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Noise-induced behavioral change driven by transient chaos

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ABSTRACT

We study behavioral change in the context of a *stochastic, non-linear consumption model* with preference adjusting, interdependent agents. Changes in long-run consumption behavior are modelled as noise induced transitions between coexisting attractors. A particular case of multistability is considered: two fixed points, whose immediate basins have smooth boundaries, coexist with a periodic attractor, with a fractal immediate basin boundary. If a trajectory leaves an immediate basin, it enters a set of complexly intertwined basins for which final state uncertainty prevails. The standard approach to predicting transition events rooted in the stochastic sensitivity function technique due to Mil'shtein and Ryashko (1995) does not apply since the required exponentially stable attractor, for which a confidence region could be constructed, does not exist. To solve the prediction problem we propose a heuristic based on the idea that a vague manifestation of a non-attracting chaotic set (chaotic repellor) - could serve as a surrogate for an attractor. A representation of the surrogate is generated via an algorithm for generating the boundary of an absorbing area due to Mira et al. (1996). Then a confidence domain for the surrogate is generated using the approach due to Bashkirtseva and Ryashko (2019). The intersections between this confidence region and the immediate basins of the coexisting attractors can then be used to make predictions about transition events. Preliminary assessments show that the heuristic indeed explains the transition probabilities observed in numerical experiments.

1. Introduction

We study behavioral change modelled as a transition between coexisting attractors in the context of a *stochastic, non-linear consumption model* with interacting and preference adjusting agents. The notion that consumer behavior evolves in time and the perception of consumption as a social activity can be traced back to the seminal contributions of social scientists as [13,22,33,45,56].

These intriguing theoretical and empirical contributions motivated economists to embrace the idea of interdependence among consumers. Examples from the resulting strand of the consumption literature are [1,3,12,14,17,18,34,38,39,52,57]. These research efforts tend to focus on the existence of equilibria and their characterization.

A second strand of the literature focuses on the time evolution of consumption pattern. Examples include the early habit formation work due to [15,32,51], as well as the efforts by [26,37,47,50,58]. Based on [19] in which a general theory of experience dependent choice was developed, [11] present a deterministic non-linear model of endogenous preference change in discrete time. In subsequent modelling efforts [27–30] analyze a non-linear model of interdependent consumer choice with endogenous preference change. As subsequent modelling efforts, for example, due to [9,21,23,25] show that complex consumption dynamics may arise as a consequence of rational choice behavior. [16] give conditions under which chaos occurs in a cash-in-advance economy and [53] give conditions under which consumers will exhibit a preference for variety.

Our current work on the stochastic consumption model draws in various ways on existing research that has revealed a multitude of noise induced phenomena. Given the scope of this paper, an account of a few recent examples from this active research area has to suffice. Noise-induced escapes from attractors have been analyzed by [20]. [31] introduce the concept of "noise-induced point-overspreading route to noisy chaos" in stochastically perturbed quasi-Hamiltonian systems. The effect of noise on closed curve attractors has been studied among others by [36,46]. The authors rely on variants of the stochastic sensitivity analysis proposed by [48]. This technique and its spin-offs have been implemented successfully in various research contexts by [5–8,10]. More recent examples include [59] who discuss noise induced extinction in a model of bacterial infection and [55] who study the effect of noise on attractors in a dynamic model of the cardiac action potential. The same approach is utilized in [54] to study noise-induced transitions in the context of coupled chaotic oscillators. Moreover, the stochastic sensitivity function (SSF) methodology has been used to study the

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sensitivity of attractors in economic and financial models formulated in discrete time in [40,42].

In previous work involving a stochastic version of [28] we paid special attention to scenarios in which the influence of other consumers' past consumption plays a significant role in the preference adjustment process of an individual. It turns out, that in the region of the parameter space, which corresponds to this case, we find plenty of evidence for *bi*-stability as well as for multistability. Thus, for a given set of model parameters several long-run consumption behaviors coexist. In the presence of shocks, transitions between the coexisting attractors might occur leading to intricate dynamics of the consumption process.

In particular, we studied cases in which two attractors such as fixed points, *k*-cycles, or closed invariant curves coexisted. In these *bi*-stability scenarios the corresponding immediate basins had smooth boundaries [41–43]. Relying on the indirect approach to the analysis of a stochastic dynamic system, we describe the potentially existing transition scenarios, and identify conditions in terms of behavioral and environmental parameters under which such transitions are likely to occur.

The cases of *bi*-stability we studied so far, exhibited the following structure. Consider an attractor Γ with an immediate basin $\mathcal{B}(\Gamma)$ that has a smooth boundary $\partial\mathcal{B}(\Gamma)$. A transition from Γ occurs when the stochastic trajectory escapes $\mathcal{B}(\Gamma)$ to move into another immediate basin, i.e. into a set that has the same properties as the one it escaped from. This latter aspect is not given in the case of multistability we scrutinize in this paper.

In the current effort we study a scenario in which two fixed points coexist with a periodic attractor. While the immediate basins of the fixed points possess *smooth* boundaries, the basin boundary of the periodic attractor is of a fractal nature. Whenever a trajectory escapes an immediate basin it enters a set in state space that is formed by the complexly intertwined basins of the three coexisting attractors. Once a trajectory enters this set *transient chaos* [44] occurs. One typically observes a long complex temporal evolution of consumption states before the process eventually converges to one of the fixed points or the periodic attractor. The situation is characterized by final (asymptotic) state uncertainty. In this case, we cannot apply the strategies outlined above to predict which of the coexisting attractors will be reached.

To overcome this problem we propose an unorthodox procedure. First, we identify a geometric object in state space which is traversed by trajectories before they converge to one of the coexisting attractors. Noting that features of this object could be informative with respect to the nature of the transition process, we represent its shape (Gestalt) using the concept of critical lines following [49]. Next, using the technique due to [4] we generate a confidence region for this object. Focussing on the intersections of the confidence region with the immediate basins of the coexisting attractors one can predict the outcome of the transition process. Both of the techniques we instrumentalize have been designed to work on attractors. But despite the fact that the object of interest is not an attractor - it is in fact associated with a non-attracting chaotic set - our heuristic can be used to predict behavioral transitions in the context of coexisting long-run consumption behaviors.

The stochastic consumption model is reiterated in Section 2. Section 3 contains an in-depth discussion of the multistability scenario of interest and a motivation for the heuristic that is introduced in Section 4. A basic assessment of operational characteristics of the heuristic is given in Section 4 and a short conclusion formulated in Section 6 finalizes the paper.

2. The stochastic consumption model

The ideas and arguments presented in the sequel of the paper concern modelling transition phenomena in non-linear stochastic systems characterized by multistability. Our discussion involves a particular 2D non-invertible map arising naturally in the dynamic theory of household consumption. It originates from a modelling effort by [28,29] that combines the concepts of endogenous preference adjustment, learning, and social interaction between consumers. The stochastic variant of the system captures economic as well as the social dimension of consumption in noisy environments.

The model considers two myopic utility-maximizing individuals, indexed by $i = 1, 2$, who consume amounts of two non-storable commodities x and y. At every time period, each individual equipped with an idiosyncratic preference order (modelled by a Cobb-Douglas type utility function) is endowed with a fixed exogenous income b_i and faces a time-invariant price system $p = (p_x, p_y)$. Embedded in such a stable economic environment, each individual chooses to consume (x_{it}, y_{it}) units of the respective commodities given the preferences held at time t. Between two consumption decisions, individuals adjust their preference in response to its own past consumption experience (learning) as well as to the past consumption of the other individual. While the original stochastic variant of the model encompasses different types of exogenous shocks, we focus solely on the case of *additive* noise in this exposition.

The demand for commodity x at time t, $x_t = (x_{1t}, x_{2t})^T$, evolves according to the non-linear stochastic difference equation[1]

$$x_{t+1} = f(x_t) + \varepsilon\xi_t, \tag{1}$$

where f represents the 2D noninvertible map $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$

$$f(x) = \begin{pmatrix} \dfrac{b_1}{p_x p_y}(\alpha_1 x_1 (b_1 - p_x x_1) + D_{12} x_2 (b_2 - p_x x_2)) \\ \dfrac{b_2}{p_x p_y}(\alpha_2 x_2 (b_2 - p_x x_2) + D_{21} x_1 (b_1 - p_x x_1)) \end{pmatrix}. \tag{2}$$

The real, strictly positive, parameters α_1, α_2 and D_{12}, D_{21} are referred to as *learning* and *influence* parameters respectively and the scalar $\varepsilon \geq 0$ functions as a noise intensity. The additive shocks $\xi_t = (\xi_{1t}, \xi_{2t})^T$ are assumed to follow a *bi*-variate Gaussian white-noise process $\xi_t \sim MVN_2$ $(0, I_{(2,2)})$.

The feasible region for the deterministic skeleton ($\varepsilon = 0$) is given in the following lemma.

Lemma 1. *If* $\alpha_1 b_1^2 + D_{12} b_2^2 < 4 p_x p_y, \quad \alpha_2 b_2^2 + D_{21} b_1^2 < 4 p_x p_y$ *holds, then* $f(S) \subset S$ *where*

$$S = \left(0, \frac{b_1}{p_x}\right) \times \left(0, \frac{b_2}{p_x}\right) \tag{3}$$

is the feasible phase region.

The *fixed economic environment* of prices and incomes is specified as $p = \left(p_x, p_y\right) = \left(\frac{1}{4}, 1\right)$, $b = (b_1, b_2) = (10, 20)$ assigning the role of the high income consumer to individual 2.

Despite this difference, the preference adjustment processes for the individuals are qualitatively similar: for both, the main driver of preference adjustment is the past consumption of the other individual. The effect of the individuals' own past consumption is relatively weak. We capture such a scenario by assigning the values $\alpha_1 = 0.0002$, $\alpha_2 = 0.00052$ to the learning parameters. According to Lemma 1 this choice allows us to vary the influence parameters over the set

$$D^e = \{(D_{12}, D_{21}) | 0 \leq D_{12} \leq 0.00245 \wedge 0 \leq D_{21} \leq 0.00792\}. \tag{4}$$

The parameter D_{21} will be fixed at 0.0073 and D_{12} will take the role of a bifurcation parameter throughout the remainder of the paper.

[1] In line with standard assumptions of household theory, we assume that an individual's consumption expenditure for units of the two commodities exhausts its income. Thus it suffices to describe the consumption dynamics only in terms of one commodity. Here, we consider commodity x.

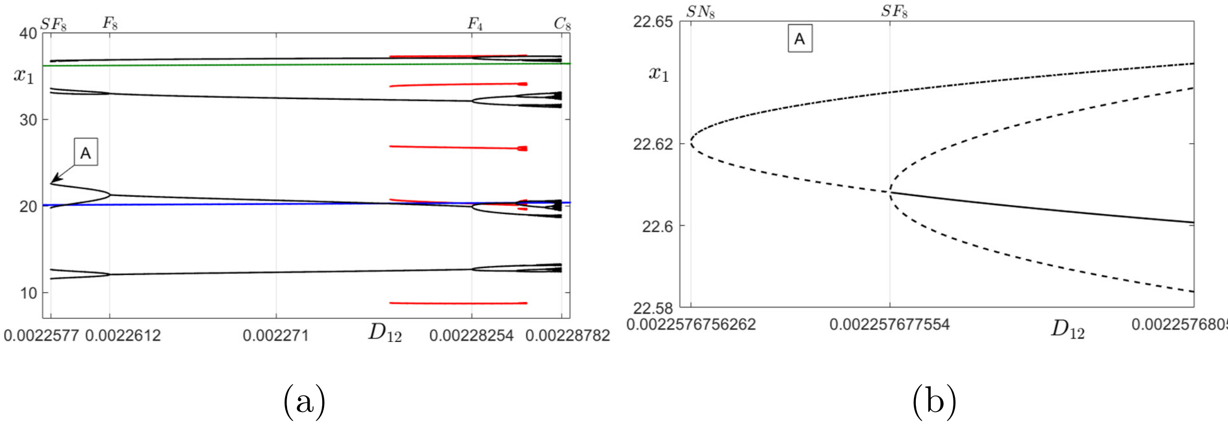

Fig. 1. (a) 1D-bifurcation diagram for $D_{12} \in (0.0022577, 0.00228782)$ and (b) Region A magnified.

3. Multistability

Multistability defined as the coexistence of several attractors for specific values of the bifurcation parameters has been established for various types of deterministic and stochastic systems. Early evidence for the use of the concept can be found in [2] in the context of perception. Deterministic multistable systems can exhibit complex dynamic behavior due to intricate interaction among the attractors [24]. In the case of stochastic systems noise induced escapes from attractors can lead to intricate transitions between attractors. Our consumption model (Eq. (1)) can be viewed as a system of coupled logistic functions with peculiar coupling terms. There is evidence in the literature that for such a system the phenomenon of multistability can be expected to exists.

In the course of previous analyses we have singled out areas of the parameter space for which the deterministic skeleton of the stochastic consumption system (Eq. (1)) possess possibly complex coexisting attractors [41–43]. These past research efforts revealed (i) conditions in terms of social influence and noise under which transitions between attractors, i.e. alternative long-run behaviors, occur and (ii) factors which determine the nature (dynamic/statistical properties) of the transition process. Moreover, focussing on situations of *bi*-stability (two attractors coexisting) our efforts demonstrated the versatility of concepts and tools of sensitivity analysis of attractors for the study of transitions.

Scrutinizing the 1D-bifurcation diagram shown in Fig. 1a one readily concludes that the sole focus on *bi*-stability should be transcended. For values of the influence parameter D_{12} in the parameter window $(0.0022577, 0.00228782)$ the plot establishes the coexistence of up to four attractors. For $D_{12} \in (0.0022577, 0.0022612)$ two fixed points x_b (blue) and x_g (green) coexist with an 8-cycle. Increasing D_{12} beyond 0.0022612, leaves us again with three coexisting attractors as x_b and x_g coexist with a 4-cycle C_4. Once the influence parameter exceeds $D_{12} = 0.0022777$, four attractors coexist. The two fixed points coexist with C_4 and a 5-cycle colored in red. To exemplify the challenges one is facing when trying to analyze and model transitions between more than two coexisting attractors, we turn to the scenario prevailing at $D_{12} = 0.002271$ in which two fixed points coexist with the periodic attractor C_4.

To facilitate our argument, it is constructive to adopt a state space perspective of the situation prevailing at $D_{12} = 0.002271$. Fig. 2a shows the three coexisting attractors: the fixed points x_b (●) and x_g (●) as well as the periodic attractor C_4 whose periodic points are indicated by the symbol ★. The basins of the fixed points x_b and x_g are colored in light blue and light green respectively. In the sequel, the corresponding immediate basins are denoted as $\mathcal{B}(x_b)$ and $\mathcal{B}(x_g)$. Finally, we use the color yellow to indicate the basin of the periodic attractor. Since at the given resolution the immediate basin $\mathcal{B}(C_4)$ is

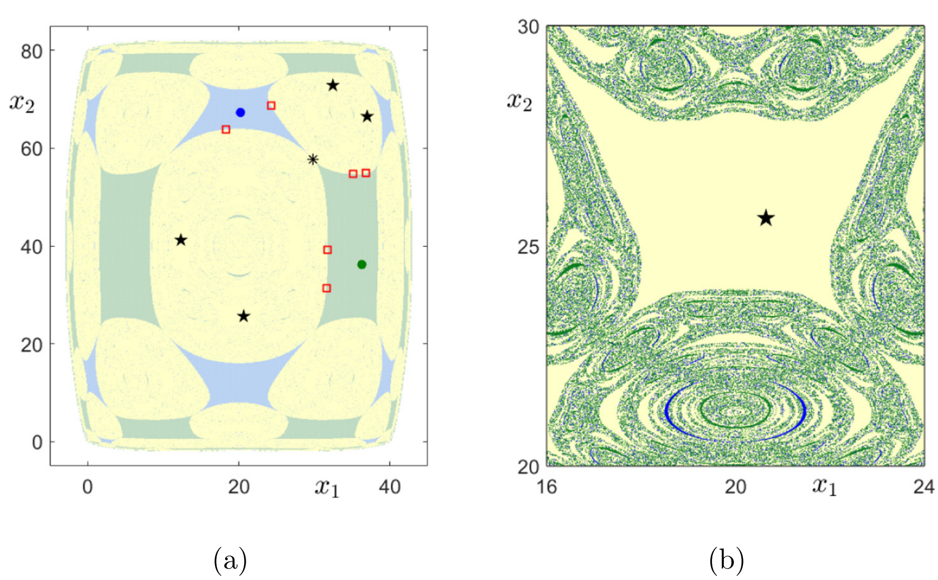

Fig. 2. For $D_{12} = 0.002271$ and $\varepsilon = 0$ (a) Coexisting attractors x_b (●), x_g (●), and C_4 (★), periodic points of saddle (□) and (b) immediate basin of a periodic point.

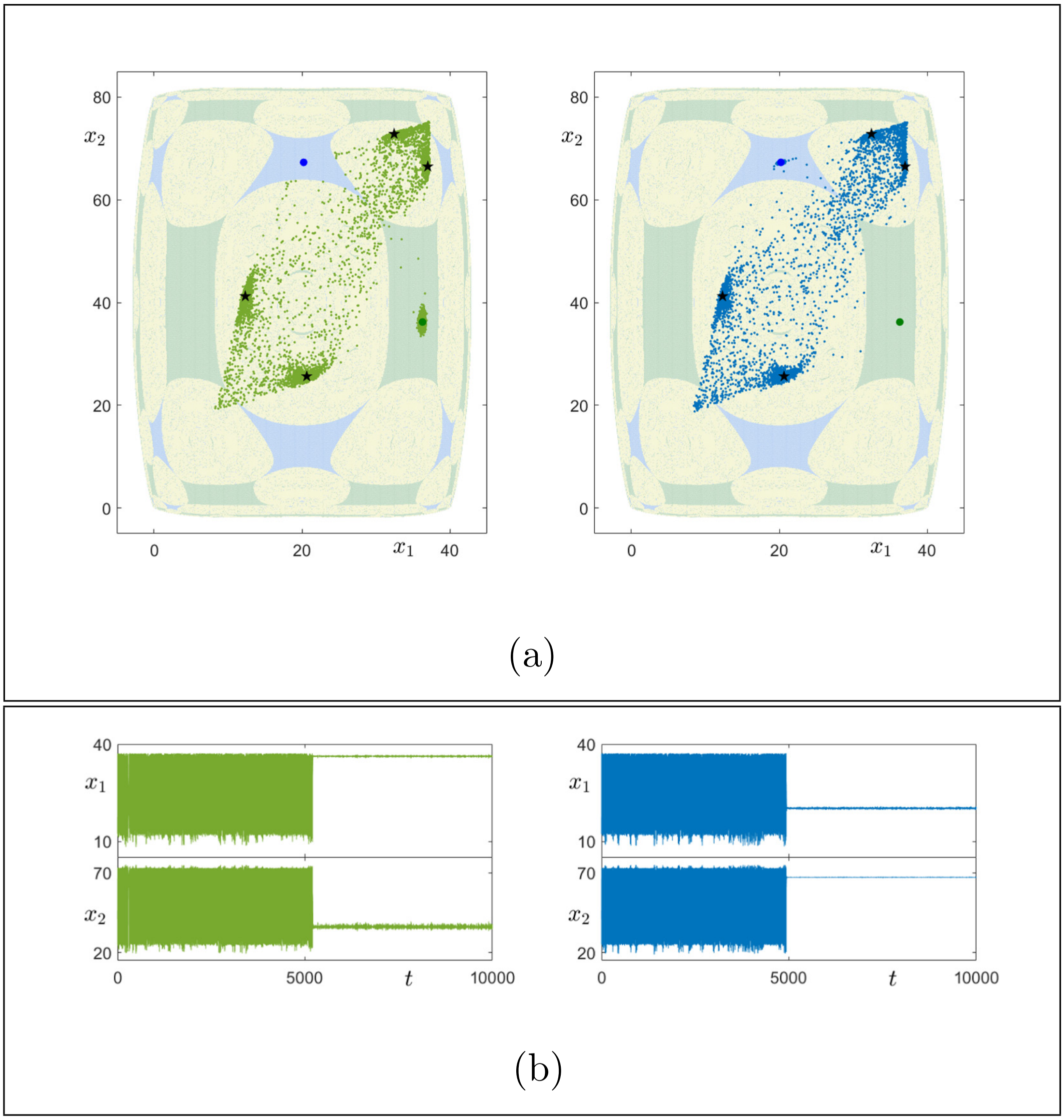

Fig. 3. Noise-induced transition from C_4 to x_g and to x_b (a) shown in state space and (b) in the time domain at $D_{12} = 0.002271$ with $\varepsilon = 0.1$.

not visible, the immediate open neighborhood surrounding one of the periodic points is shown in Fig. 2b.

The stable fixed point x_b is born in the course of a transcritical bifurcation. Since this bifurcation occurs at a value of $D_{12} < 0.002$ the event is not visible in Fig. 2a. The smooth boundary of its immediate basin $\partial\mathcal{B}(x_b)$ is generated by the stable manifold of the 2-saddle cycle whose periodic points are indicated by the symbol $\square$ in Fig. 1 a. Initially unstable, the fixed point x_g emerges as a stable fixed point from a subcritical-flip bifurcation at $D_{12} = 0.00215315$ (which lies outside the window shown in Fig. 1a). Also in this case, the stable manifold of the 4-saddle cycle ($\square$) creates the smooth boundary of its immediate basin $\partial\mathcal{B}(x_b)$. Finally, it should be noted that the boundaries $\partial\mathcal{B}(x_b)$ and $\partial\mathcal{B}(x_g)$ "touch" in a repelling fixed point represented by the symbol *.

As an inspection of Fig. 1a shows, an 8-cycle is born in a subcritical-flip bifurcation in a small neighborhood of $D_{12} = 0.0022577$. Subsequently "period halving" occurs as D_{12} assumes the value 0.0022612, leading to the periodic attractor C_4. Apparently, the way in which the boundary $\partial\mathcal{B}(C_4)$ corresponding to the open neighborhoods of the periodic points emerges, differs from the way in which the smooth boundaries $\partial\mathcal{B}(x_b)$ and $\partial\mathcal{B}(x_b)$ materialize. The open sets surrounding the periodic points of C_4 border on a set in which the basins of the three attractors x_b, x_g and C_4 are complexly interwoven. This apparently fractal set extends over a significant part of the state space. Evidently, the boundary $\partial\mathcal{B}(C_4)$ has the characteristics of a fractal. Thus we observe a case in which the boundaries of immediate basins of attraction belonging to different coexistent attractors do not have the same structure.

This observation has important ramifications for the stochastic dynamics ($\varepsilon > 0$). Whenever a trajectory escapes an immediate basin, it enters a set that is characterized by *final state sensitivity*. If one chooses an initial state x_0 in a subset of the state space possessing this property – initially formulated by [35] – then the associated trajectory will converge to one of the coexisting attractors (here x_b, x_g, or C_4). Iterating from an initial condition in an arbitrarily small neighborhood of x_0, the trajectory might reach another attractor.[2] Suppose, in our scenario, the process escapes one of the open immediate neighborhoods around a periodic point of C_4. Then we have to consider the following three elementary events: the stochastic trajectory approaches x_b (ω_b), or x_g (ω_g) or C_4 (ω_C) asymptotically. Details of two simulation runs originating on C_4 are shown in Fig. 3. In each case, the color of the

[2] [35] also proposed a procedure to measure the degree of sensitivity (uncertainty coefficient α). Finding alternative operationalizations of the concepts is still a matter of ongoing research.

trajectory corresponds to the color of the fixed point x_b or x_g it eventually approaches.

In each subfigure of panel (a), we observe the emergence of a cloud of points (consumption states) exhibiting a peculiar shape. The trajectory traverses this "object" before it enters either $\mathcal{B}(x_b)$ or $\mathcal{B}(x_g)$. The respective - temporarily chaotic - time series shown in Fig. 3b, suggests that these specific realizations of the stochastic process do not differ significantly with respect to the time spend moving on the object. It takes approximately 5000 iterations before convergence to x_g or x_b occurs.

The observed phenomenon appears to be consistent with the concept of *transient chaos* which is linked to the existence of a *non-attracting chaotic set* in phase space of the deterministic skeleton.[3] That is, a *chaotic repellor* – recall that the map (Eq. (2)) is non-invertible – is embedded in the basin. Such a chaotic set is an invariant set with Lebesgue measure 0. (So the probability for the event "a randomly chosen initial condition lies on the set (chaotic repellor)" equals 0.) But if the initial condition lies close to the set, it will stay in the vicinity of the set (repellor) for a long but finite time, before it eventually converges to some asymptotic state (x_g or x_b in our case). Hence, non-attracting chaotic sets cannot be observed directly. Since the trajectory (at best) traverses a neighborhood of the set, the objects shown in Fig. 2 will reflect a small neighborhood of the chaotic repellor. In the case at hand we conjecture that the chaotic repellor is related to a chaotic attractor that disappeared in a boundary crises at D_{12} = 0.00217563. Since the repellor appears to us only in vague, approximate form, we refer to the object visible in Fig. 3 as a *ghost* in the remainder of the paper.

Even though we do not particularly focus on more intricate aspects of time evolution in the sequel, two remarks seem to be in order. First of all, in both cases (Fig. 3a) the event "convergence to a fixed point occurs within 10000 iterations" materializes. For the two cases considered, the waiting times until the process enters an immediate basin are roughly identical. But, of course, this does not have to be the case. Even though the distribution of the waiting time is of eminent interest, it is not essential for the point we are trying to make. Thus, to maintain the focus of the paper we do not discuss the issue further. Second, the longer the stochastic process traverses the object, the clearer the emergence of its contours. It is manifested under the iteration of the stochastic system (Eq. (1)). Moreover, visual inspection suggests that a major part of the object is embedded in the set of complexly intertwined basins, while it also overlaps with the immediate basins $\mathcal{B}(x_b)$ and $\mathcal{B}(x_g)$.

Fig. 3 only shows the outcomes of two simulation runs. Of course, we could rely on a simulation strategy to estimate the probabilities of the elementary events ω_b, ω_g and ω_C given above. Estimates of transition probabilities for various levels of the noise intensity based on such an approach are given in Fig. 4. For very small noise intensities a trajectory starting on C_4 will stay close to the periodic attractor. As ε grows, the probabilities for reaching a fixed point increase. Once the trajectory escapes $\mathcal{B}(C_4)$, it is more probable to convergence to x_g than to x_b for all levels of the noise intensity considered. In particular, at $\varepsilon = 0.1(0.44)$ in 20% (10%) of all cases the trajectory reaches x_b, while the x_g is reached in about 80% (90%) of the simulation runs.

In [41–43] a strategy to analyze and describe transitions between attractors in the case of *bi*-stability has been proposed and implemented. In those cases, the attractors considered were born in the course of saddle node- or Neimark–Sacker bifurcations and the boundaries of immediate basins were smooth. Under such circumstances the intersections of the confidence domain of one attractor – derived on the basis of the sensitivity function for the attractor – with the immediate basin of the other attractor would give a lead to nature (direction) and the likelihood of transitions at given levels of the noise intensity.

[3] Numerical experiments showed that traces of the structure also become visible if we choose an initial condition for the deterministic process in the set of intertwined basin boundaries.

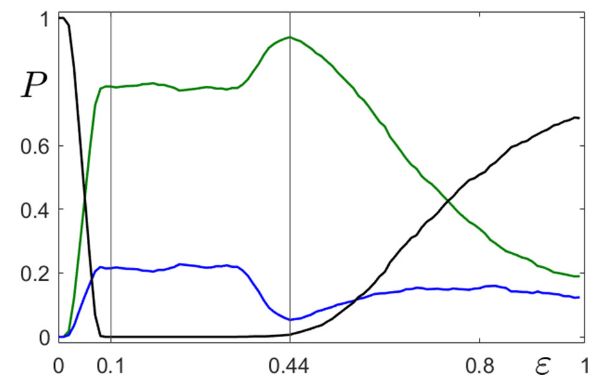

Fig. 4. Estimated transition probabilities $\widehat{p}_b(\varepsilon)$ (blue), $\widehat{p}_g(\varepsilon)$ (green), and $\widehat{p}_c(\varepsilon)$ (black) versus $\varepsilon \in [0,1]$ for $D_{12} = 0.002271$.

In the case of three coexisting attractors with basins that become complexly intertwined, this strategy could only be useful to predict the level of noise at which escapes from the immediate basins of x_b and x_g and C_4 become likely events. Once the trajectory enters the set of intertwined basin boundaries, we cannot apply the strategy outlined above to predict which of the attractors will be reached. There is no exponentially stable attractor in sight that is required for a standard sensitivity analysis to work. One simply lacks the basis for which to build a confidence domain. Under this orthodox view, there seems to be no way to explain the estimates of transition probabilities displayed in Fig. 4. But as it will be argued below, there exists an unorthodox approach that facilitates the analysis of transition between coexisting attractor in the situation portrayed above.

4. A heuristic procedure

The evidence shown in Fig. 3 demonstrates that once it has escaped from $\mathcal{B}(C_4)$, under the evolution of the stochastic trajectory (transients) a ghost structure appears that shows a vague resemblance to a non-attracting chaotic set embedded in the state space. This ghost is definitely not an attractor in the conventional sense. On the basis of Fig. 3 one could speculate that the appearance (gestalt) of the ghost is related to the convergence of the process to the fixed point x_b and x_g. We see that in both cases the structure comes very close to (possibly overlaps with) the immediate basins $\mathcal{B}(x_b)$ and $\mathcal{B}(x_g)$.

Next, we will demonstrate that it is possible to describe or represent the ghost by determining the *minimal absorbing area* based on critical lines, i.e. by a technique usually applied in the context of chaotic attractors. The resulting representation of the ghost can be combined with a confidence region for the ghost (based on the sensitivity function for a chaotic attractor) to obtain precise information concerning the intersections between the ghost and the immediate basins of x_b and x_g at any given level of noise ε. The extend of the intersection between the *ghost domain* and $\mathcal{B}(x_b)$ and $\mathcal{B}(x_g)$ will enable us to predict which type of transitions occur at given levels noise and how likely such transition events are. That is, we will be able to explain the outcome of the simulations shown in Fig. 4.

Letting $\varepsilon = 0$, the map f in Eq. (1), $(x_1', x_2') = f(x_1, x_2)$ is non-invertible since the preimage of rank 1 $(x_1, x_2) = f(x_1', x_2')$ is not unique. The critical curve of rank 1 referred to as LC is the geometrical locus of points that have two or more coinciding rank 1 preimages that are located on the curve of merging preimages LC_{-1}. Suppose both component functions of the endomorphism f are continuous and differentiable, then the curve of merging preimages is given by $LC_{-1} = \{(x_1,x_2) \| J(x_1,x_2)| = 0\}$ where J denotes the Jacobian of the map f. For example, in our case, the Jacobian is given by

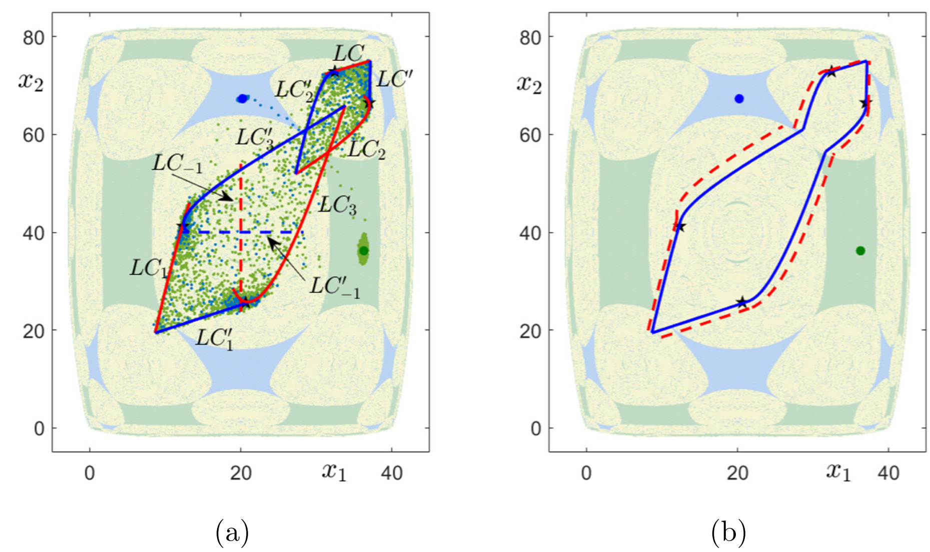

Fig. 5. At $D_{12} = 0.002271$ (a) pseudo "minimal absorbing area" for the ghost and (b) the confidence domain for the ghost $\mathcal{C}(G, 0.1, 0.9972)$.

$$J = \begin{pmatrix} \alpha_1\left(\frac{b_1^2}{p_x p_y} - 2\frac{b_1}{p_y}x_1\right) & D_{12}\left(\frac{b_1 b_2}{p_x p_y} - 2\frac{b_1}{p_y}x_2\right) \\ D_{21}\left(\frac{b_1 b_2}{p_x p_y} - 2\frac{b_2}{p_y}x_1\right) & \alpha_2\left(\frac{b_2^2}{p_x p_y} - 2\frac{b_2}{p_y}x_2\right) \end{pmatrix} \tag{5}$$

and therefore

$$LC_{-1} = \left\{(x_1, x_2) | \left(x_1 = \frac{1}{2}\frac{b_1}{p_x} \wedge 0 \le x_2 \le \frac{b_1}{p_x}\right) \vee \left(0 \le x_1 \le \frac{b_1}{p_x} \wedge x_2 = \frac{1}{2}\frac{b_2}{p_x}\right)\right\}. \tag{6}$$

In any neighborhood of a point $(x_1, x_2) \in LC_{-1}$ there are at least two *distinct* points that are mapped into the same point. Thus f is not locally invertible in the points lying on LC_{-1}. Apparently, the critical set of rank 1 is the image of rank 1 of LC_{-1}, i.e. $LC = F(LC_{-1})$. The critical sets of rank r are the images of rank r of LC_{-1} that is $LC_{r-1} = F^r(LC_{-1}) = F^{r-1}(LC)$ for $r = 1, 2, \ldots$ and $LC_0 = LC$. An absorbing area A is an area in state space whose boundary is constituted by segments of critical curves (segments of critical curve LC and its images) such that there exists a neighborhood $U \supset A$ whose points enter in A and never leave under the action of the map f, i.e. $f(A) \subseteq A$.

To generate the boundary of an absorbing area A in practice, one implements an algorithm that has been suggested in the [49]. A stylized version of the procedure, reflecting only the basic elements of the approach, is given below.

[step 1] Choose a subset (part, piece, segment) of LC_{-1} lying in the area of the state space that is of interest; call it LC_{-1}^{int}.[4]

[step 2] For $r = 1,2,3,\ldots$, generate images of increasing rank of LC_{-1}^{int}, $LC_{r-1} \leftarrow F^r(LC_{-1}^{i})$ until a closed region A appears.

[step 3] Initialize $LC_{-1}^{int} \leftarrow A \cap LC_{-1}$. If it can be established that the union of the iterates of LC_{-1}^{i} covers the entire boundary of A, then A is invariant.

Faced with the structure evolving in state space shown in Fig. 3, steps 1 and 2 of the procedure were implemented. Since it had been observed that the trajectory would eventually converge to one of the attractors x_b or x_g, the area occupied by the ghost could not be an invariant set. Therefore, the 3rd step of the procedure would clearly be irrelevant. Nonetheless, performing the first two steps, the respective images of increasing ranks of LC_{-1}^{int} describe a closed region indicated in Fig. 5 which approximates the area containing the ghost very well.

Despite the fact that the procedure we used so far was developed in the context of deterministic dynamics, it successfully describes the ghost revealed through a trajectory generated by the stochastic system (Eq. (1)) with $\varepsilon = 0.1$. But there should be a potential for improving the representation even further. This potential can be exploited by considering the confidence domain associated with the region we identified. In the case at hand, we exploit this potential by using a technique for generating confidence domains for a strange attractor due to [4]. The authors show how - based on the stochastic sensitivity technique, the concept of Gaussian approximation and the notion of an absorbing area - one can deduce a *confidence set* containing a chaotic attractor. For a given noise intensity ε the stochastic states will lie in this subset of the state space with probability π. In the sequel such confidence sets will be denoted as $\mathcal{C}(A, \varepsilon, \pi)$.

We apply (misuse) their technique to generate a confidence region $\mathcal{C}(G, \varepsilon, \pi)$ for the ghost which is, of course, not an attractor. The resulting 99.72% confidence region $\mathcal{C}(G, 0.1, 0.9972)$ has been superimposed on the region G (ghost) in Fig. 5b (indicated by red broken lines). The visual inspection of Fig. 5b suggest the following relation

$$\mu\{\mathcal{C}(G, 0.1, 0.9972) \cap \mathcal{B}(x_g)\} > \mu\{\mathcal{C}(G, 0.1, 0.9972) \cap \mathcal{B}(x_b)\}. \tag{7}$$

where $\mu\{\}$ measures the size of the respective sets. Thus we find, that the probability of observing a state that belongs to G and to the immediate basin of a fixed point is higher in the case of x_g than for x_b. Based on the procedure outlined above, we predict that under a moderate noise intensity ($\varepsilon = 0.1$), a transition from the long-run periodic relatively volatile behavior (C_4) to a less volatile and simpler long-run behavior in which both individuals are consuming equal amounts of the commodity x is more likely than the transition to the stochastic fixed point in which individual 2 consumes much more of x than individual 1. This prediction is in line with the evidence exhibited in Fig. 4.

5. Basic assessment of the heuristic

According to Fig. 4 the levels $\hat{p}_b(\varepsilon), \hat{p}_g(\varepsilon)$ are fairly stable over noise intensities between 0.1 and 0.4. Once ε exceeds that threshold,

[4] In practice, the initial segment of LC_{-1} typically emerges via a trial-and-error approach.

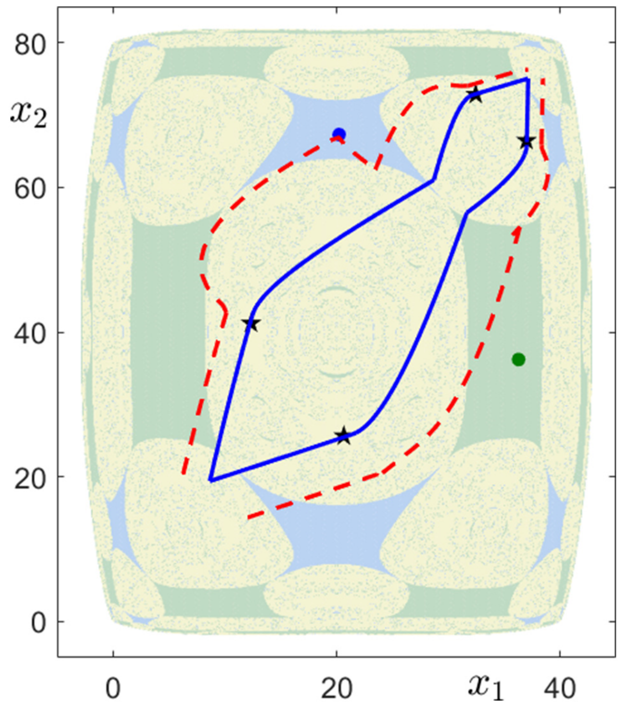

Fig. 6. The confidence region for the ghost $\mathcal{C}(G, 0.44, 0.9972)$ at $D_{12} = 0.002271$ resulting from an increase in the noise intensity to $\varepsilon = 0.44$.

convergence to x_g (x_b) becomes more (less) likely and $\widehat{p}_g(\varepsilon)$ reaches its maximum (minimum) at 0.44.

One way to assess the performance of our heuristic is to check whether it is able to capture this extreme situation. The confidence region $\mathcal{C}(G, 0.44, 0.09972)$ has been calculated and superimposed on the state space. Comparing the resulting Fig. 6 with Fig. 5b, we find that the area of the intersection between the confidence region for the ghost and the immediate basin of x_g has increased considerably relative to the baseline (i.e. the respective area in Fig. 5b). Moreover, it is considerably larger than the intersection corresponding to x_b. On the basis of Fig. 6 one could still arrive at a reasonable prediction as to which type of behavioral transition one could expect given an escape from C_4 has occurred.

On the one hand the comparison between the scenarios depicted in Figs. 6 and 5b attests a certain degree of validity to our heuristic. But, of course, a more rigid assessment strategy is called for. On the other hand, the appearance of the confidence region $\mathcal{C}(G, 0.44, 0.9972)$ in Fig. 6 provokes thoughts about the limits of our approach.

6. Conclusion

Studying behavioral change in the context of consumption we analyze transition phenomena in a non-linear stochastic consumption system characterized by multistability. In the scenario we focus on two fixed points coexist with a periodic attractor. We argue that in this case an approach to prediction of transitions between attractors that exploits basins of attraction and confidence regions based on the stochastic sensitivity function technique is not feasible.

Three attractors coexist such that the immediate basin of one attractor has a fractal boundary and the complement of the immediate basins (formed by complexly intertwined boundaries) exhibits final state uncertainty. Due to the fact that there is no exponentially stable attractor in sight that is required for a standard sensitivity analysis approach which relies on the interplay between attractor basins and confidence domains for attractors. In the case at hand, one simply lacks the basis for which to build a confidence domain.

We propose an unorthodox approach to predicting transition pattern in such a case. The novel idea consists in using the ghost - a vague manifestation of a non-attracting chaotic set (chaotic repellor) - as a surrogate for an attractor.

Our two-step heuristic combined two techniques which are based on critical lines. First, we apply an algorithm for generating the boundary of an absorbing area due to [49] to provide a representation of the ghost (approximate representation of the chaotic repellor). On the second step, we use the technique for generating confidence domains for a strange attractor due to [4] to build a confidence domain for the ghost. The intersections between this confidence region and the immediate basins of the coexisting attractors can then be used to make predictions about transition events.

A first basic assessment of the heuristic rendered promising results. In future work, we need to evaluate the performance of the heuristic in a more refined way. Naturally, such an assessment should include various other instances of multistability. Moreover, more work needs to be done to understand why this approach works and where is merits and limitations lie.

CRediT authorship contribution statement

Jochen Jungeilges: conceptualization, methodology, validation, writing original and review.
Tatiana Perevalova: conceptualization, methodology, software, visualization.
Makar Pavletsov: software, technical assistance.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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