H}^N = \Big\{u \in \mathcal{H} : u = \sum_{i=1}^N v_i\chi_i, \quad v_i \in \mathbb{R}^m\Big\},$$

where χ_i denote the characteristic function of $[t_i, t_{i-1})$ for $i \in 1 : N$ and $t_i = -ih/N$, $i \in 0 : N$. The subspace $\mathcal{H}^N$ is isometrically isomorphic to $\mathbb{R}^{mN}$ by means of the embedding $\gamma^N : \mathbb{R}^{mN} \to \mathcal{H}^N$ such that $(\gamma^N g)(s) = v_i$, $s \in [t_i, t_{i-1})$, $i \in 1 : N$, where $g = [v_1; \ldots; v_N]$. On $\mathbb{R}^{mN}$, we define the induced inner product

$$\langle f, g\rangle_N = f'Q^N g, \quad f, g \in \mathbb{R}^{mN},$$

where

$$Q^N = \operatorname{diag}[I_m, \ldots, I_m]h/N \in \mathbb{R}^{mN\times mN}.$$

The corresponding vector and matrix norms will be denoted by $\|\cdot\|_N$. The dual mapping $\gamma^{N*} : \mathcal{H}^N \to \mathbb{R}^{mN}$ has the natural extension $\pi^N : \mathcal{H} \to \mathbb{R}^{mN}$ defined by

$$\pi^N u = [v_1; \ldots; v_N], \quad v_i = \int_{t_i}^{t_{i-1}} u(s)ds N/h, \quad i \in 1 : N.$$

We have that $P^N = \gamma^N\pi^N$ is an self-adjoint orthogonal projector onto $\mathcal{H}^N$ and $\pi^N\gamma^N = I_{mN}$. Introduce the the following matrices:

$$B_i^N = \lim_{s\uparrow t_i}(B(s + h/N) - B(s)) = B(t_{i-1}) - B(t_i), \quad i \in 1 : N.$$

Note that the matrix $B(s)$ is left-continuous. For $\phi \in \mathcal{H}$, let $\pi^N\phi = g = [v_1; \ldots; v_N] \in \mathbb{R}^{mN}$. Then we can approximate the infinite-dimensional operators as follows:

$$\mathbb{B}\phi \approx \mathbb{B}P^N\phi = \sum_{i=1}^N B_i^N v_i; \quad D\phi \approx \nabla P^N\phi = \sum_{i=1}^N N(v_{i-1} - v_i)\chi_i/h, \quad v_0 = 0;$$

$$\mathbf{B}u \approx [B_0u; N\chi_1 u/h].$$

Denote by Z^N the space $\mathbb{R}^n\times\mathcal{H}^N$. Introduce the approximating operators $\mathbf{A}^N = [A, \mathbb{B}P^N; 0, \nabla P^N] : Z^N \to Z^N$ and $\mathbf{B}^N = [B_0; N\chi_1/h] : \mathbb{R}^m \to Z^N$. Let $\mathbb{T}_t^N$ denote the C_0-semigroup generated by $\mathbf{A}^N$ on Z^N and let $\bar{\pi}^N = [I_n, 0; 0, \pi^N]$, $\bar{\gamma}^N = [I_n, 0; 0, \gamma^N]$ be the operators on Z and on $\mathbb{R}^{n+mN}$, respectively. The following theorem is true.

Theorem 2 [3, Theorem 3.1]. *Let the matrix $B(s)$ have the form*

$$B(s) = -\textstyle\sum_{i=0}^{q} B_i\chi_{(-\infty,-h_i]}(s) - \displaystyle\int_s^0 B_{01}(r)dr, \quad 0 = h_0 < \cdots < h_q = h, \tag{5.1}$$

where $B_{01}(\cdot) \in L_2^{n\times m}[-h,0]$. Then there exist constants M and ω independent of N such that

$$\|e^{\bar{\pi}^N \mathbf{A}^N \bar{\gamma}^N t}\|_N \le Me^{\omega t}.$$

It follows from definitions of operators that

$$\bar{\pi}^N \mathbf{A}^N \bar{\gamma}^N = [A, \mathbb{B}\gamma^N; 0, \pi^N \nabla \gamma^N] \in \mathbb{R}^{(n+mN)\times(n+mN)}.$$

Therefore, the finite-dimensional approximation for (4.4), (4.5) is written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \mathbb{B}\gamma^N g(t) + B_0 u(t), \\ \dot{g}(t) &= \pi^N \nabla \gamma^N g(t) + N[u(t); 0; \ldots; 0]/h. \end{aligned} \tag{5.2}$$

Since $g(t) = [v_1(t); \ldots; v_N(t)]$ we can write the matrices of system (5.2), where the state vector is $[x(t); g(t)]$, in the following form

$$\begin{aligned} \mathbf{A}^N &= \left[A, B_1^N, \ldots, B_N^N; 0_{mN\times n}, (Q^N)^{-1}V\right], \\ V &= \left[\begin{array}{ccccc} -I_m, & 0, & \ldots & ,0, & 0; \\ I_m, & -I_m, & \ldots & ,0, & 0; \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \ldots & ,I_m, & -I_m \end{array}\right], \\ \mathbf{B}^N &= [B_0; I_m N/h; \ldots; 0; 0]. \end{aligned} \tag{5.3}$$

By Trotter–Kato theorem and Theorem 2 the following estimates are true [3, Theorems 4.4 and 4.10].

(i) If $z \in \mathbf{b}(\mathbf{A})$, then

$$\left\|[I_n; P^N]\mathbb{T}_t z - \mathbb{T}_t^N [I_n; P^N] z\right\| \le \alpha_1 e^{\alpha t}(h/N)\|z\|_{\mathbb{R}^n\times W}, \quad \forall N \in \mathbb{N}, \quad t > 4h.$$

(ii) For $t > 5h$ and $\forall N > N_0$,

$$\left\|[I_n; P^N]\mathbb{T}_t - \mathbb{T}_t^N [I_n; P^N]\right\| \le \alpha_2 e^{\alpha t}(h/N).$$

(iii) There exists a positive constant α_3, dependent on t but independent on N, such that for every $u(\cdot) \in L_2^m[0,t]$ and all $N \in \mathbb{N}$, we have

$$\left\| \int_0^t \mathbb{T}^N(t-r)\mathbf{B}^N u(r)dr \right\|_{Z^N} \le \alpha_3 \|u\|_{L_2^m[0,t]}.$$

From (iii) it follows that

$$\lim_{N\to\infty} \int_0^t \mathbb{T}^N(t-r)\mathbf{B}^N u(r)dr = \int_0^t \mathbb{T}(t-r)\mathbf{B}u(r)dr.$$

It is unknown whether estimates in Theorem 2 and in (i)–(iii) without an assumption (5.1) are true.

6. Application to averaged controllability and examples

For the flock (1.3), Lemma 1 can be reformulated in the following way.

Lemma 4. *The flock of systems (1.3) is controllable in average for the weights $p_i > 0$ iff there is a segment $[a, b]$, $-h \le a < b \le T$, such that*

$$\operatorname{rank}\left(\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha\right) = n, \tag{6.1}$$

where

$$\mathcal{B}(T,\alpha) = \sum_{i=1}^{d} p_i \int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_i(T-\theta)}dB_i(\alpha-\theta).$$

Of course, the condition (6.1) holds iff the equality

$$l'\mathcal{B}(T,\alpha) = 0 \quad \text{a.e. on } [a,b] \quad \text{implies that} \quad l = 0. \tag{6.2}$$

Corollary 2 has the form.

Corollary 3. *The function $\mathcal{B}(T,\alpha)$ from (6.1) can be expressed in the form*

$$\mathcal{B}(T,\alpha) = \sum_{i=1}^{d} p_i e^{A_i(T-\alpha)}\boldsymbol{b}_i(T,\alpha), \quad \textit{where} \quad \boldsymbol{b}_i(T,\alpha) = \int_{(\alpha-T)\vee(-h)}^{\alpha\wedge 0} e^{A_i s}dB_i(s).$$

If $T > h$ and $a = 0$, $b = T - h$, then

$$\boldsymbol{b}_i(T,\alpha) = \int_{-h}^{0} e^{A_i s}dB_i(s) = \text{const}$$

on $[a,b]$. Hence, the implication (6.2) is equivalent to the rank condition

$$\operatorname{rank}\Big[\sum_{i=1}^{d} p_i \int_{-h}^{0} e^{A_i(s+h)}dB_i(s),\ \sum_{i=1}^{d} p_i A_i \int_{-h}^{0} e^{A_i(s+h)}dB_i(s), \\ \dots, \sum_{i=1}^{d} p_i A_i^{nd-1} \int_{-h}^{0} e^{A_i(s+h)}dB_i(s)\Big] = n. \tag{6.3}$$

Let us pass to the property of LTAC. For the sake of example, we restrict the analysis to the case of the null control, i.e. the goal is to steer and keep the average equal to zero. We also consider the case with $d = 2$ components, and we chose $p_1 = p_2 = 1/2$, and $B_1(s) = B_2(s) = B(s)$. We do the remark.

Remark 2. All the statements in Section 3 are still valid, if at step $k + 1$ in (3.4) we consider any matrix $C_{k+1} = [RC; \tilde{L}_k]$, with $R \in \mathbb{R}^{p\times p}$, $\det R \ne 0$, and $\tilde{L}_k$ is a matrix of n columns such that $\ker C \cap \ker L_k = \ker C \cap \ker \tilde{L}_k$, where L_k is defined by (3.4). With this modification, $\bar{y}$ has to be modified in $R\bar{y}$.

Let $\mathcal{U}$ be the subspace defined by (3.3). In what follows, P denotes the orthogonal projector of $\mathbb{R}^n$ on $\mathcal{U}^\perp$, and we set $E = (A_1 - A_2)/2$, $F = (A_1 + A_2)/2$.

Instead of the sequence C_k introduced in (3.4), we use the sequence Ξ_k defined by

$$\Xi_k = \begin{bmatrix} I_n, & I_n; \\ PE, & -PE; \\ PEF, & -PEF; \\ \vdots & \vdots \\ PEF^{k-1}, & -PEF^{k-1} \end{bmatrix} \in \mathbb{R}^{(k+1)n\times 2n}. \tag{6.4}$$

We can note the following.

- For $k = 0$ $\quad \Xi_0 = 2C = [I_n, I_n]$.
- For $k = 1$, let P_0 be the orthogonal projector of $\mathbb{R}^n$ on $\Xi_0[\mathcal{U};\mathcal{U}]^\perp = \mathcal{U}^\perp$. We see $P_0 = P$. Then we set $\tilde{L}_1 = [\Xi_0; P\Xi_0 A] = [I_n, I_n; PA_1, PA_2]$. Since $\ker \tilde{L}_1 = \ker \Xi_1$, matrix Ξ_1 is suitable, according to Remark 2.
- Assume that at step k the matrix Ξ_k given by (6.4) is suitable. We define P_k, the orthogonal projector of $\mathbb{R}^{(k+1)n}$ on $\Xi_k[\mathcal{U};\mathcal{U}]^\perp = \operatorname{diag}[P, I_n, \dots, I_n]$. Then we set

$$\tilde{L}_{k+1} = [\Xi_0; P_k\Xi_k A] = \begin{bmatrix} I_n, & I_n; \\ PA_1, & PA_2; \\ PEA_1, & -PEA_2; \\ \vdots & \vdots \\ PEF^{k-1}A_1, & -PEF^{k-1}A_2 \end{bmatrix}.$$

  It is obvious that $\ker \tilde{L}_{k+1} = \ker \Xi_{k+1}$. So, Ξ_{k+1} is suitable.

As in Lemma 2, we have $\ker \Xi_{k+1} \subset \ker \Xi_k \subset \ker \Xi_0 \subset \mathbb{R}^{2n}$. Since $\dim(\ker \Xi_0) = n$ there exists $K \in 0 : n$ such that $\ker \Xi_{K+1} = \ker \Xi_K$, and we have $\ker \Xi_K = \ker \Xi_n$ (see Lemma 2).

As a consequence of Theorem 1 and the above considerations, we obtain the following result.

Corollary 4. *Let $d = 2$ and let $A_1, A_2 \in \mathbb{R}^{n\times n}$, and $B_1(s) = B_2(s)$. Then for every $x_{10}, x_{20} \in \mathbb{R}^n$ the flock of systems (1.3) is LTAC to 0 for $p_1 = p_2 = 1/2$ if and only if the condition like (2.2) holds for some $0 \le a < b \le T - h$ with*

$$\mathcal{B}(T,\alpha) = \Xi_n \operatorname{diag}\left[\int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_1(T-\theta)} dB(\alpha-\theta),\ \int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_2(T-\theta)} dB(\alpha-\theta)\right]$$

and $\operatorname{rank}\Xi_n$, *where the matrix Ξ_n is given by (6.4) for $k = n$.*

Remark 3. The Corollary 4 ensures that the solutions $x_1(t)$ and $x_2(t)$ of (1.3) (with $d = 2$ and $B_1(s) = B_2(s) = B(s)$) can be steered to some $[x_1(T); x_2(T)] \in \ker \Xi_n$. This condition can be equivalently rewritten as

$$x_1(T) + x_2(T) = 0,$$
$$x_1(T) - x_2(T) \in \left\{g \in \mathbb{R}^N : EF^k g = 0 \quad \forall k \in 0 : n-1\right\}.$$

Let $g = (x_1 - x_2)/2$ and $f = (x_1 + x_2)/2$. Then for every control $v(t) \in \mathcal{U}$ we have

$$\begin{cases} \dot f = Ff + Eg + v(t), \\ \dot g = Ef + Fg, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot x_1 = A_1x_1 + v(t), \\ \dot x_2 = A_2x_2 + v(t). \end{cases}$$

Now it becomes obvious that $f(t) = 0$ for $t \geq T$ if and only if $v(t) = -Eg(t)$ and $g(t)=e^{F(t-T)}g(T)$ such that $Eg(t) = 0$ for $t \geq T$. Note that $v(t) \in \mathcal{U}$.

Of course, we need a control $u(t)$, $t \geq T$, such that

$$\int_{(-h)\vee(T-t)}^{0} dB(s)u(t+s) = v(t), \quad t \geq T, \tag{6.5}$$

similarly to (3.8).

Example 2. Let us return to the flock in the Example 1. We have $A_1 = [0, -1; 1, 0]$, $A_2 = [1, 2; 0, 1]$. The system has 8 parameters. Let $b_{111} = b_{112} = b_{211} = b_{212} = 1$. Other parameters are equal to zero. It corresponds to one control with one delay in the form $[u(t) + u(t-1); 0]$. The flock is controllable in average for every $T > 1$ in the sense of the Remark 1, as condition (6.3) is fulfilled. Here we have the projector $P = [0, 0; 0, 1]$. It was shown in [2] that the systems with one scalar ordinary control:

$$\dot{x}_1 = A_1x_1 + [v(t); 0], \quad \dot{x}_2 = A_2x_2 + [v(t); 0],$$

are controllable in average, but not simultaneously controllable. Moreover, this system has the long-time averaged controllability property. Hence, there is a control $v(t)$, $t \geq T$, owing to the Remark 3. We can find a control $u(t)$, $t \in [0, T-1]$, such that $x_1(T) + x_2(T) = 0$ due to controllability. Equation (6.5) is $u(t) + u(t-1) = v(t)$, $t \geq T$. As in (3.9), it can be resolved step-by-step on segments $[T+i-1, T+i]$:

$$u_i(t) = v(t) - u_{i-1}(t-1), \quad t \in [T+i-1, T+i], \quad i \in \mathbb{N},$$

where $u_0(t) = 0$.

We can also analyze the property of LTAC for the case $B_1(s) \neq B_2(s)$ when $d = 2$. Then we use the general considerations of Section 3. Note that equation (3.9) can be easily resolved only if the matrix B_0 is square and $\det B_0 \neq 0$. For our examples, it corresponds to the condition $\det [b_{111}, b_{121}; b_{211}, b_{221}] \neq 0$. This determinant equals zero in Example 2, but, nevertheless, we found the $u(t)$.

Example 3. Let $b_{112} = b_{211} = 1$ and others parameters equal zero. It corresponds to one control with one delay in the forms $[u(t-1); 0]$ for the first system and $[u(t); 0]$ for the second one. The average controllability for every $T > 1$ is easily verified due to condition (6.3). Introduce $B_1 = [0, 1; 0, 0]$, $B_2 = [1, 0; 0, 0]$, and $v(t) = [v_1(t); v_2(t)]$. The corresponding systems with ordinary controls have the form:

$$\dot{x}_1 = A_1x_1 + B_1v(t), \quad \dot{x}_2 = A_2x_2 + B_2v(t).$$

This system has the LTAC property with $v_1(t) \neq v_2(t)$. We cannot solve the equation $u(t-1) = v_2(t)$, $u(t) = v_1(t)$. It may be solved only if $v_1(t-1) = v_2(t)$. Let us pass to the approximation from Section 5. Let b=[1; 0], then our flock of systems is written as

$$\dot{x}_1(t) = A_1x_1(t) + bu(t-1), \quad \dot{x}_2(t) = A_2x_2(t) + bu(t),$$
$$y(t) = (x_1(t) + x_2(t))/2.$$

We need to approximate only the first system. Here $m = 1$ and $t_i = -i/N$, $i \in 0 : N$. As $B(s) = -b\chi_{(-\infty,-1]}(s)$, the matrix $B_N^N = b$ and $B_i^N = 0$, $i \in 1 : N-1$. Therefore, matrices (5.3) have the form

$$\mathbf{A}_1^N = \left[A_1, 0_{2\times(N-1)}, b; 0_{N\times 2}, (Q^N)^{-1}V\right],$$

$$V = \left[\begin{array}{ccccc} -1, & 0, & \dots & ,0, & 0; \\ 1, & -1, & \dots & ,0, & 0; \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \dots & ,1, & -1 \end{array}\right],$$

$$\mathbf{b}^N = [0_{2\times 1}; N; \dots; 0; 0],$$

where $\mathbf{A}_1^N \in \mathbb{R}^{(2+N)\times(2+N)}$, $V \in \mathbb{R}^{N\times N}$, and $\mathbf{b}^N \in \mathbb{R}^{2+N}$. We compose the matrices $A = \mathrm{diag}\left[\left[\mathbf{A}_1^N\right], A_2\right]$, $B = \left[\left[\mathbf{b}^N\right]; b\right]$, and $C = [I_2, 0_{2\times N}, I_2]/2$. For the obtained system $\dot{x} = Ax + Bu$, we verify the property of C-LTOC. It does not hold for any N. The flock has not the property of LTAC for the case $b_{112} = b_{211} = 1$ and others equal zero.

It can be verified that the approximating system in Example 2 has the LTAC property.

7. Conclusion and open problems

In this paper, we considered the notion of output controllability for ordinary systems with retarded controls and gave the necessary and sufficient condition for that. For the notion of long-time output controllability, we obtained only sufficient conditions. This notions were applied for the investigation of averaged controllability of mentioned systems. The general approach for that is to approximate the systems by the ordinary ones. In connection with the results obtained, a number of interesting open questions arise.

- Assume that there exists a number $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ the approximating system has the C-LTOC property. Is it sufficient for C-LTOC property of the original system? And vice versa, if the original system has C-LTOC property, whether it is sufficient for C-LTOC of the approximating system?
- How to obtain any rank conditions for output controllability of systems with delays in the state and control? The same question about the C-LTOC property of such a systems.
- We considered the LTAC property for flocks with finite number of members. Can the results be extended for flocks with infinite members?
- Does output controllability imply output feedback stabilisation? Suppose that the system is output controllable, does it exist a feedback control $u(t) = Ky(t)$ such that y(t) goes to zero as t goes to ∞?

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