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The functional characterizations of the Rothberger and Menger properties

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ABSTRACT

For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. In this paper we continue to study different selectors for sequences of dense sets of $C_p(X)$ started to study in the paper [14].
A set $A \subseteq C_p(X)$ will be called 1-*dense* in $C_p(X)$, if for each $x \in X$ and an open set W in $\mathbb{R}$ there is $f \in A$ such that $f(x) \in W$.
We give the characterizations of selection principles $S_1(\mathcal{A}, \mathcal{A})$, $S_{fin}(\mathcal{A}, \mathcal{A})$ and $S_1(\mathcal{S}, \mathcal{A})$ where

- $\mathcal{A}$ — the family of 1-dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_p(X)$, through the selection principles of a space X. In particular, we give the functional characterizations of the Rothberger and Menger properties.

1. Introduction

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0, 1] \subset \mathbb{R}$, and $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0.

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Basic open sets of $C_p(X)$ are of the form $[x_1, ..., x_k, U_1, ..., U_k] = \{f \in C(X) : f(x_i) \in U_i,\ i = 1, ..., k\}$, where each $x_i \in X$ and each U_i is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point f as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon\ \forall x \in A\}$, A is a finite subset of X and $\epsilon > 0$.

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup\mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every γ-cover contains a countably γ-cover.

For a topological space X we denote:

- $\mathcal{O}$ — the family of open covers of X;
- Γ — the family of countable open γ-covers of X;
- Ω — the family of open ω-covers of X;
- $\mathcal{D}$ — the family of dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_p(X)$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup\mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Many equivalences hold among these properties, and the surviving ones appear in the Diagram (Fig. 1) (where an arrow denotes implication), to which no arrow can be added except perhaps from $U_{fin}(\Gamma, \Gamma)$ or $U_{fin}(\Gamma, \Omega)$ to $S_{fin}(\Gamma, \Omega)$ [7].

The papers [7,8,19,22,24] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

In papers [1–5,8,9,12–14,16–21,24] (and many others) were investigated the applications of selection principles in the study of the properties of function spaces. In particular, the properties of the space $C_p(X)$ were investigated. In this paper we continue to study different selectors for sequences of dense sets of $C_p(X)$.

2. Main definitions and notation

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

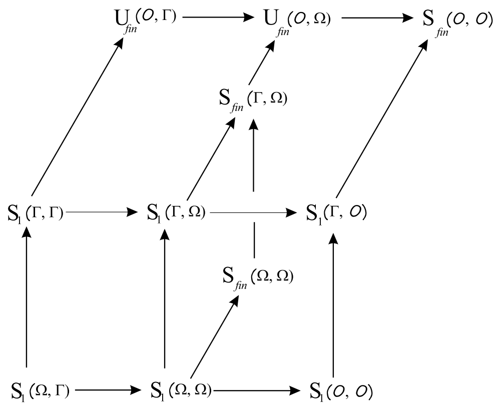

Fig. 1. The Scheepers Diagram for Lindelöf spaces.

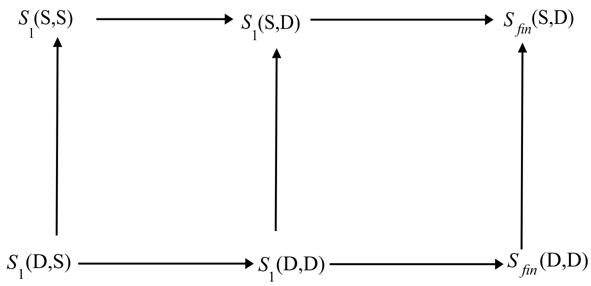

Fig. 2. The Diagram of selectors for sequences of dense sets of $C_p(X)$.

Theorem 2.1. *(Noble [11]) A space $C_p(X)$ is separable iff $iw(X) = \aleph_0$.*

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

So we have three types of topological properties described through the selection principles:

- local properties of the form $S_*(\Phi_x, \Psi_x)$;
- global properties of the form $S_*(\Phi, \Psi)$;
- semi-local of the form $S_*(\Phi, \Psi_x)$.

In paper [14], we investigated different selectors for sequences of dense sets of $C_p(X)$. We gave the characteristics of selection principles $S_1(\mathcal{P}, \mathcal{Q})$, $S_{fin}(\mathcal{P}, \mathcal{Q})$ for $\mathcal{P}, \mathcal{Q} \in \{\mathcal{D}, \mathcal{S}\}$ through the selection principles of a space X.

So for some selectors for sequences of dense sets of $C_p(X)$ (Fig. 2) we obtained the corresponding characteristics through the selection principles of a space X (see Fig. 3 for a metrizable separable space X).

Our main goal is to describe the remaining topological properties of X of the Scheepers Diagram in terms of local, global and semi-local properties of $C_p(X)$.

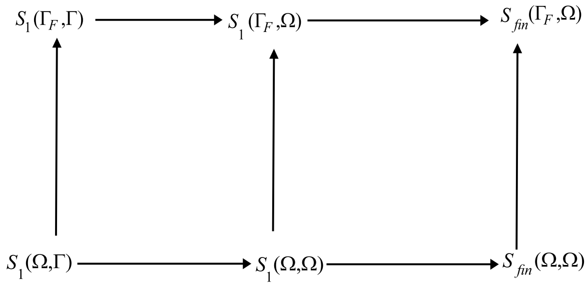

Fig. 3. The Diagram of selection principles for metrizable separable space X corresponding to selectors for sequences of dense sets of $C_p(X)$.

3. The Rothberger and Menger properties

A space X is said to be Rothberger [15] (or, [10]) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each n, $V_n \in \mathcal{U}_n$, and $\{V_n : n \in \mathbb{N}\}$ is an open cover of X.

Definition 3.1. A set $A \subseteq C_p(X)$ will be called *n-dense* in $C_p(X)$, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, ..., W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1, n}$.

Obviously, that if A is a n-dense set of $C_p(X)$ for each $n \in \mathbb{N}$ then A is a dense set of $C_p(X)$.
For a space $C_p(X)$ we denote:
$\mathcal{A}_n$ — the family of a n-dense subsets of $C_p(X)$.
If $n = 1$, then we denote $\mathcal{A}$ instead of $\mathcal{A}_1$.

Definition 3.2. Let $f \in C(X)$. A set $B \subseteq C_p(X)$ will be called *n-dense* at point f, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ and $\epsilon > 0$ there is $h \in B$ such that $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for $i \in \overline{1, n}$.

Obviously, that if B is a n-dense at point f for each $n \in \mathbb{N}$ then $f \in \overline{B}$.
For a space $C_p(X)$ we denote:
$\mathcal{A}_{n,f}$ — the family of a n-dense at point f subsets of $C_p(X)$.
If $n = 1$, then we denote $\mathcal{A}_f$ instead of $\mathcal{A}_{1,f}$.
Let $\mathcal{U}$ be an open cover of X and $n \in \mathbb{N}$.

- $\mathcal{U}$ is an n-cover of X if for each $F \subset X$ with $|F| \leq n$, there is $U \in \mathcal{U}$ such that $F \subset U$ [23].

Denote by $\mathcal{O}_n$ — the family of open n-covers of X.

- $S_1(\mathcal{O}, \mathcal{O}) = S_1(\Omega, \mathcal{O})$ [19].
- $S_1(\Omega, \mathcal{O}) = S_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathcal{O})$ [23].

Theorem 3.3. *For a space X, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{A}, \mathcal{A})$;
2. X *satisfies* $S_1(\mathcal{O}, \mathcal{O})$ *[Rothberger property];*
3. $C_p(X)$ *satisfies* $S_1(\mathcal{A}_f, \mathcal{A}_f)$;
4. $C_p(X)$ *satisfies* $S_1(\mathcal{A}, \mathcal{A}_f)$;
5. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \mathcal{A})$;
6. $C_p(X)$ *satisfies* $S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$;

7. *$C_p(X)$ satisfies $S_1(\{\mathcal{A}_{n,f}\}_{n\in\mathbb{N}}, \mathcal{A}_f)$;*
8. *$C_p(X)$ satisfies $S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A}_f)$.*

Proof. (1) ⇒ (2). Let $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ be a sequence of open covers of X. We set $A_n = \{f \in C(X) : f \upharpoonright (X\setminus U) = 1$ and $f \upharpoonright K = q$ for some $U \in \mathcal{O}_n$, a finite set $K \subset U$ and $q \in \mathbb{Q}\}$. It is not difficult to see that each A_n is a 1-dense subset of $C_p(X)$ because $\mathcal{O}_n$ is a cover of X and X is Tychonoff.

By the assumption there exists $f_n \in A_n$ such that $\{f_n : n \in \mathbb{N}\} \in \mathcal{A}$.

For each f_n we take $U_n \in \mathcal{O}_n$ such that $f_n \upharpoonright (X \setminus U_n) = 1$.

Set $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$. For $x \in X$ we consider the basic open neighborhood $[x, W]$ of $\mathbf{0}$, where $W = (-\frac{1}{2}, \frac{1}{2})$.

Note that there is $m \in \mathbb{N}$ such that $[x, W]$ contains $f_m \in \{f_n : n \in \mathbb{N}\}$. This means $x \in U_m$. Hence $\mathcal{U}$ is a cover of X.

(2) ⇒ (3). Let $B_n \in \mathcal{A}_f$ for each $n \in \mathbb{N}$. We renumber $\{B_n\}_{n\in\mathbb{N}}$ as $\{B_{i,j}\}_{i,j\in\mathbb{N}}$. Since $C(X)$ is homogeneous, we may think that $f = \mathbf{0}$. We set $\mathcal{U}_{i,j} = \{g^{-1}(-1/i, 1/i) : g \in B_{i,j}\}$ for each $i, j \in \mathbb{N}$. Since $B_{i,j} \in \mathcal{A}_0$, $\mathcal{U}_{i,j}$ is an open cover of X for each $i, j \in \mathbb{N}$. In case the set $M = \{i \in \mathbb{N} : X \in \mathcal{U}_{i,j}\}$ is infinite, choose $g_m \in B_{m,j}$ $m \in M$ so that $g^{-1}(-1/m, 1/m) = X$, then $\{g_m : m \in \mathbb{N}\} \in \mathcal{A}_f$.

So we may assume that there exists $i' \in \mathbb{N}$ such that for each $i \geq i'$ and $g \in B_{i,j}$ $g^{-1}(-1/i, 1/i)$ is not X.

For the sequence $\mathcal{V}_i = (\mathcal{U}_{i,j} : j \in \mathbb{N})$ of open covers there exist $f_{i,j} \in B_{i,j}$ such that $\mathcal{U}_i = \{f^{-1}_{i,j}(-1/i, 1/i) : j \in \mathbb{N}\}$ is a cover of X. Let $[x, W]$ be any basic open neighborhood of $\mathbf{0}$, where $W = (-\epsilon, \epsilon)$, $\epsilon > 0$. There exists $m \geq i'$ and $j \in \mathbb{N}$ such that $1/m < \epsilon$ and $x \in f^{-1}_{m,j}(-1/m, 1/m)$. This means $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{A}_f$.

(3) ⇒ (4) is immediate.

(4) ⇒ (1). Let $A_n \in \mathcal{A}$ for each $n \in \mathbb{N}$. We renumber $\{A_n\}_{n\in\mathbb{N}}$ as $\{A_{i,j}\}_{i,j\in\mathbb{N}}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_i : i \in \mathbb{N}\}$. Fix $i \in \mathbb{N}$. By the assumption there exists $f_{i,j} \in A_{i,j}$ such that $\{f_{i,j} : j \in \mathbb{N}\} \in \mathcal{A}_{q_i}$ where q_i is the constant function to q_i. Then $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{A}$.

(1) ⇒ (5). Since a dense set of $C_p(X)$ is a 1-dense set of $C_p(X)$, we have $C_p(X)$ satisfies $S_1(\mathcal{D}, \mathcal{A})$.

(5) ⇒ (6). Let $D_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$. We renumber $\{D_n\}_{n\in\mathbb{N}}$ as $\{D_{i,j}\}_{i,j\in\mathbb{N}}$. Then $P_j = \{D_{i,j} : i \in \mathbb{N}\}$ is a dense subset of $C_p(X)$ for each $j \in \mathbb{N}$. By (5), there is $p_j \in P_j$ for each $j \in \mathbb{N}$ such that $\{p_j : j \in \mathbb{N}\} \in \mathcal{A}$. Hence, we have $C_p(X)$ satisfies $S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$.

(6) ⇒ (8) is immediate.

(8) ⇒ (2). Claim that X satisfies $S_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathcal{O})$. Fix $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$ a set $\mathcal{S}_n = \{f \in C(X) : f \upharpoonright (X \setminus U) = 1$ and $f(x_i) \in \mathbb{Q}$ for each $i = \overline{1, n}$ for $U \in \mathcal{O}_n$ and a finite set $K = \{x_1, ..., x_n\} \subset U\}$. Note that $\mathcal{S}_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$. By (8), there is $f_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ such that $\{f_n : n \in \mathbb{N}\} \in \mathcal{A}_0$. Then $\{U_n : n \in \mathbb{N}\} \in \mathcal{O}$.

(3) ⇒ (7) is immediate.

(7) ⇒ (2). The proof is analogous to proof of implication (8) ⇒ (2). □

A space X is said to be Menger [6] (or, [17]) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a cover of X.

Every σ-compact space is Menger, and a Menger space is Lindelöf.

Theorem 3.4. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{A}, \mathcal{A})$;*
2. *X satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$ [Menger property];*
3. *$C_p(X)$ satisfies $S_{fin}(\mathcal{A}_f, \mathcal{A}_f)$;*
4. *$C_p(X)$ satisfies $S_{fin}(\mathcal{A}, \mathcal{A}_f)$;*
5. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{A})$;*
6. *$C_p(X)$ satisfies $S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$;*

7. $C_p(X)$ *satisfies* $S_{fin}(\{\mathcal{A}_{n,f}\}_{n\in\mathbb{N}}, \mathcal{A}_f)$*;*
8. $C_p(X)$ *satisfies* $S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A}_f)$*.*

Proof. The proof is analogous to proof of Theorem 3.3. □

4. $S_1(\mathcal{S}, \mathcal{A})$

Definition 4.1. (Sakai) A γ-cover $\mathcal{U}$ of co-zero sets of X is γ_F**-shrinkable** if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:

- Γ_F — the family of γ_F-shrinkable γ-covers of X.

Lemma 4.2. *(Lemma 6.5 in [14]) Let* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *be a* γ_F*-shrinkable co-zero cover of a space* X*. Then the set* $S = \{f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1$ *for some* $n \in \mathbb{N}\}$ *is sequentially dense in* $C_p(X)$*.*

Theorem 4.3. *For a space* X*, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{A})$*;*
2. X *satisfies* $S_1(\Gamma_F, \mathcal{O})$*;*
3. $C_p(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \mathcal{A}_f)$*;*
4. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{A}_f)$*.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_i : i \in \mathbb{N}\} \subset \Gamma_F$, $\mathcal{U}_i = \{U_i^m : m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ we consider a set $\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \mathbb{N}\}$.

Since $\mathcal{U}_i$ is a γ-cover of cozero subsets of X, then, by Lemma 4.2, $\mathcal{S}_i$ is a sequentially dense subset of $C_p(X)$ for each $i \in \mathbb{N}$.

Since $C_p(X)$ satisfies $S_1(\mathcal{S}, \mathcal{A})$, there is a set $\{f_i^{m(i)} : i \in \mathbb{N}\}$ such that for each i, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \mathbb{N}\}$ is an element of $\mathcal{A}$.

Consider a set $\{U_i^{m(i)} : i \in \mathbb{N}\}$.

(a). $U_i^{m(i)} \in \mathcal{U}_i$.
(b). $\{U_i^{m(i)} : i \in \mathbb{N}\}$ is a cover of X.

Let $x \in X$ and $U = \langle \mathbf{0}, x, \frac{1}{2} \rangle$ be a base neighborhood of $\mathbf{0}$, then there is $f_{i_{j_0}}^{m(i)_{j_0}} \in U$ for some $j_0 \in \mathbb{N}$. It follows that $x \in U_{i_{j_0}}^{m(i)_{j_0}}$. We thus get X satisfies $S_1(\Gamma_F, \mathcal{O})$.

(3) $\Rightarrow$ (2). Let $\{\mathcal{U}_i\} \subset \Gamma_F$. For each $i \in \mathbb{N}$ we consider the set $\mathcal{S}_i = \{f \in C(X) : f \upharpoonright F(U) = 0$ and $f \upharpoonright (X \setminus U) = 1$ for $U \in \mathcal{U}_i\}$.

Since $\mathcal{F}_i = \{F(U) : U \in \mathcal{U}_i\}$ is a γ-cover of X, we have that $\mathcal{S}_i$ converge to $\mathbf{0}$, i.e. $\mathcal{S}_i \in \Gamma_0$ for each $i \in \mathbb{N}$.

Since $C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \mathcal{A}_f)$, there is a sequence $\{f_i\}_{i\in\mathbb{N}}$ such that for each i, $f_i \in \mathcal{S}_i$, and $\{f_i : i \in \mathbb{N}\} \in \mathcal{A}_{\mathbf{0}}$.

Consider $\mathcal{V} = \{U_i : U_i \in \mathcal{U}_i$ such that $f_i \upharpoonright F(U_i) = 0$ and $f_i \upharpoonright (X \setminus U_i) = 1\}$. Let $x \in X$ and $W = [x, (-1, 1)]$ be a neighborhood of $\mathbf{0}$, then there exists $i_0 \in \mathbb{N}$ such that $f_{i_0} \in W$.

It follows that $x \in U_{i_0}$ and $\mathcal{V} \in \mathcal{O}$. We thus get X satisfies $S_1(\Gamma_F, \mathcal{O})$.

(2) $\Rightarrow$ (3). Fix $\{S_n : n \in \mathbb{N}\} \subset \Gamma_0$. We renumber $\{S_n : n \in \mathbb{N}\}$ as $\{S_{i,j} : i, j \in \mathbb{N}\}$.

For each $i, j \in \mathbb{N}$ and $f \in S_{i,j}$, we put $U_{i,j,f} = \{x \in X : |f(x)| < \frac{1}{i+j}\}$, $Z_{i,j,f} = \{x \in X : |f(x)| \leq \frac{1}{i+j+1}\}$.

Each $U_{i,j,f}$ (resp., $Z_{i,j,f}$) is a cozero-set (resp., zero-set) in X with $Z_{i,j,f} \subset U_{i,j,f}$. Let $\mathcal{U}_{i,j} = \{U_{i,j,f} : f \in S_{i,j}\}$ and $\mathcal{Z}_{i,j} = \{Z_{i,j,f} : f \in S_{i,j}\}$. So without loss of generality, we may assume $U_{i,j,f} \neq X$ for each $i, j \in \mathbb{N}$ and $f \in S_{i,j}$.

We can easily check that the condition $S_{i,j} \in \Gamma_0$ implies that $\mathcal{Z}_{i,j}$ is a γ-cover of X.

Since X satisfies $S_1(\Gamma_F, \mathcal{O})$ for each $j \in \mathbb{N}$ there is a sequence $\{U_{i,j,f_{i,j}} : i \in \mathbb{N}\}$ such that for each i, $U_{i,j,f_{i,j}} \in \mathcal{U}_{i,j}$, and $\{U_{i,j,f_{i,j}} : i \in \mathbb{N}\} \in \mathcal{O}$. Claim that $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{A}_0$. Let $x \in X$, $\epsilon > 0$, and $W = [x, (-\epsilon, \epsilon)]$ be a base neighborhood of $\mathbf{0}$, then there exists $j' \in \mathbb{N}$ such that $\frac{1}{1+j'} < \epsilon$. It follow that there exists i' such that $f_{i',j'}(x) \in (-\epsilon, \epsilon)$. So $C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \mathcal{A}_f)$.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $S_n \in \mathcal{S}$ for each $n \in \mathbb{N}$. We renumber $\{S_n\}_{n\in\mathbb{N}}$ as $\{S_{i,j}\}_{i,j\in\mathbb{N}}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_i : i \in \mathbb{N}\}$. Fix $i \in \mathbb{N}$. By the assumption there exist $f_{i,j} \in S_{i,j}$ such that $\{f_{i,j} : j \in \mathbb{N}\} \in \mathcal{A}_{q_i}$ where q_i is the constant function to q_i. Then $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{A}$. □

Corollary 4.4. *Suppose that a space X satisfies $S_1(\Gamma, \mathcal{O})$. Then a space $C_p(X)$ satisfies $S_1(\mathcal{S}, \mathcal{A})$.*

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