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ON THE TRANSFORM OF A GUIDANCE GAME

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Abstract. This paper is devoted to the study of the structure of solutions of the differential guidance game in the case where the objective set is contained in the position space and is the controllability set for some control system.

CONTENTS

1. Introduction

This problem is devoted to the study of the structure of solutions of the differential guidance game in the case where the objective set is contained in the position space and is the controllability set for some control system. To such a system, we put in correspondence an equivalent problems of guidance "at moment" for the transformed system. A guidance game is called a problem "at moment" if one must guide the system to a set in the phase space at the final instant of time.

The structure of a nonlinear guidance game is exhaustively characterized by the theorem on alternative proved by Krasovskii and Subbotin (see [9–11]), which asserts that, under the condition of information consistency, there exists a saddle point in the class of the corresponding feedback strategies. If the condition of information consistency does not hold, a saddle point exists in the class of pairs *counterstrategy/strategy* (see [10, 11], where the existence of a saddle point in the pairs *strategy/counterstrategy* and *mixed strategy/mixed strategy* was established [10, 11]). The form of an optimal strategy for guidance problems [10, 11] is known: a strategy (in the case where the condition of information consistency of the counterstrategy does not hold) can be constructed by the method of extremal shift by some set; this set is the maximal u-stable bridge in the sense of Krasovskii. Thus, the solution of a guidance game is reduced to the construction of the maximal u-stable bridge. If the problem is considered in the classes *mixed strategy/mixed strategy* or *strategy/counterstrategy*, then the corresponding optimal control can be obtained by the method of extremal shift by a maximal $\tilde{u}$-stable or u_*-stable bridge, respectively (see [10, 11]).

Together with guidance problems, game problems of minimization of functionals are also studied in the theory of differential games. For such problems, based on the theorem on alternative, Krasovskii and Subbotin proved the existence of the value function (see [10, 11]).

The specific construction of the set of positional absorption or the value function can be performed by using the method of program iterations proposed by Chentsov (see [3–5, 14] and also [7, 12, 15]).

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An important case of a set invariant with respect to a control system is a cylinder set; it is invariant with respect to the identity transformation of the phase space. Guidance problems on the cylinder set are also called guidance "to moment" problems.

In this paper, the transformation of a problem is performed by a peculiar extension of the control system. We prove that for the case of an autonomous conflict-controlled system, the problem of guidance to the maximal invariant set is equivalent to the problem of guidance "at moment" of the extended system. The equivalence of these problems follows from the fact that iterations constructed by the method of program iterations for both these problems coincide. The method of transformation of problems proposed here can be applied to game problems of minimization of the path functional.

2. Basic Definitions and Notation

In this paper, we consider convergence/divergence problems with the set M of autonomous conflict-controlled systems of the form

$$\dot{x} = f(x, u, v), \quad u \in P, \quad v \in Q, \tag{1}$$

on the time interval $[0, \vartheta]$. Assume that the first player disposing of the control u wants to lead the system on the set M, $M \subset [0, \vartheta] \times \mathbb{R}^n$, and the second player disposing of the control v aims to prevent this.

We assume that f is a continuous, locally Lipschitz (with respect to the phase variable) function satisfying the condition of the sublinear growth, $P \subset \mathbb{R}^p$, $Q \subset \mathbb{R}^q$, P and Q are compact sets, and M is closed . This differential game is considered in the class *counterstrategy/strategy*, i.e., we assume that the first player (who disposes of the control u) exerts its control in the class of counterstrategies and the second player (who disposes of the control v) exerts its control in the class of pure feedback strategies (see [10, 11]). We state the strict definitions of counterstrategies, feedback strategies, and motions for autonomous games, following the definitions introduced in [10, 11] for the general case.

A *counterstrategy* of the first player is an arbitrary function $U : [0, \vartheta] \times \mathbb{R}^n \times Q \to P$ measured with respect to the third argument. A *feedback strategy* of the second player is an arbitrary function $V : [0, \vartheta] \times \mathbb{R}^n \to Q$. A *motion* generated by a counterstrategy $U(t, x, v)$ on the interval $[t_*, t^*]$ and started at a point x_* is defined (following [10, 11]) as the limit of Euler polygonal approximations as the fineness of the partition tends to zero. The Euler polygonal approximations are constructed as follows. Let $\Delta = \{\tau_k\}_{k=0}^r$ be a partition of the interval $[t_*, t^*]$ and $v(\cdot)$ be a control of the second player. On any interval $[\tau_{k-1}, \tau_k)$, $k = \overline{1, r}$, we define the Euler polygonal approximation as the solution of the equation

$$x_k[t] = x_{k-1}[\tau_{k-1}] + \int\limits_{\tau_{k-1}}^{t} f\Big(x_k[\theta],\ U\big(\tau_{k-1},\ x_{k-1}[\tau_{k-1}],\ v(\theta)\big),\ v(\theta)\Big) d\theta, \quad x_0[\tau_0] \triangleq x_*.$$

The second player generates his control as follows. Let $\Xi = \{\xi_j\}_{j=0}^m$ be a partition of the interval $[t_*, t^*]$. The control of the second player is constant on the intervals $[\xi_{j-1}, \xi_j)$, $j = \overline{1, m}$, and is equal to $V(\xi_{j-1}, x_{j-1})$, where x_{j-1} is the state of the system at the time instant ξ_{j-1}.

By the theorem on alternative (see [10, Theorem 82.2]), the solution of a convergence/divergence problem is completely defined by the solvability set (the maximal u-stable bridge) $\mathfrak{W}$. In this case, the optimal counterstrategy $U(t, x, v)$ is constructed by the rule of extremal shift by $\mathfrak{W}$ (see [10, 11]).

A set is called a *u-stable bridge* (see [10, 11]) if for any $v \in Q$ and any position $(t_*, x_*) \in W$, there exist a solution $y(\cdot)$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(x, u, v) : u \in P\}$$

and an instant $\xi \in [t_*, \vartheta]$ such that $y(\xi) \in M[\xi]$ and for all $t \in [t_*, \xi]$ the inclusion $y(t) \in W[t]$ holds.

If the Isaacs condition (the saddle-point condition in a small game)

$$\forall s, x \in \mathbb{R}^n \quad \min_{u\in P}\max_{v\in Q}\langle s, f(x,u,v)\rangle = \max_{v\in Q}\min_{u\in P}\langle s, f(x,u,v)\rangle$$

holds, it suffices to consider a differential game in the class of feedback strategies (see [10, 11]); in this case, strategies of the first player depend only on the realized position.

Except for conflict-controlled systems, in this paper we also use ordinary control systems. Let Λ be a compact set in a finite-dimensional metric space. Consider the control system

$$\dot{x} = h(x,b), \quad b \in \Lambda. \tag{2}$$

For fix $b \in \Lambda$, we denote by $S^{\tau}_{h,b}$ the flow generated by the vector field $h(\cdot,b)$ for time τ (assume that $\tau \in \mathbb{R}$). Let $(t_*,x_*) \in [0,\vartheta]\times\mathbb{R}^n$ and $b(\cdot):[0,\vartheta]\to\Lambda$ be a measurable function. A *motion* of system (2) generated by the control $b(\cdot)$ is a solution of the equation

$$x(t) = x_* + \int_{t_*}^{t} h(x(\xi), b(\xi))d\xi.$$

Assume that t is an arbitrary time instant from the interval $[0,\vartheta]$. Denote the motion of the system starting at the position (t_*,x_*) and generated by the control $b(\cdot)$ by $x_h\big(t,t_*,x_*,b(\cdot)\big)$. Consider the case where $b(\cdot)$ is a piecewise constant, right-continuous function. Assume that $t \geq t_*$. Since $b(\cdot)$ is a piecewise constant, right-continuous function, there exist a collection of numbers $\tau_1,\ldots,\tau_k \in [0,\infty)$ and controls $b_1,\ldots,b_k \in \Lambda$ such that

$$t = t_* + \tau_1 + \ldots + \tau_k, \quad b(\xi) = b_i$$

for $\xi \in \Big[t_* + \tau_1 + \ldots + \tau_{i-1},\ t_* + \tau_1 + \ldots + \tau_{i-1} + \tau_i\Big]$. In this case,

$$x_h\big(t,t_*,x_*,b(\cdot)\big) = S^{\tau_k}_{h,b_k}\circ\ldots\circ S^{\tau_1}_{h,b_1}(x_*).$$

This formula is also valid in the case where $t < t_*$; then $\tau_i < 0$.

Except for ordinary controls, we consider generalized controls (control measures). Consider the Borel σ-algebra of subsets of $[0,\vartheta]\times\Lambda$. A *generalized control* is an arbitrary measure defined on this σ-algebra. We denote the set of all measures by $\mathcal{R}_\Lambda$. Let $(t_*,x_*)\in[0,\vartheta]\times\mathbb{R}^n$, $\mu\in\mathcal{R}_\Lambda$. Then a *motion* starting at the position (t_*,x_*) and generated by a generalized control μ is the solution of the equation

$$x(t) = x_* + \int\limits_{[t_*,t]\times\Lambda} h\big(x(\xi),b\big)\mu\big(d(\xi,b)\big)$$

for $t \geq t_*$ and the solution of the equation

$$x(t) = x_* - \int\limits_{[t,t_*]\times\Lambda} h\big(x(\xi),b\big)\mu\big(d(\xi,b)\big)$$

for $t < t_*$. We denote this motion by $\varphi_h(\cdot,t_*,x_*,\mu)$. It is known (see [8]) that for any generalized control $\mu\in\mathcal{R}_\Lambda$, there exists a sequence of piecewise constant, right-continuous, measurable functions such that

$$x_h\big(\cdot,t_*,x_*,b^k(\cdot)\big) \rightrightarrows \varphi_h(\cdot,t_*,x_*,\mu), \quad k\to\infty.$$

Moreover, the set of ordinary controls can be embedded in the space of generalized controls, namely, for any measurable control $b(\cdot)$, there exists a measure $\mu_{b(\cdot)}$ such that

$$\int\limits_{[0,\vartheta]\times\Lambda} \psi(t,b)\mu_{b(\cdot)}(d(t,\mu)) = \int_0^{\vartheta}\psi(t,b(t))dt$$

for all $\psi \in C([0,\vartheta] \times \Lambda)$. Note that

$$x_h\big(\cdot,\cdot,\cdot,b(\cdot)\big) = \varphi_h\big(\cdot,\cdot,\cdot,\mu_{b(\cdot)}\big).$$

The problem of the solution of a differential game can be reduced to a sequence of control problems by the method of program iterations proposed by A. G. Chentsov. We define the operator of program absorption (see [6, 14]), which acts in the space of closed sets by keeping in this set all positions from which the first player knowing the constant control of the antagonist can lead the system to the objective set such that the motion does not leave the given set. Consider the conflict-controlled system (1). For any $v \in Q$, we define the control system

$$f_v(x,u) \triangleq f(x,u,v).$$

The image of the closed set $E \subset [0,\vartheta] \times \mathbb{R}^n$ is the set

$$A_f(E) = \Big\{(t_*,x_*) \in E : \forall v \in Q \ \exists \mu \in \mathcal{R}(P) \ \exists \xi \in [t_*,\vartheta] : \\ \varphi_{f_v}\big(\xi,t_*,x_*,\mu\big) \in M[\xi] \ \& \ \big(\varphi_{f_v}(t,t_*,x_*,\mu) \in E[t] \ \forall t \in [t_*,\xi]\big)\Big\},$$

where $E[t]$ is the section of the set E:

$$E[t] \triangleq \{x : (t,x) \in E\}.$$

Consider the sequence

$$W_0 \triangleq [0,\vartheta] \times \mathbb{R}^n, \quad W_k = A_f(W_{k-1}) \quad \forall k \in \mathbb{N}.$$

Chentsov proved that the solvability set of the guidance problem can be represented in the form

$$\mathfrak{W} = \bigcap_{k=0}^{\infty} W_k.$$

We also note that the condition of the u-stability can be stated by using the operator of program absorption: a set W is a u-stable bridge if $A(W) = W$.

3. Statement of the Main Result

Consider the convergence/divergence problem for a conflict-controlled system of the form (1) with the objective set M. Introduce the notation $F \triangleq M[\vartheta]$. We assume that M is the controllability set with the objective set $M^* \triangleq \{\vartheta\} \times F$ with respect to some control system $g(x,\omega)$, $\omega \in \Omega$:

$$M = \Big\{(t,x) \in [0,\vartheta] \times \mathbb{R}^n : \ \exists x_* \in F \ \exists \mu \in \mathcal{R}(\Omega) : \ x = \varphi_g(t,\vartheta,x_*,\mu)\Big\};$$

here Ω is a compact set in a finite-dimensional arithmetic space.

To construct an equivalent game "at moment," we broaden opportunities of the first player who creates the control u by introducing two additional controls: ν that takes values 0 or 1, and ω that takes values in the set Ω. Consider the conflict-controlled system

$$\dot{x} = f^*(x,\nu,u,\omega,v), \quad x \in \mathbb{R}^n, \quad \nu \in \{0,1\}, \quad u \in P, \quad \omega \in \Omega, \quad v \in Q, \tag{3}$$

on the interval $[0,\vartheta]$, where

$$f^*(x,\nu,u,\omega,v) = \nu \cdot f(x,u,v) + (1-\nu) \cdot g(t,\omega) = \begin{cases} f(x,u,v), & \nu = 0, \\ g(x,\omega), & \nu = 1. \end{cases}$$

In the new system (3), the first player disposes of the controls ν, u, and ω whereas the second player, as in system (1), disposes of the control v.

For system (3), we consider the problem of guidance to the set $M^* \triangleq \{\vartheta\} \times F$.

Similar methods of transformation of systems were considered earlier in the class of program controls for transforming problems of guidance to a cylindrical set (see [1, 13]).

Introduce the following notation:

$$P^* = \{0,1\} \times P \times \Omega;$$

if $h = f_v$ and $u \in P$, then we denote $S^\tau_{h,u}$ by $\mathcal{F}^\tau_{u,v}$; if $h = f^*_v$ and $u^* = (\nu, u, \omega) \in P^*$, then we denote S^τ_{h,u^*} by $\mathcal{F}^{*\tau}_{u^*,v}$. Introduce the notation

$$\mathcal{G}^\tau_\omega \triangleq S^\tau_{g,\omega}.$$

Let $(t_*, x_*) \in [0,\vartheta] \times \mathbb{R}^n$, $t \in [0,\vartheta]$; if $h = f_v$ and $\mu \in \mathcal{R}_P$, then we denote $\varphi_h(t, t_*, x_*, \mu)$ by $\phi(t, t_*, x_*, \mu, v)$. Similarly, in the case where $h = f^*_v$ and $\mu \in \mathcal{R}_{P^*}$, we denote $\varphi(t, t_*, x_*, \mu)$ by $\phi^*(t, t_*, x_*, \mu, v)$. The operator of program absorption for the initial problem is denoted by A, and for the transformed problem, by A^*. Elements of the sequence constructed by the method of program iterations for the initial system are denoted by W_k, $k \in \mathbb{N}_0$; elements of the sequence constructed by the method of program iterations for the transformed problem are denoted by W^*_k, $k \in \mathbb{N}_0$. The solvability sets for the initial and transformed problems are denoted by $\mathfrak{W}$ and $\mathfrak{W}^*$, respectively.

Theorem. *Let M be the controllability set with the objective set $M^* = \{\vartheta\} \times F$ for a control system $g(x,\omega)$. If for all $u \in P$, $V \in Q$, $\omega \in \Omega$, and $\tau', \tau'' \geq 0$, the flows $\mathcal{F}^{\tau'}_{u,v}$ and $\mathcal{G}^{\tau''}_\omega$ commute, i.e.,*

$$\mathcal{F}^{\tau'}_{u,v} \circ \mathcal{G}^{\tau''}_\omega = \mathcal{G}^{\tau''}_\omega \circ \mathcal{F}^{\tau'}_{u,v}, \tag{4}$$

then the following assertions hold:

(1) $W_k = W^*_k$ *for all* $k \in \mathbb{N}$;

(2) *the problem of guidance of system* (1) *to the cylindrical set* $[0,\vartheta] \times F$ *is equivalent to the problem of guidance of system* (3) *to the set* $\{\vartheta\} \times F$;

(3) *if system* (1) *satisfies the Isaacs condition, then system* (3) *also satisfies the Isaacs condition.*

Note that in the case where $f(\cdot, u, v)$ and $g(\cdot, \omega)$ are smooth vector fields, condition (4) can be expressed through the commutator of vector fields $[\cdot, \cdot]$:

$$[f(\cdot, u, v), g(\cdot, \omega)] = 0 \quad \forall u \in P,\ \forall v \in Q,\ \forall \omega \in \Omega.$$

In particular, if the objective set has the form of a cylinder $M = [0,\vartheta] \times F$ (such a problem is called a problem of guidance to the set F "up to a moment"), the choice of $g(x,\omega) \equiv 0$ as an auxiliary control system reduces the guidance problem to the problem with the objective set $\{\vartheta\} \times F$ (the problem of guidance to the set F "at moment"). This problem was considered in [2, 13].

4. Properties of the Operator of Program Absorption

Let $E \subset [0,\vartheta] \times \mathbb{R}^n$. We say that E does not increase by sections with respect to a control system $g(x,\omega)$, $\omega \in \Omega$, if for all (t_*, x_*), $t \in [0, t_*]$, and $\sigma \in \mathcal{R}_\Omega$ the inclusion

$$\varphi_g(t, t_*, x_*, \sigma) \in E[t]$$

holds.

Lemma 1. *If a closed set $E \subset [0,\vartheta] \times \mathbb{R}^n$ is such that $M \subset E$ and E does not increase by sections with respect to a control system $g(x,\omega)$, $\omega \in \Omega$, then the set $A^*(E)$ also possesses these properties.*

Proof. First, we prove that E does not increase by sections with respect to the control system $g(x,\omega)$, $\omega \in \Omega$. Let $(t_*, x_*) \in A^*(E)$, $t \in [0, t_*]$, and $\sigma \in \mathcal{R}_\Omega$. We prove that

$$\varphi_g(t, t_*, x_*, \sigma) \in (A^*(E))[t].$$

Since $(t_*, x_*) \in A^*(E) \subset E$, we have

$$\Big(\tau,\ \varphi_g\big(\tau, t_*, x_*, \sigma\big)\Big) \in E \quad \forall \tau \in [t, t_*]. \tag{5}$$

For any $v \in Q$, there exists a measure $\mu \in \mathcal{R}_{P^*}$ such that

$$\phi^*(\xi, t_*, x_*, \mu, v) \in M[\xi]$$

for some $\xi \in [t_*, \vartheta]$, and for all $\tau \in [t_*, \xi]$, the inclusion

$$\phi^*(\tau, t_*, x_*, \mu, v) \in E[\tau]$$

holds.

There exists a measure $\tilde{\sigma} \in \mathcal{R}_{P^*}$ such that

$$\int\limits_{[0,\vartheta]\times P^*} \psi(t,\omega)\tilde{\sigma}(d(t,\nu,\omega,u)) = \int\limits_{[0,\vartheta]\times\Omega} \psi(t,\omega)\sigma(d(t,\omega))$$

for all $\psi \in C([0,\vartheta] \times \Omega)$.

We denote by $\tilde{\mu}$ a measure such that

$$\int\limits_{[t,\vartheta]\times P^*} \psi(t,u^*)\tilde{\mu}(d(t,u^*)) = \int\limits_{[t,t_*]\times P^*} \psi(t,u^*)\tilde{\sigma}(d(t,u^*)) + \int\limits_{[t_*,\vartheta]\times P^*} \psi(t,u^*)\mu(d(t,u^*)).$$

Introduce the notation $\bar{x} = \varphi_g(t, t_*, x_*, \sigma)$. Note that

$$\varphi_g(\tau, t_*, x_*, \sigma) = \varphi_g(\tau, t, \bar{x}, \sigma) = \phi^*(\tau, t, \bar{x}, \tilde{\sigma}, v) = \phi^*(\tau, t, \bar{x}, \tilde{\mu}, v).$$

Equation (5) and this chain of equalities imply that

$$\phi^*\big(\tau, t, \bar{x}, \tilde{\mu}, v\big) \in E[\tau] \tag{6}$$

and

$$x_* = \varphi_g\big(t_*, t, \bar{x}, \sigma\big) = \phi^*\big(\tau, t, \bar{x}, \tilde{\mu}, v\big).$$

This implies that

$$\phi^*(\tau, t_*, x_*, \mu, v) = \phi^*(\tau, t_*, x_*, \tilde{\mu}, v) = \phi^*(\tau, t, \bar{x}, \tilde{\mu}, v).$$

Since $(t_*, x_*) \in A^*(E)$, we conclude that

$$\phi^*(\xi, t, \bar{x}, \tilde{\mu}, v) \in M[\xi]$$

and

$$\phi^*(\tau, t, \bar{x}, \tilde{\mu}, v) \in E[\tau]$$

for all $\tau \in [t_*, \xi]$. These inclusions and inclusion (6) imply that $(t, \bar{x}) \in A^*(E)$.

Now we prove that $M \subset A^*(E)$. Indeed, if $M \subset E$, then the definition of the operator of program absorption implies that $M \subset A^*(E)$. $\square$

Now let $u^*(\cdot) : [0, \vartheta] \to P^*$ be a piecewise constant control constructed for the second system. Let $t_*, \xi \in [0, \vartheta]$, $t_* \le \xi$. The semi-interval $[t_*, \xi)$ is the union of semi-intervals $[\xi_{i-1}, \xi_i)$, $i = \overline{1, k}$, such that on any semi-interval the control $u^*(\cdot)$ is constant and is equal to u_i^*. Introduce the notation $\tau_i \triangleq \xi_i - \xi_{i-1}$. We have

$$u_i^* = (\nu_i, u_i, \omega_i), \qquad \nu_i \in \{0, 1\}, \quad u_i \in P, \quad \omega_i \in \Omega.$$

Introduce the sets

$$J' = \{i : \nu_i = 1\} = \{r_1, \ldots, r_l\}, \quad J'' = \{i : \nu_i = 0\}.$$

Let $\hat{\xi}_0 = t_*$, $\hat{\xi}_i = \hat{\xi}_j + \tau_{r_j}$, and

$$\bar{\xi} = \xi_{r_l}. \tag{7}$$

We define a piecewise constant control

$$u(\cdot):[t_*,\bar{\xi})\to P$$

as follows:

$$u(t)=u_{r_j},\quad t\in\Big[\hat{\xi}_{r_{j-1}},\ \hat{\xi}_{r_j}\Big). \tag{8}$$

Now let $t\in[t_*,\bar{\xi}]$. Either there exists j such that $t\in\big[\hat{\xi}_{r_{j-1}},\hat{\xi}_{r_j}\big)$ or $t=\bar{\xi}$. In the first case, we set

$$\gamma(t)\triangleq\xi_{r_j-1}+t-\hat{\xi}_{r_{j-1}};$$

in the second case we set $\gamma(t)\triangleq\bar{\xi}$.

Introduce the notation

$$J''_t=\big\{i\in J'':\xi_i<\gamma(t)\big\}=\{s_1,\dots,s_m\}.$$

Lemma 2. *There exists a piecewise constant control*

$$\omega(\cdot):[0,\vartheta]\to\Omega,$$

such that for all $t\in[t,\bar{\xi}]$ we have

$$\mathbf{x}^*\Big(\gamma(t),t_*,x_*,u^*(\cdot),v\Big)=x_g\Big(\gamma(t),\ t,\ \mathbf{x}\big(t,t_*,x_*,u(\cdot),v\big),\ \omega(\cdot)\Big).$$

Proof. We have

$$\mathbf{x}^*\Big(\gamma(t),t_*,x_*,u^*(\cdot),v\Big)=\mathcal{F}^{*\,\gamma(t)-\xi_{r_j-1}}_{u^*_{s_j},v}\circ\ldots\circ\mathcal{F}^{*\,\tau_i}_{u^*_i,v}\circ\ldots\circ\mathcal{F}^{*\,\tau_1}_{u^*_1,v}(x_*).$$

Therefore,

$$\mathcal{F}^{*\,\gamma(t)-\xi_{r_j-1}}_{u^*_{r_j},v}=\mathcal{F}^{t-\hat{\xi}_{r_j-1}}_{u^*_{r_j},v}.$$

We have

$$\mathcal{F}^{*\,\tau_i}_{u^*_i,v}=\mathcal{F}^{\tau_i}_{u_i,v}$$

for $i\in J'$ and

$$\mathcal{F}^{*\,\tau_i}_{u^*_i,v}=\mathcal{G}^{\tau_i}_{\omega_i}$$

for $i\in J''$.

Since the flows $\mathcal{F}^{\tau'}_{u,v}$ and $\mathcal{G}^{\tau''}_{\omega}$ commute, we obtain

$$\mathbf{x}^*\Big(\gamma(t),t_*,x_*,u^*(\cdot),v\Big)=\mathcal{G}^{\tau_{s_m}}_{\omega_m}\circ\ldots\circ\mathcal{G}^{\tau_{s_1}}_{\omega_1}\circ\mathcal{F}^{t-\hat{\xi}_{r_j-1}}_{u^*_{r_j},v}\circ\ldots\circ\mathcal{F}^{\tau_{r_1}}_{u_{r_1},v}(x_*),$$

which proves the lemma. □

Lemma 3. *Let E not increase by sections with respect to the control system $g(x,\omega)$, $\omega\in\Omega$ and let $M\subset E$. Then $A(E)=A^*(E)$.*

Proof. First, we prove the inclusion $A(E)\subset A^*(E)$. Let $(t_*,x_*)\in A(E)$; this means that for all $v\in Q$ there exist $\mu\in\mathcal{R}_P$ and $\xi\in[t_*,\vartheta]$ such that

$$\phi(\xi,t_*,x_*,\mu,v)\in M[\xi],\quad \phi(t,t_*,x_*,\mu,v)\in E[t]$$

for all $t\in[t_*,\xi]$. Since M is the controllability set with the objective set $\{\vartheta\}\times F$ for the control system $g(x,\omega)$, there exists a measure $\sigma\in\mathcal{R}_\Omega$ such that

$$\varphi_g(t,\xi,\phi(\xi,t_*,x_*,\mu,v),\sigma,v)\in M[t],\quad t\in[\xi,\vartheta].$$

By the Riesz theorem, there exist measures $\hat{\mu}\in\mathcal{R}_{P^*}$ and $\tilde{\sigma}\in\mathcal{R}_{P^*}$ such that

$$\int\limits_{[0,\vartheta]\times P}\psi(t,u)\mu(d(t,u))=\int\limits_{[0,\vartheta]\times P^*}\psi(t,u)\hat{\mu}(d(t,\nu,u,\omega))$$

for all $\psi \in C([0,\vartheta]\times P)$ and

$$\int\limits_{[0,\vartheta]\times\Omega} \psi(t,\omega)\sigma(d(t,\omega)) = \int\limits_{[0,\vartheta]\times P^*} \psi(t,\omega)\tilde{\sigma}(d(t,\nu,u,\omega))$$

for $\psi \in C([0,\vartheta]\times\Omega)$.

We denote by $\beta \in \mathcal{R}_{P^*}$ a measure such that for any function $\psi \in C([0,\vartheta]\times P^*)$, the following relations hold:

$$\int\limits_{[0,\xi]\times P^*} \psi(t,\nu,u,\omega)\beta(d(t,\nu,u,\omega)) = \int\limits_{[0,\xi]\times P^*} \psi(t,\nu,u,\omega)\hat{\mu}(d(t,\nu,u,\omega)),$$

$$\int\limits_{[\xi,\vartheta]\times P^*} \psi(t,\nu,u,\omega)\beta(d(t,\nu,u,\omega)) = \int\limits_{[\xi,\vartheta]\times P^*} \psi(t,\nu,u,\omega)\tilde{\sigma}(d(t,\nu,u,\omega)).$$

We have

$$\phi^*(t,t_*,x_*,\beta,v) = \begin{cases} \phi(t,t_*,x_*,\mu,v),\sigma,v), & t\in[t_*,\xi], \\ \varphi_g(t,\xi,\phi(\xi,t_*,x_*,\mu,v),\sigma,v), & t\in[\xi,\vartheta]. \end{cases}$$

Since $M[t]\subset E[t]$ $M[\vartheta]=F$, we conclude that

$$(\vartheta,\phi^*(\vartheta,t_*,x_*,\beta,v)\in\{\vartheta\}\times F$$

and for all $t\in[t_*,\vartheta]$ the inclusion

$$\phi^*(t,t_*,x_*,\beta,v)\in E[t]$$

holds.

Now we prove the opposite inclusion.

Let $(t_*,x_*)\in A^*(E)$ and $C>0$ be a number such that for all $t_1,\ t_2\in[0,\vartheta]$, $t_2\le t_1$, $x',\ x''\in\mathcal{G}$, and $\sigma\in\mathcal{R}_\Omega$, the inequality

$$\Big\|\varphi_g\big(t_2,t_1,x',\sigma\big)-\varphi_g\big(t_2,t_1,x'',\sigma\big)\Big\|\le C\|x'-x''\|$$

holds, where $\mathcal{G}$ is the set of positions reachable from the segment $[0,\vartheta]\times\{x_*\}$ by the control system $\dot{x}=f^*(x,\nu,u,\omega,v)$, $\nu\in\{0,1\}$, $u\in P$, $\omega\in\Omega$, $v\in Q$.

The inclusion $(t_*,x_*)\in A^*(E)$ means that for any $v\in Q$, there exists a measure $\beta\in\mathcal{R}_{P^*}$ such that

$$\phi^*(\vartheta,t_*,x_*,\beta,v)\in F$$

and

$$\phi(t,t_*,x_*,\beta,v)\in E[t]$$

for all $t\in[t_*,\vartheta]$. There exists a sequence of piecewise constant controls for the transformed system $\{\zeta^\alpha(\cdot)\}_{\alpha=1}^\infty$, $\zeta^\alpha(\cdot):[t_*,\vartheta]\to P^*$, such that

$$\varepsilon^\alpha \triangleq \sup_{t\in[t^*,\vartheta]}\Big\|\mathbf{x}^*\big(t,t_*,x_*,\zeta^\alpha(\cdot),v\big)-\phi^*(t,t_*,x_*,\beta,v)\Big\|\to 0,\quad \alpha\to\infty.$$

Consider the sequence of controls u^α constructed by rule (8), and moments ξ^α defined by (7). For any α, the functions $\gamma^\alpha(\cdot)$ are defined.

There exists a subsequence $\{\alpha_k\}$ such that $\xi^{\alpha_k}\to\xi$, $\mu_{\zeta^{\alpha_k}}\rightharpoondown\mu$. We assume that $\{\alpha_k\}$ coincides with the sequence $\{\alpha\}$.

We have $\gamma^\alpha(\xi^\alpha)=\vartheta$ for all $\alpha\in\mathbb{N}$. Lemma 2 implies that for some control $\omega^\alpha(\cdot)$ we have

$$\mathbf{x}^*\big(\vartheta,t_*,x_*,\zeta^\alpha(\cdot),v\big)=x_g\Big(\vartheta,\xi,\mathbf{x}\big(\xi,t_*,x_*,u^\alpha(\cdot),v\big),\omega^\alpha(\cdot)\Big),$$

which is equivalent to the equality

$$\mathbf{x}\big(\xi, t_*, x_*, u^\alpha(\cdot), v\big) = x_g\Big(\xi,\ \vartheta,\ \mathbf{x}^*\big(\vartheta, t_*, x_*, \zeta^\alpha(\cdot), v\big),\ \omega^\alpha(\cdot)\Big).$$

Thus,

$$\Big\|\mathbf{x}\big(\xi^\alpha, t_*, x_*, u^\alpha(\cdot), v\big) - x_g\Big(\xi,\ \vartheta,\ \phi^*\big(\vartheta, t_*, x_*, \beta, v\big),\ \omega^\alpha(\cdot)\Big)\Big\|$$
$$\leq \Big\|x_g\Big(\xi^\alpha,\ \vartheta,\ \mathbf{x}^*\big(\vartheta, t_*, x_*, \zeta^\alpha(\cdot), v\big),\ \omega^\alpha(\cdot)\Big) - x_g\Big(\xi,\ \vartheta,\ \phi^*\big(\vartheta, t_*, x_*, \beta, v\big),\ \omega^\alpha(\cdot)\Big)\Big\| \leq C\varepsilon^\alpha.$$

Since M is the controllability set in $\{\vartheta\} \times F$ for the control system $g(x, \omega)$, $\omega \in \Omega$, we have

$$\Big(\xi^\alpha,\ x_g\big(\xi^\alpha, \vartheta, \phi^*(\vartheta, t_*, x_*, \beta, v), \omega^\alpha(\cdot)\big)\Big) \in M.$$

Therefore,

$$\big(\xi, \mathbf{x}(\xi, t_*, x_*, \mu, v)\big) \in M.$$

Now let $t \in [t_*, \xi]$. For sufficiently large α we have $t \leq \xi^\alpha$ and

$$\phi^*\big(\gamma^\alpha(t), t_*, x_*, \beta, v\big) \in E[\gamma(t)].$$

Lemma 2 also implies that for some control $\omega(\cdot)$

$$\mathbf{x}\big(t, t_*, x_*, u^\alpha(\cdot), v\big) = x_g\Big(t,\ \gamma^\alpha(t),\ \mathbf{x}^*\big(\gamma^\alpha(t), t_*, x_*, \zeta^\alpha(\cdot), v\big),\ \omega(\cdot)\Big).$$

From this we conclude that

$$\Big\|\mathbf{x}\big(t, t_*, x_*, u^\alpha(\cdot), v\big) - x_g\Big(t,\ \gamma^\alpha(t),\ \phi^*\big(\gamma^\alpha(t), t_*, x_*, \beta, v\big),\ \omega(\cdot)\Big)\Big\|$$
$$\leq \Bigg\|x_g\Big(t,\ \gamma^\alpha(t),\ \mathbf{x}^*\big(\gamma^\alpha(t), t_*, x_*, \zeta^\alpha(\cdot), v\big),\ \omega(\cdot)\Big)$$
$$- x_g\Big(t,\ \gamma^\alpha(t),\ \phi^*\big(\gamma^\alpha(t), t_*, x_*, \beta, v\big),\ \omega(\cdot)\Big)\Bigg\| \leq C\varepsilon^\alpha.$$

Since E does not increase by sections with respect to the control system $g(x, \omega)$, $\omega \in \Omega$,

$$\phi^*\big(\gamma^\alpha(t), t_*, x_*, \beta, v\big) \in E[\gamma^\alpha(t)], \quad x_g\Big(t,\ \gamma^\alpha(t),\ \phi^*\big(\gamma^\alpha(t), t_*, x_*, \beta, v\big),\ \omega(\cdot)\Big) \in E[t].$$

This implies that

$$\phi(t, t_*, x_*, \mu, v) \in E[t].$$

Since $v \in Q$ is arbitrary, $(t_*, x_*) \in A(E)$. Thus, we have proved that $A^*(E) \subset A(E)$. $\square$

Proof of the theorem. Proof of item 1 immediately follows from Lemmas 1 and 3 since $M \subset [0, \vartheta] \times \mathbb{R}^n$ and $[0, \vartheta] \times \mathbb{R}^n$ does not increase by sections with respect to $g(x, \omega)$, $\omega \in \Omega$.

Item 2 follows from item 1 and the representation of the solvability set.

Item 3 is proved directly. Obviously,

$$\max_{v \in Q} \min_{(\nu, u, \omega) \in P^*} \big\langle s,\ f^*(x, \nu, u, \omega, v)\big\rangle \leq \min_{(\nu, u, \omega) \in P^*} \max_{v \in Q} \big\langle s,\ f^*(x, \nu, u, \omega, v)\big\rangle.$$

Prove the inverse inequality. Denote by (u_*, v_*) the saddle point in the small game for the initial system. Let

$$\min_{(\nu, u, \omega) \in P^*} \max_{v \in Q} \big\langle s,\ f^*(x, \nu, u, \omega, v)\big\rangle = \min_{u \in P, \omega \in \Omega} \max_{v \in Q} \big\langle s,\ f^*(x, 1, u, \omega, v)\big\rangle.$$

In particular, this implies that

$$\max_{v \in Q} \min_{u \in P} \big\langle s,\ f(x, u, v)\big\rangle = \min_{u \in P} \max_{v \in Q} \big\langle s,\ f(x, u, v)\big\rangle \leq \min_{\omega \in \Omega} \big\langle s,\ g(x, \omega)\big\rangle.$$

In this case

$$\min_{(\nu,u,\omega)\in P^*}\max_{v\in Q}\big\langle s,\ f^*(x,\nu,u,\omega,v)\big\rangle = \min_{u\in P}\max_{v\in Q}\big\langle s,\ f(x,u,v)\big\rangle$$

$$= \max_{v\in Q}\min_{u\in P}\big\langle s,\ f(x,u,v)\big\rangle = \min_{u\in P}\big\langle s,\ f(x,u,v_*)\big\rangle$$

$$= \min\Big\{\min_{u\in P}\big\langle s,\ f(x,u,v_*)\big\rangle,\ \min_{\omega\in\Omega}\big\langle s,\ g(x,\omega)\big\rangle\Big\} =$$

$$= \min_{(\nu,u,\omega)\in P^*}\big\langle s,\ f^*(x,\nu,u,\omega,v_*)\big\rangle \le \max_{v\in Q}\min_{(\nu,u,\omega)\in P^*}\big\langle s,\ f^*(x,\nu,u,\omega,v)\big\rangle.$$

Now let

$$\min_{(\nu,u,\omega)\in P^*}\max_{v\in Q}\big\langle s,\ f^*(x,\nu,u,\omega,v)\big\rangle = \min_{u\in P,\omega\in\Omega}\max_{v\in Q}\big\langle s,\ f^*(x,0,u,\omega,v)\big\rangle = \min_{\omega\in\Omega}\big\langle s,\ g(x,\omega)\big\rangle.$$

Then

$$\min_{\omega\in\Omega}\big\langle s,\ g(x,\omega)\big\rangle = \min\Big\{\min_{\omega\in\Omega}\big\langle s,\ g(x,\omega)\big\rangle,\ \min_{u\in P}\max_{v\in Q}\big\langle s,\ f(x,u,v)\big\rangle\Big\}$$

$$= \min\Big\{\min_{\omega\in\Omega}\big\langle s,\ g(x,\omega)\big\rangle,\ \min_{u\in P}\big\langle s,\ f(x,u,v_*)\big\rangle\Big\} = \min_{(\nu,u,\omega)\in P^*}\big\langle s,\ f(x,\nu,u,\omega,v_*)\big\rangle \le$$

$$\le \max_{v\in Q}\min_{(\nu,u,\omega)\in P^*}\big\langle s,\ f(x,\nu,u,\omega,v)\big\rangle.$$

The theorem is proved. □

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