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BESSEL POLYNOMIALS AND SOME CONNECTION FORMULAS IN TERMS OF THE ACTION OF LINEAR DIFFERENTIAL OPERATORS

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Abstract: In this paper, we introduce the concept of the $\mathbb{B}_\alpha$-classical orthogonal polynomials, where $\mathbb{B}_\alpha$ is the raising operator $\mathbb{B}_\alpha := x^2 \cdot d/dx + \big(2(\alpha-1)x+1\big)\mathbb{I}$, with nonzero complex number α and $\mathbb{I}$ representing the identity operator. We show that the Bessel polynomials $B_n^{(\alpha)}(x),\ n\geq 0$, where $\alpha\neq -m/2,\ m\geq -2,\ m\in\mathbb{Z}$, are the only $\mathbb{B}_\alpha$-classical orthogonal polynomials. As an application, we present some new formulas for polynomial solution.

Keywords: Classical orthogonal polynomials, Linear functionals, Bessel polynomials, Raising operators, Connection formulas.

1. Introduction

Let $\{B_n^{(\alpha)}\}_{n\geq 0}$ be the monic Bessel polynomial sequence. It satisfies the following explicit expression [10, 23]

$$B_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)} x^\nu, \quad n\geq 0, \tag{1.1}$$

for $\alpha\neq -m/2,\ m\in\mathbb{N}$. To complete the definition, $B_n^{(\alpha)}(0)$ is set equal to

$$B_n^{(\alpha)}(0) = 2^n \frac{\Gamma(n+2\alpha-1)}{\Gamma(2n+2\alpha-1)}, \quad n\geq 0. \tag{1.2}$$

It is well known that the monic Bessel polynomial sequence is classical and satisfies the following relations [8, 10, 16, 23]:

–The Second-Order Differential Equation (SODE)

$$x^2 B_n^{(\alpha)''}(x) + 2\big(\alpha x+1\big) B_n^{(\alpha)'}(x) = n(n+2\alpha-1)B_n^{(\alpha)}(x), \quad n\geq 0. \tag{1.3}$$

–The Lowering Relation (LR)

$$DB_n^{(\alpha)}(x) = nB_{n-1}^{(\alpha+1)}(x), \quad n\geq 1, \tag{1.4}$$

where $D := d/dx$ is the standard derivate operator.

After a simple calculation, the SODE can be written for $n \geq 0$ as follows

$$\Big(x^2 B_n^{(\alpha)\prime}(x)\Big)' + \Big(2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x)\Big)' = (n+1)(n+2\alpha-2)B_n^{(\alpha)}(x). \tag{1.5}$$

Using the LR (1.4), the equation (1.5) becomes for $n \geq 0$

$$\Big(x^2 B_n^{(\alpha)\prime}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x)\Big)' = (n+2\alpha-2)B_{n+1}^{(\alpha-1)\prime}(x).$$

Using the primitive of the last equation, we get

$$x^2 B_n^{(\alpha)\prime}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x) + K,$$

with $(\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z})$, and where, using (1.2), we have

$$K = 2B_n^{(\alpha)}(0) - (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(0) = 0.$$

Then we finally obtain the following *Raising Relation* (RR) satisfied by the monic Bessel polynomials

$$\mathbb{B}_\alpha B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x), \tag{1.6}$$

where $\mathbb{B}_\alpha := x^2 D + 2\big((\alpha-1)x+1\big)\mathbb{I}$ is called the degree raising shift operator for the Bessel polynomials with $\mathbb{I}$ representing the identity operator. For more details see also the degree raising shift operator for the family of classical orthogonal polynomials [13].

In view of (1.6), we can say that $\{B_n^{(\alpha)}\}_{n\geq 0}$ is an $\mathbb{B}_\alpha$-classical polynomial sequence, since it satisfies the Hahn's property with respect to the operators $\mathbb{B}_\alpha$, i.e., it is an orthogonal polynomial sequence whose sequence of $\mathbb{B}_\alpha$-derivatives is also orthogonal. Note that an orthogonal polynomial sequence $\{p_n\}_{n\geq 0}$ is called classical, if $\{p_n'\}_{n\geq 0}$ is also orthogonal (see [16–19]). This characterization is essentially the Hahn–Sonine characterization (see [11, 21]) of the classical orthogonal polynomials.

In the same context, a natural question arises about the characterization of $\mathbb{B}_\alpha$-classical orthogonal polynomials. The purpose of this paper is to introduce the concept of the $\mathbb{B}_\alpha$-classical polynomial sequence and to give a complete description of this family of orthogonal polynomials. Note that many researches have been devoted to these topics where lowering, transfer and raising operators have been used (see for example [1–7, 9, 11, 12, 20]).

The paper is organized as follows: Section 2 gives the basic notations and tools that will be used throughout the paper. Section 3 deals with $\mathbb{B}_\alpha$-classical orthogonal polynomial sequence. In Section 4, we put in evidence some differential relations satisfied by the polynomials solution of our problem. In Section 5, we give a conclusion.

2. Preliminaries

Let $\mathcal{P}$ be linear space of polynomials in one variable with complex coefficients and $\mathcal{P}'$ be its dual space, whose elements are linear functionals. We write $\langle u, p\rangle := u(p)$ $(u \in \mathcal{P}',\ p \in \mathcal{P})$. In particular, we denote by $(u)_n := \langle u, x^n\rangle,\ n \geq 0$, the moments of u. Let us define the following operations on $\mathcal{P}'$. For any linear functional u, any polynomial f and any $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, let $Du := u',\ fu,\ h_a u$ and $\tau_b u$ be the linear functionals defined by the duality [15, 16]

$$\langle fu, p\rangle := \langle u, fp\rangle, \quad \langle u', p\rangle := -\langle u, p'\rangle,$$

$$\langle h_a u, p\rangle := \langle u, h_a p\rangle = \langle u, p(ax)\rangle, \quad \langle \tau_b u, p\rangle := \langle u, \tau_{-b} p\rangle = \langle u, p(x+b)\rangle.$$

A linear functional u is called normalized if it satisfies $(u)_0 = 1$. We assume that the linear functionals used in this paper are normalized.

Let $\{p_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg p_n = n$, $n \geq 0$ (MPS in short) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, p_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the MPS $\{p_n\}_{n\geq 0}$ (see [16–18]).

Let us recall the following result.

Lemma 1 [16, 17]. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*:

(i) $\langle u, p_{m-1}\rangle \neq 0, \quad \langle u, p_n\rangle = 0, \quad n \geq m,$

(ii) $\exists \lambda_\nu \in \mathbb{C}, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0$ *such that* $\displaystyle u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{p_n^{[1]}\}_{n\geq 0}$ where

$$p_n^{[1]}(x) := (n+1)^{-1} D p_{n+1}(x), \quad n \geq 0,$$

is given by [16, 19] as

$$D u_n^{[1]} = -(n+1) u_{n+1}, \quad n \geq 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\tilde{p}_n\}_{n\geq 0}$, where

$$\tilde{p}_n(x) := a^{-n} p_n(ax+b)$$

with $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by [16, 19]

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n, \quad n \geq 0.$$

A linear functional u is called *regular* if we can associate with it a MPS $\{p_n\}_{n\geq 0}$ such that [16, 19] as

$$\langle u, p_n p_m\rangle = r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{p_n\}_{n\geq 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS in short) with respect to u. Note that $u = (u)_0 u_0 = u_0$, since u is normalized.

Proposition 1. [16]. *Let $\{p_n\}_{n\geq 0}$ be a MPS and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent*:

(i) *$\{p_n\}_{n\geq 0}$ is orthogonal with respect to u_0,*

(ii) *$\{p_n\}_{n\geq 0}$ satisfies the linear recurrence relation of order two*

$$\begin{cases} p_0(x) = 1, \quad p_1(x) = x - \beta_0, \\ p_{n+2}(x) = (x - \beta_{n+1}) p_{n+1}(x) - \gamma_{n+1} p_n(x), \quad n \geq 0, \end{cases}$$

where

$$\beta_n = \langle u_0, x p_n^2\rangle \langle u_0, p_n^2\rangle^{-1}, \quad n \geq 0,$$

and

$$\gamma_{n+1} = \langle u_0, p_{n+1}^2\rangle \langle u_0, p_n^2\rangle^{-1} \neq 0, \quad n \geq 0,$$

(iii) *the dual sequence $\{u_n\}_{n\geq 0}$ satisfies*:

$$u_n = \langle u_0, p_n^2\rangle^{-1} p_n u_0, \quad n \geq 0.$$

A MOPS $\{p_n\}_{n\geq 0}$ is called D-classical, if $\{Dp_n\}_{n\geq 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*) [19]. Moreover, if $\{p_n\}_{n\geq 0}$ is orthogonal with respect to u_0, then there exists a monic polynomial ϕ with $\deg\phi \leq 2$ and a polynomial ψ with $\deg\psi = 1$ such that u_0 satisfies the *Pearson's equation* (PE) [19]

$$D(\phi u_0) + \psi u_0 = 0.$$

A second characterization of these polynomials is that they are the only polynomial solutions of the SODE [8, 19],

$$\phi(x)p''_{n+1}(x) - \psi(x)p'_{n+1}(x) = \lambda_n p_{n+1}(x), \quad n \geq 0,$$

where

$$\lambda_n = (n+1)\Big(\frac{1}{2}\phi''(0)n - \psi'(0)\Big) \neq 0, \quad n \geq 0.$$

Note that if $p_n(x) = B_n^{(\alpha)}(x),\ n \geq 0,\ (\alpha \neq -n/2,\ n \geq 0)$ is the monic Bessel polynomial and we write $\mathcal{B}^{(\alpha)}$ for u_0, then the regular form $\mathcal{B}^{(\alpha)}$ satisfies the following PE [16, 19]

$$D\big(x^2\mathcal{B}^{(\alpha)}\big) - 2(\alpha x + 1)\mathcal{B}^{(\alpha)} = 0, \tag{2.1}$$

and $B_n^{(\alpha)}(x),\ n \geq 0$ satisfies the SODE (1.3).

3. The $\mathbb{B}_\alpha$-classical polynomials

Recall the operator

$$\begin{aligned} \mathbb{B}_\alpha : \mathcal{P} &\longrightarrow \mathcal{P}, \\ f &\longmapsto \mathbb{B}_\alpha(f) := x^2 f' + 2\big((\alpha-1)x + 1\big)f, \end{aligned}$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

Clearly, the operator $\mathbb{B}_\alpha$ raises the degree of any polynomial. Such an operator is called *raising operator* [14, 22].

Definition 1. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathbb{B}_\alpha$-classical if $\{\mathbb{B}_\alpha P_n\}_{n\geq 0}$ is also orthogonal.*

For any MPS $\{P_n\}_{n\geq 0}$ we define

$$Q_{n+1}(x;\alpha) := \frac{1}{n+2\alpha-2}\,\mathbb{B}_\alpha P_n(x), \quad n \geq 0,$$

or equivalently

$$(n+2\alpha-2)Q_{n+1}(x;\alpha) := x^2 P'_n(x) + 2\big((\alpha-1)x+1\big)P_n(x), \quad n \geq 0, \tag{3.1}$$

with initial value $Q_0(x;\alpha) = 1$.

Clearly, $\{Q_{n+1}(.;\alpha)\}_{n\geq 0}$ is a MPS and

$$\deg Q_{n+1}(x;\alpha) = n+1.$$

In the sequel, we write

$$Q_n(x) := Q_n(x;\alpha), \quad n \geq 0,$$

if there is no ambiguity. Our next goal is to describe all the $\mathbb{B}_\alpha$-classical polynomial sequences. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$P_{n+2}(x) = (x - \varpi_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \tag{3.2}$$

with initial values $P_0(x) = 1,\ P_1(x) = x - \varpi_0$, and

$$Q_{n+2}(x) = (x - \theta_{n+1})Q_{n+1}(x) - \zeta_{n+1}Q_n(x), \quad n \geq 0, \tag{3.3}$$

with initial values $Q_0(x) = 1,\ Q_1(x) = x - \theta_0$.

Next, a first result will be deduced as a consequence of relations (3.1), (3.2) and (3.3).

Proposition 2. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$x^2 P_n(x) = Q_{n+2}(x) + s_n Q_{n+1}(x) + t_n Q_n(x), \quad n \geq 0,$$

where

$$s_n = (n + 2\alpha - 2)\big(\varpi_n - \theta_{n+1}\big), \quad n \geq 0,$$
$$t_n = (n + 2\alpha - 3)\gamma_n - (n + 2\alpha - 2)\zeta_{n+1}, \quad n \geq 0,$$

with the convention $\gamma_0 = 0$.

P r o o f. Differentiating (3.2), we obtain

$$P'_{n+2}(x) = (x - \varpi_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x) + P_{n+1}(x), \quad n \geq 0.$$

We multiply the last equation by x^2 and the relation (3.2) by $2\big((\alpha - 1)x + 1\big)$, take the sum of the two resulting equations, and substitute (3.1). Then, we get

$$(n + 2\alpha)Q_{n+3}(x) = (n + 2\alpha - 1)(x - \varpi_{n+1})Q_{n+2}(x)$$
$$-(n + 2\alpha - 2)\gamma_{n+1}Q_{n+1}(x) + x^2 P_{n+1}(x), \quad n \geq 0.$$

Using the relation (3.3), we get

$$x^2 P_{n+1}(x) = Q_{n+3}(x) + (n + 2\alpha - 1)\big(\varpi_{n+1} - \theta_{n+2}\big)Q_{n+2}(x)$$
$$+\big((n + 2\alpha - 2)\gamma_{n+1} - (n + 2\alpha - 1)\zeta_{n+2}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid if $n + 1$ is replaced by n with the convention $\gamma_0 = 0$. Hence we got the desired result. □

Note that, for $n = 0$, the Proposition 2 gives

$$x^2 = Q_2(x) + (2\alpha - 2)(\varpi_0 - \theta_1)Q_1(x) - (2\alpha - 2)\zeta_1 Q_0(x), \tag{3.4}$$

and using the fact that

$$Q_1(x) = x - \theta_0 = x + \frac{1}{\alpha - 1},$$

we obtain

$$Q_2(x) = x^2 + (2\alpha - 2)(\theta_1 - \varpi_0)x + (2\alpha - 2)\zeta_1 + 2(\theta_1 - \varpi_0).$$

It gives by comparing with (3.3) for $n = 0$

$$\theta_1 = \frac{-\theta_0 + 2(\alpha - 1)\varpi_0}{2\alpha - 1} = \frac{1}{(\alpha - 1)(2\alpha - 1)} + \frac{2(\alpha - 1)}{2\alpha - 1}\,\varpi_0,$$
$$\zeta_1 = \frac{\theta_0\theta_1 + 2(\varpi_0 - \theta_1)}{2\alpha - 1} = \frac{-1}{(\alpha - 1)^2}.$$

Denote by u_0 and v_0 the regular forms (linear functionals) in $\mathcal{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ respectively. Then we can state the following result.

Lemma 2. *The following algebraic relation between the regular forms u_0 and v_0 holds*

$$x^2 v_0 = \frac{2}{(\alpha - 1)}\, u_0.$$

P r o o f. According to Proposition 2, we obtain

$$\big\langle x^2 v_0, P_n(x)\big\rangle = 0, \quad n \geq 1. \tag{3.5}$$

On the other hand, by (3.4) we have

$$\begin{aligned}\big\langle x^2 v_0, P_0(x)\big\rangle &= \langle v_0, Q_2(x)\rangle + 2(\alpha-1)(\varpi_0 - \theta_1)\langle v_0, Q_1(x)\rangle - 2(\alpha-1)\zeta_1\langle v_0, Q_0(x)\rangle r \\ &= -2(\alpha-1)\zeta_1 = \frac{2}{(\alpha-1)},\end{aligned} \tag{3.6}$$

since $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the normalized form v_0. According to Lemma 1 and using (3.5) and (3.6), we obtain the desired result. □

Based on PE satisfied by the linear functional of $\mathcal{B}^{(\alpha)}$, we can state the following theorem.

Theorem 1. *The sequence of Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$, with $\alpha \neq -m/2,\ m \geq -2,$ $m \in \mathbb{Z}$, is the only $\mathbb{B}_\alpha$-classical orthogonal sequence. More precisely, $P_n(x) = B_n^{(\alpha)}(x)$ and $Q_n(x) = B_n^{(\alpha-1)}(x)$, $n \geq 0$.*

P r o o f. If we apply v_0 in (3.1), we get for $n \geq 0$

$$\big\langle v_0, (n+2\alpha-2)Q_{n+1}(x)\big\rangle = \Big\langle v_0, x^2 P_n'(x) + 2\big((\alpha-1)x+1\big)P_n(x)\Big\rangle = 0.$$

But the right hand side may be read as

$$\Big\langle -D\big(x^2 v_0\big) + 2\big((\alpha-1)x+1\big)v_0, P_n(x)\Big\rangle = 0, \quad n \geq 0.$$

Hence we have for all polynomials P, expanding P in the basis $\{P_n\}_{n\geq 0}$, the following relation

$$\Big\langle -D\big(x^2 v_0\big) + 2\big((\alpha-1)x+1\big)v_0, P(x)\Big\rangle = 0.$$

In other words we have

$$\big(x^2 v_0\big)' - 2\big((\alpha-1)x+1\big)v_0 = 0. \tag{3.7}$$

This implies that v_0 is the Bessel functional $\mathcal{B}^{(\alpha-1)}$ according to the corresponding PE (2.1), i.e.,

$$Q_n(x) = B_n^{(\alpha-1)}(x), \quad n \geq 0,$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

Multiplying (3.7) by x^2 and using Lemma 2, we obtain

$$\big(x^2 u_0\big)' - 2\big(\alpha x + 1\big)u_0 = 0. \tag{3.8}$$

Essentially (3.8) corresponds to the PE of linear functional $\mathcal{B}^{(\alpha)}$ of the sequence of Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$. Hence, $P_n(x) = B_n^{(\alpha)}(x),\ n \geq 0$. □

In conclusion, we give the following relation, which is satisfied by Bessel polynomials

$$x^2 B_n^{(\alpha)\prime}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x), \quad n \geq 0$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

4. Representations of Bessel polynomials in terms of the action of linear differential operators

In this section, we prove some higher order differential relations between the Bessel polynomials (solution of our problem). First, we need the following fundamental relation

$$\big(xD + (n+\alpha-1)\mathbb{I}\big)B_n^{(\alpha/2)}(x) = (2n+\alpha-1)B_n^{((\alpha+1)/2)}(x), \tag{4.1}$$

which is obtained after a simple calculation from (1.1).

Theorem 2. *The representation of Bessel polynomials $B_n^{((\alpha+m)/2)}(x)$ in terms of action of linear differential operators on the Bessel polynomials $B_n^{(\alpha/2)}(x)$ is given by*

$$B_n^{((\alpha+m)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m-1)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)}x^{m-k}D^{m-k}B_n^{(\alpha/2)}(x), \quad n\geq 0, \quad m \geq 0. \tag{4.2}$$

P r o o f. We prove this by induction on $m\in\mathbb{N}$. For $m=0$ this is obvious. Now, suppose (4.2) holds and prove the same for $m+1$ instead of m. Indeed, by differentiating both sides of (4.2) and using (1.4), we get, for all $n\geq 1$,

$$B_{n-1}^{((\alpha+m+2)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m-1)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times\big[(m-k)x^{m-k-1}D^{m-k-1} + x^{m-k}D^{m-k}\big]B_{n-1}^{((\alpha+2)/2)}(x), \quad n\geq 1.$$

Replacing $\alpha+1$ by α, $n-1$ by n and using the identity (4.1) we obtain for all $n\geq 0$

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times\big[(m-k)x^{m-k-1}D^{m-k-1} + x^{m-k}D^{m-k}\big]\big(xD+(n+\alpha-1)\mathbb{I}\big)B_n^{(\alpha/2)}(x), \quad n\geq 0.$$

Equivalently

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times\Big[(m-k)(n+\alpha+m-k-2)x^{m-k-1}D^{m-k-1} + (n+\alpha+2m-2k-1)x^{m-k}D^{m-k} + x^{m+1-k}D^{m+1-k}\Big]B_n^{(\alpha/2)}(x), \quad n\geq 0.$$

After some calculations, we finally obtain for all $n\geq 0$

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)}\sum_{k=0}^{m+1}\binom{m+1}{k}\frac{\Gamma(n+\alpha+m)}{\Gamma(n+\alpha+m-k)} \times x^{m+1-k}D^{m+1-k}B_n^{(\alpha/2)}(x), \quad m\geq 0.$$

Hence the desired result is proved. □

5. Conclusion

We have described the $\mathbb{B}_\alpha$-classical orthogonal polynomials using the Pearson's equation that the corresponding linear functionals satisfy. More precisely, we have proved that the Bessel polynomial sequence $\{B_n^{(\alpha)}(x)\}_{n\geq 0}$, where $\alpha \neq -m/2$, $m \geq -2$, $m \in \mathbb{Z}$, is the only $\mathbb{B}_\alpha$-classical sequence. As a consequence, some connection formulas between the corresponding polynomials are deduced.

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Appendix

Table A. Bessel polynomials.

$$\{\mathbf{B_n}\}_{\mathbf{n\geq 0}} \perp \mathcal{B}(\alpha)$$

$$\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x + 1),$$

$$\beta_0 = -\frac{1}{\alpha}, \quad \beta_{n+1} = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}, \quad n \geq 0,$$

$$\gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}, \quad n \geq 0,$$

$$x^2 B''_{n+1}(x) + 2(\alpha x + 1)B'_{n+1}(x) - (n+1)(n+2\alpha)B_{n+1}(x) = 0,$$

$$x^2 B'_{n+1}(x) = (n+1)\left(x - \frac{1}{n+\alpha}\right) B_{n+1}(x) - (2n+2\alpha+1)\gamma_{n+1}B_n(x),$$

$$\langle \mathcal{B}(\alpha), f\rangle = J(\alpha)^{-1}\int_0^{+\infty} x^{2\alpha-2}e^{-2/x}\left(\int_x^{+\infty} \xi^{-2\alpha}e^{2/\xi}s(\xi)d\xi\right)f(x)dx,$$

$$J(\alpha) := 4\int_0^{+\infty} t^{3-8\alpha}e^{2/t^4}e^{-t}\sin(t)\left(\int_0^{t^4} x^{2\alpha-2}e^{-2/x}dx\right)dt,$$

$$s(x) = \begin{cases} 0, & x \leq 0, \\ e^{-x^{1/4}}\sin x^{1/4}, & x > 0. \end{cases}$$