SPECIAL ISSUE PAPER

Analytical solution of integro-differential equations describing the process of intense boiling of a superheated liquid

Irina V. Alexandrova | Alexander A. Ivanov | Dmitri V. Alexandrov

Department of Theoretical and Mathematical Physics, Laboratory of Multi-Scale Mathematical Modeling, Ural Federal University, Ekaterinburg, Russian Federation

Correspondence
Irina V. Alexandrova, Department of Theoretical and Mathematical Physics, Laboratory of Multi-Scale Mathematical Modeling, Ural Federal University, Ekaterinburg, Russian Federation.
Email: irina.alexanderova@urfu.ru

Funding information
Ministry of Science and Higher Education of the Russian Federation, Grant/Award Number: FEUZ-2020-0057; Russian Foundation for Basic Research, Grant/Award Number: 20-08-00199

1 | INTRODUCTION

It is well known that the phase and structural transformations, which are widely encountered in various natural phenomena and numerous technological processes, completely determine the dynamics of the system, and are responsible for its properties, structure and the final result (state of the system) after the phase transition. As this takes place, the dynamics of a metastable system is determined by the intensity of the formation of nuclei of a new phase (e.g., centers of crystallization or vaporization, the nucleation of which depends on the intensity of fluctuations). We also especially note that the dynamic behavior of a system is often sensitive to changes in its parameters responsible for the formation of different evolutionary scenarios (e.g., self-oscillations and instability). This, in particular, explains the need to develop approximate analytical methods for analyzing such systems.

Mathematical models of such processes of structural-phase transformations consist of a kinetic equation for the particle size distribution function and a balance equation (for temperature, concentration of a dissolved impurity, etc.). Such a system of equations is integro-differential and depends on the kinetics of the evolution of nuclei. Note that the rate of growth/reduction of nuclei, generally speaking, is a solution to a separate problem with moving boundaries. Therefore, there are no general methods for solving such a closed model of a process with moving boundaries of phase transformations. The solution to each problem is a separate study, which often requires the development of special mathematical methods.

This study is devoted to the development of an analytical solution to an integro-differential model of intense boiling of a liquid. For simplicity of the model, we assume that the liquid is homogeneous in all spatial directions, and its temperature...
depends only on the time variable \( t \). For simplicity of the model, we also assume that the properties of the liquid and the radius of critical nuclei are constant throughout the volume, and the bulk concentration of bubbles is assumed to be insignificant. The solution of the integro-differential model under consideration is based on the Laplace integral transform method and the saddle point technique used to approximate the Laplace-type integral. The applied method was previously used for the analytical description of phase transformations in supercooled and supersaturated liquids.\(^{30-34}\)

This article is organized as follows. The system of integro-differential equations supplemented with the corresponding initial and boundary conditions is given in Section 2. Its analytical solutions are presented in Section 3. Our main conclusions are discussed in Section 4.

### 2 | THE MODEL

Let us introduce the bubble-size distribution function \( f(r, t) \) and current temperature \( T(t) \) of the liquid, which describe the time-dependent state of intense boiling. Here, \( r \) and \( t \) designate the bubble radius and time. The distribution function satisfies the kinetic equation

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial r} \left( \frac{dr}{dt} f \right) + \gamma f = 0, \quad r > r_*, \quad t > 0,
\]

where the number of withdrawing bubbles is considered to be proportional to the concentration of bubbles with a constant coefficient \( \gamma \). Here, \( dr/dt \) represents the growth rate of bubbles and \( r_* \) is the critical radius of nucleating bubbles capable to further growth. The flux of such bubbles is equal to the rate \( J \) of their appearance, that is,

\[
\frac{dr}{dt} f = J \left( \frac{T - T_0}{T_0} \right), \quad r = r_*.
\]

where \( T_0 \) stands for the boiling temperature.

The temperature balance equation depends on the bubble-size distribution function and heat exchange with external medium of temperature \( T_m \) and takes the form

\[
\rho c \frac{dT}{dt} = \alpha(T_m - T) - 4\pi \rho' L \int_{r_*}^{\infty} r^2 \frac{df}{dt} f(r, t) dr, \quad t > 0.
\]

Here, \( \rho' \) is the density of two-phase system, \( c \) is its thermal capacity, \( \alpha \) is the heat exchange coefficient, and \( L \) is the latent heat of phase transition.

The initial distribution function and system temperature should be regarded as known

\[
f(r, t) = f_0(r), \quad T(t) = T(0), \quad t = 0.
\]

For the sake of definiteness, let us assume that the growth rate of bubbles is given by

\[
\frac{dr}{dt} = \mu w \left( \frac{T - T_0}{T_0} \right), \quad w \left( \frac{T - T_0}{T_0} \right) = \left( \frac{T - T_0}{T_0} \right)^n,
\]

where \( \mu \) is a constant coefficient and \( w \) represents the dimensionless function of rescaled temperature \( (T - T_0)/T_0 \). Let us especially note that we use here the power law with the constant exponent \( n \) frequently met in applications. So, for example, if the bubble growth is limited by the inertia of liquid, one obtains \( n = 1/2 \).\(^3\)\(^5\) In addition, if the growth of bubbles is controlled by the heat supply rate of evaporation, one can get \( n = 1 \).\(^3\)

The rate \( J \) of bubble appearance satisfies the Dering–Volmer and Frenkel–Zeldovich–Kagan nucleation models assuming the following law\(^3\):

\[
J(u) = J_0 R(u), \quad R[u(\tau)] = \exp \left[ -\frac{\kappa}{u^2(\tau)[u(\tau) + 1]} \right], \quad \kappa = \frac{16\pi \rho^3 \Psi}{3L^2 \rho''^2 k T_0},
\]

where...
where \( u = (T - T_0)/T_0 \) stands for the rescaled temperature, \( J_0 \) represents the constant factor, \( \sigma \) is the surface tension, \( \Psi \) is the constant coefficient, \( \rho'' \) is the density of vapour, and \( k \) is the Boltzmann constant.

Model (1)–(6) is a closed system of integro-differential equations, boundary and initial conditions for studying the evolution of the process of intense boiling with polydisperse bubbles.

For the convenience of solving the formulated nonlinear model, we introduce dimensionless variables and parameters as follows:

\[
y = \frac{(r - r_s)\gamma}{\mu w(u_0)}, \quad \tau = \gamma t, \quad \Phi(y, \tau) = \int_0^t f_r(r, t), \quad \Phi_0(y) = \int_0^t f_0(r), \quad l_0 = \frac{\mu}{\gamma}, \quad y_s = \frac{r_s \gamma}{\mu w(u_0)}.
\]

Here, \( u_0 \) represents a characteristic system temperature introduced as

\[
\tau = \int_0^\tau w[u(\tau_1)]d\tau_1 = w(u_0)\tau.
\]

Using the dimensionless variables and parameters (7), we rewrite the model (1)–(6) in the form of

\[
\frac{\partial \Phi}{\partial \tau} + \frac{w(u)}{w(u_0)} \frac{\partial \Phi}{\partial y} + \Phi = 0, \quad y > 0, \quad \tau > 0,
\]

\[
\Phi(y, \tau) = Y[u(\tau)], \quad y = 0; \quad Y[u(\tau)] = \frac{\nu R[u(\tau)]}{w[u(\tau)]},
\]

\[
\frac{du}{d\tau} = \frac{u_m - u}{a} - hw(u)\Lambda[u(\tau), \tau], \quad \tau > 0; \quad \Lambda[u(\tau), \tau] = \int_0^\infty (y + y)^2 \Phi(y, \tau) dy,
\]

\[
\Phi(y, \tau) = \Phi_0(y), \quad u(\tau) = u(0), \quad \tau = 0.
\]

As this takes place, the growth rate of bubbles takes the form

\[
\frac{dy}{d\tau} = \frac{w(u)}{w(u_0)}.
\]

Let us consider below the method of analytical solution of the dimensionless integro-differential model (9)–(13).

### 3 Analytical Solutions

Applying the Laplace transform with respect to the spatial variable \( y \) to the kinetic equation (9) and the corresponding initial condition (12) and keeping in mind the boundary condition (10), we come to

\[
\frac{d\Phi_s}{d\tau} + \frac{w(u)}{w(u_0)} [s\Phi_s - Y[u(\tau)]] + \Phi_s = 0, \quad \tau > 0; \quad \Phi_s = \Phi_{0s}, \quad \tau = 0,
\]

where \( \Phi_s = \Phi_s(s, \tau) \) and \( s \) is the Laplace transform parameter.

Taking into account expression (8), we arrive at the solution to Equation (14) in the form of

\[
\Phi_s(s, \tau) = \exp \left[ -(s + 1)\tau \right] \left\{ \Phi_{0s} + \int_0^\tau \frac{w[u(\tau_1)]Y[u(\tau_1)]}{w(u_0)} \exp [(s + 1)\tau_1] d\tau_1 \right\}.
\]
Now applying the inverse Laplace transform to (15), we get

\[ \Phi(y, \tau) = \begin{cases} \Phi_0(y - \tau) \exp(-\tau), & y \geq \tau \\ \nu R[u(\tau - y)] / w(u_0) \exp(-y), & y < \tau \end{cases}, \]  

(16)

where \( \Phi_0(0) = \nu R[u(0)] / w(u_0) \). This expression determines the bubble-size distribution function in dimensionless form.

Now combining expressions (11) and (16), we obtain

\[
\Lambda[u(\tau), \tau] = N[u(\tau), \tau] + M(\tau),
\]

\[
N[u(\tau), \tau] = \nu \int_0^\infty \frac{\nu}{w(u_0)} (y + y)^2 R[u(\tau - y)] \exp(-y) \, dy, ~ M(\tau) = \int_0^\infty (y + y)^2 \Phi_0(y - \tau) \exp(-\tau) \, dy.
\]

As is easily seen, the temperature dynamics is defined by the integro-differential equation (11), where its right-hand side is given by the function \( N[u(\tau), \tau] \) and \( M(\tau) \) represents the known dependence.

To approximately evaluate the integral \( N[u(\tau), \tau] \), let us introduce the new variable \( \xi = \tau - y \). In this case, \( N[u(\tau), \tau] \) can be written in the form of

\[
N[u(\tau), \tau] = \nu \frac{\nu}{w(u_0)} \int_0^\tau \tilde{f}(\xi, \tau) \exp\{\kappa S[u(\xi)]\} \, d\xi,
\]

(18)

\[
\tilde{f}(\xi, \tau) = (y + \tau - \xi)^2 \exp[-(\tau - \xi)], \quad S(u) = -\frac{1}{u^2(u + 1)}.
\]

The Laplace-type integral (18) can be evaluated using the saddle-point technique. Namely, the function \( S[u(\xi)] \) attains the maximum value at maximal \( u(\xi) \), that is, at the upper limit of integration \( \xi = \tau \). Taking this into account, let us write out the main contribution of the Laplace-type integral (18), which takes the form

\[
N[u(\tau), \tau] = \nu \frac{\nu^2}{w(u_0)} \frac{\exp\{\kappa S[u(\tau)]\}}{\kappa w(u_0) S'[u(\tau)]},
\]

(19)

where \( \kappa \) is considered large enough and

\[
S'[u(\tau)] = \frac{3u(\tau) + 2}{u^3(u + 1)^2} u'(\tau).
\]

(20)

To calculate \( u'(\tau) \), we use Equation (11), where its right-hand side \( \Lambda \) is given by expression (17). As a result, we have from (19)

\[
N[u(\tau), \tau] \approx \nu \frac{\nu^2}{w(u_0)} \frac{\exp\{\kappa S[u(\tau)]\}}{\kappa w(u_0) [3u(\tau) + 2] \{P[u(\tau), \tau] - hw(u)N[u(\tau), \tau]\}},
\]

(21)

where

\[
P[u(\tau), \tau] = \frac{u_m - u(\tau)}{a} - hw(u)M(\tau).
\]

Now expressing \( N[u(\tau), \tau] \) from (21), we come to the following approximation:

\[
N[u(\tau), \tau] \approx \frac{P[u(\tau), \tau] \pm \sqrt{P^2[u(\tau), \tau] - 4hw(u)Q[u(\tau)]}}{2hw(u)},
\]

(22)
where

\[ Q[u(\tau)] = \frac{\nu y^2 \exp \{kS[u(\tau)]\} u^3(\tau)[u(\tau) + 1]^2}{\kappa w(u_0)[3u(\tau) + 2]} . \]

Now, expression (22) defines \( N[u(\tau), \tau] \), and the right-hand side of temperature equation (11) represents a function of \( u(\tau) \) and \( \tau \). This is meant that the temperature dynamics can be found from the following Cauchy problem:

\[
\frac{du}{d\tau} = \frac{u_m - u}{a} - hw(u)\Lambda[u(\tau), \tau],
\]

\[ u(\tau) = u(0), \quad \tau = 0, \tag{23} \]

where \( \Lambda[u(\tau), \tau] \) and \( M(\tau) \) are defined by expressions (17), and \( N[u(\tau), \tau] \) is given by (22).

Determining the temperature from (23), we completely know the bubble-size distribution function (16) at \( y < \tau \). Thus, the approximate analytical solution of the integro-differential problem on intense boiling is described by solutions (16), (17), (22), and (23).

To illustrate the analytical solutions obtained, let us introduce the normal initial distribution function

\[ f_0(r) = \frac{A}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{1}{2} \left( \frac{r - \mu_f}{\sigma} \right)^2 \right], \tag{24} \]

where \( \mu \) stands for the mean of the distribution, \( \sigma \) represents the standard deviation, and \( A \) is a constant. Rewriting (24) in dimensionless form, we get

\[ \Phi_0(y) = B \exp \left\{ -\frac{1}{2} \chi (y + y_\ast - \mu_\ast) \right\}, \tag{25} \]

where

\[ B = \frac{l_0^4 A}{\sqrt{2\pi}\sigma}, \quad \chi = \frac{l_0 w(u_0)}{\sigma}, \quad \mu_\ast = \frac{y\mu_f}{w(u_0)\mu}. \]

The initial distribution function (25) at different values of \( y_\ast \) is shown in Figure 1. As is easily seen, the smaller the \( y_\ast \), the larger the initial distribution function, since bubbles are easier to be born.

Figure 2 demonstrates the solution of Cauchy’s problem (23) where \( N[u(\tau), \tau] \) is taken from (22). The temperature increases with time from its initial value \( u(0) = 5 \) up to the maximal dimensionless temperature \( u_m = 8 \). The other dimensionless parameters are estimated as follows: \( k = 3, h = 0.01, n = 1, B = 1, a = 1, \) and

\[ \frac{\nu}{w(u_0)} = \Phi_0(0) \exp \left\{ \frac{k}{u^2(0)[u(0) + 1]} \right\}. \]
Figure 3 shows the evolutionary behavior of the bubble-size distribution function (16). At all times, this function decreases with the increase of the bubble size. However, the form of this dependence changes with time. This is due to the competition of two processes: the nucleation of new bubbles and the enlargement of the existing ones. Namely, the number of small bubbles increases due to their inflow (temperature rise) and the number of large bubbles increases due to the enlargement of existing bubbles.

4 | CONCLUSION

In summary, in the present study, a system of integro-differential equations is formulated and solved, which describes the process of intense boiling of a liquid. The kinetic equation for the bubble-size distribution function takes into account the process of bubble withdrawal. As this takes place, the rate of bubble appearance is used accordingly to the Dering–Volmer and Frenkel–Zeldovich–Kagan nucleation theories. The subsequent growth of nucleated bubbles is considered as an arbitrary function of the relative superheating of the liquid. An important circumstance is a fact that the distribution function and temperature of the fluid are found using the saddle point method to calculate the Laplace-type integral. In this case, for simplicity of presentation of the theory and demonstration of the main idea of the proposed method, we limited ourselves only to the main term in the expansion (see expressions 18 and 19). Note that to improve the theory being developed, one can easily take into account terms of a higher order of smallness by analogy with the previously developed theory for the crystallization of supercooled liquids.38,39

The considered model of the boiling process of a superheated liquid can be generalized in future studies to take into account the “diffusion” of the distribution function in the space of bubble sizes,40 the dependence of the growth rate of bubbles on their size, a more general law for the rate of bubble nucleation, the possible presence of an impurity dissolved...
in the liquid, and the other processes and phenomena. Such generalizations can be made by analogy with the theories of crystallization and dissolution in metastable liquids.41–50

ACKNOWLEDGEMENTS
This study is divided into two parts, theoretical and numerical. The theoretical part is supported by the Russian Foundation for Basic Research (project no. 20-08-00199). The numerical part was made possible due to the support from the Ministry of Science and Higher Education of the Russian Federation (project no. FEUZ-2020-0057).

CONFLICT OF INTEREST
This work does not have any conflicts of interest.

AUTHOR CONTRIBUTION
The authors contributed equally to the present research article.

ORCID
Irina V. Alexandrova https://orcid.org/0000-0002-9606-4759
Alexander A. Ivanov https://orcid.org/0000-0002-2490-160X
Dmitri V. Alexandrov https://orcid.org/0000-0002-6628-745X

REFERENCES
18. Alexandrov DV, Malygin AP. Coupled convective and morphological instability of the inner core boundary of the earth. Phys Earth Planet Inter. 2011;189:134-141.

How to cite this article: Alexandrova IV, Ivanov AA, Alexandrov DV. Analytical solution of integro-differential equations describing the process of intense boiling of a superheated liquid. Math Meth Appl Sci. 2022;45(13):7954-7961. https://doi.org/10.1002/mma.7560