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IDENTITIES OF THE STYLIC MONOID

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ABSTRACT. We observe that for each $n \geq 2$, the identities of the stylic monoid with n generators coincide with the identities of n-generated monoids from other distinguished series of $\mathscr{J}$-trivial monoids studied in the literature, e.g., Catalan monoids and Kiselman monoids. This solves the Finite Basis Problem for stylic monoids.

A *monoid identity* is a pair of *words*, i.e., elements of the free monoid X^* over an alphabet X, written as a formal equality. An identity $w = w'$ with $w, w' \in X^*$ is said to *hold in a monoid* M if $w\varphi = w'\varphi$ for each homomorphism $\varphi\colon X^* \to M$; alternatively, we say that the monoid *satisfies* $w = w'$. Clearly, if M satisfies $w = w'$, then so does every homomorphic image of M.

Given any set Σ of monoid identities, we say that an identity $w = w'$ *follows* from Σ if every monoid satisfying all identities of Σ satisfies the identity $w = w'$ as well. A subset $\Delta \subseteq \Sigma$ is called a *basis* for Σ if each identity in Σ follows from Δ. The *Finite Basis Problem* for a monoid M is the question of whether or not the set of all identities that hold in M admits a finite basis.

A monoid M is said to be $\mathscr{J}$*-trivial* if every principal ideal of M has a unique generator, that is, $MaM = MbM$ implies $a = b$ for all $a, b \in M$. Finite $\mathscr{J}$-trivial monoids attract much attention because of their distinguished role in algebraic theory of regular languages [14, 15] and representation theory [7]. Several series of finite $\mathscr{J}$-trivial monoids parameterized by positive integers appear in the literature, including Straubing monoids [17, 18], Catalan monoids [16], double Catalan monoids [13], Kiselman monoids [9, 11, 12], and gossip monoids [5, 8, 10]. These monoids arise due to completely unrelated reasons and consist of elements of a very different nature. Surprisingly, studying identities of the listed monoids has revealed that the n-th monoids in each series satisfy exactly the same identities!

Recently, a new family of finite $\mathscr{J}$-trivial monoids, coined *stylic* monoids, have been introduced by Abram and Reutenauer [1], with motivation coming from combinatorics of Young tableaux. It is quite natural to ask whether the above phenomenon extends to this new family, i.e., whether the n-th stylic monoid again satisfies the same identities as do the n-th monoids in each aforementioned series. The present note aims to answers this question in the affirmative.

The combinatorial definition of stylic monoids can be found in [1], but here we only need their presentation via generators and relations established in [1, Theorem 8.1(ii)]. Thus, let the stylic monoid Styl_n be the monoid generated by $a_1, a_2, \ldots, a_n$ subject to the

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relations

$$a_i^2 = a_i \qquad \text{for each } i = 1,\dots,n; \tag{1}$$
$$a_j a_i a_k = a_j a_k a_i \qquad \text{if } 1 \le i < j < k \le n; \tag{2}$$
$$a_i a_k a_j = a_k a_i a_j \qquad \text{if } 1 \le i < j < k \le n; \tag{3}$$
$$a_j a_i a_i = a_i a_j a_i \qquad \text{if } 1 \le i < j \le n; \tag{4}$$
$$a_j a_j a_i = a_j a_i a_j \qquad \text{if } 1 \le i < j \le n. \tag{5}$$

We want to relate Styl_n to two other monoids with same generating set. The Kiselman monoid Kis_n, defined by Kiselman [11] for $n = 3$ and by Ganyushkin and Mazorchuk (unpublished) for an arbitrary $n \ge 2$, is generated by $a_1, a_2, \dots, a_n$ subject to the relations

$$a_i^2 = a_i \qquad \text{for each } i = 1,\dots,n; \tag{6}$$
$$a_i a_j a_i = a_j a_i a_j = a_j a_i \qquad \text{if } 1 \le i < j \le n. \tag{7}$$

Catalan monoids were defined by Solomon [16] as monoids of certain transformations on directed paths, but again, we only need their presentation via generators and relations from [16, Section 9]. So, for the purpose of this note, let Cat_n stand for the monoid generated by $a_1, a_2, \dots, a_n$ subject to the relations

$$a_i^2 = a_i \qquad \text{for each } i = 1,\dots,n; \tag{8}$$
$$a_i a_k = a_k a_i \qquad \text{if } |i-k| \ge 2,\ i,k = 1,\dots,n; \tag{9}$$
$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = a_{i+1} a_i \qquad \text{for each } i = 1,\dots,n-1. \tag{10}$$

Lemma 1. *The relations* (7) *hold in the monoid* Styl_n.

Proof. For any $i, j \in \{1, 2, \dots, n\}$ with $i < j$, we can deduce in Styl_n the following:

$$a_i a_j a_i \overset{(4)}{=} a_j a_i a_i \overset{(1)}{=} a_j a_i \quad \text{and} \quad a_j a_i a_j \overset{(5)}{=} a_j a_j a_i \overset{(1)}{=} a_j a_i.$$

Hence the relation $a_i a_j a_i = a_j a_i a_j = a_j a_i$ holds in Styl_n. □

Lemma 2. *The relations* (2)–(5) *hold in the monoid* Cat_n.

Proof. If $i, j, k \in \{1, 2, \dots, n\}$ are such that $i < j < k$, then $k - i \ge 2$ whence $a_j a_i a_k \overset{(9)}{=} a_j a_k a_i$ and $a_i a_k a_j \overset{(9)}{=} a_k a_i a_j$ in Cat_n. Thus, (2) and (3) hold in Cat_n.

If $1 \le i < j \le n$ and $j - i \ge 2$, we can apply (9) in a similar way: $a_j a_i a_i \overset{(9)}{=} a_i a_j a_i$ and $a_j a_j a_i \overset{(9)}{=} a_j a_i a_j$. Therefore, to prove that (4) and (5) hold in Cat_n, it remains to consider the case $j = i + 1$. In this case, we can deduce in Cat_n the following:

$$a_{i+1} a_i a_i \overset{(8)}{=} a_{i+1} a_i \overset{(10)}{=} a_{i+1} a_i a_{i+1} \quad \text{and} \quad a_{i+1} a_{i+i} a_i \overset{(8)}{=} a_{i+1} a_i \overset{(10)}{=} a_i a_{i+1} a_i.$$

Hence (4) and (5) also hold in Cat_n for all i, j with $1 \le i < j \le n$. □

Theorem 3. *For each $n \ge 2$, the monoids* Cat_n, Styl_n, *and* Kis_n *satisfy the same identities.*

Proof. We invoke Dyck's Theorem (see, e.g., [6, Theorem III.8.3]). Specialized in the case of monoids, it says that if a monoid M is generated by a set A subject to relations R and N is a monoid generated by A and such that all the relations R hold in N, then N is a homomorphic image of M. In view of this fact, Lemmas 2 and 1 imply that Cat_n is a homomorphic image of Styl_n, which in turn is a homomorphic image of Kis_n. Since identities are inherited by homomorphic images, every identity holding in Kis_n holds in Styl_n, and every identity holding in Styl_n holds in Cat_n. However, it follows from [2,

Theorem 8] that for each $n \geq 2$, the monoids Cat_n and Kis_n satisfy the same identities. Hence the same identities hold in the 'intermediate' monoid Styl_n as well. □

Since the identities of the monoids Cat_n and Kis_n have been characterized in [2,18], Theorem 3 leads to an efficient combinatorial description of the identities of stylic monoids. The description involves the notion of a scattered subword. Recall that a product $x_1 \cdots x_k$ of elements from an alphabet X is said to be a *scattered subword of length* k in a word $v \in X^*$ if there exist words $v_0, v_1, \ldots, v_{k-1}, v_k \in X^*$ such that $v = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k$; in other terms, $x_1 \cdots x_k$ is as a subsequence of v. The following is a combination of Theorem 3 with either [18, Proposition 4 and Corollary 2] or [2, Theorem 8].

Corollary 4. *An identity $w = w'$ holds in the monoid* Styl_n *if and only if the words w and w' have the same set of scattered subwords of length at most n.*

As yet another immediate application, we get a solution to the Finite Basis Problem for stylic monoids. It comes from Corollary 4 combined with results by Blanchet-Sadri [3,4].

Corollary 5. a) *The identities $xyxzx = xyzx,\ (xy)^2 = (yx)^2$ form a basis for the identities of the monoid* Styl_2.

b) *The identities $xyx^2zx = xyxzx,\ xyzx^2tz = xyxzx^2tz,\ zyx^2ztx = zyx^2zxtx,\ (xy)^3 = (yx)^3$ form a basis for the identities of the monoid* Styl_3.

c) *The identities of the monoid* Styl_n *with $n \geq 4$ admit no finite basis.*

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