# On some Frobenius groups with the same prime graph as the almost simple group PGL(2,49) 

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#### Abstract

The prime graph of a finite group $G$ is denoted by $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are adjacent in $\Gamma(G)$, whenever $G$ contains an element with order $p q$. We say that $G$ is unrecognizable by prime graph if there is a finite group $H$ with $\Gamma(H)=\Gamma(G)$, in while $H \not \approx G$. In this paper, we consider finite groups with the same prime graph as the almost simple group PGL $(2,49)$. Moreover, we construct some Frobenius groups whose their prime graph coincide with $\Gamma(\operatorname{PGL}(2,49))$, in particular, we get that $\operatorname{PGL}(2,49)$ is unrecognizable by prime graph.


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## 1 Introduction

Let $\mathbb{N}$ denotes the set of natural numbers. If $n \in \mathbb{N}$, then we denote by $\pi(n)$, the set of all prime divisors of $n$. Let $G$ be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of $G$ is denoted by $\pi_{e}(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_{e}(G)$ under the divisibility relation. The prime graph of $G$ is a graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (and we write $p \sim q$ ), whenever $G$ contains an element of order $p q$. The prime graph of $G$ is denoted by $\Gamma(G)$. A finite group $G$ is called unrecognizable by prime graph if for every finite group $H$ such that $\Gamma(H)=\Gamma(G)$, however $H \not \approx G$.

In [10, it is proved that if $p$ is a prime number which is not a Mersenne or Fermat prime and $p \neq 11,19$ and $\Gamma(G)=\Gamma(\operatorname{PGL}(2, p))$, then $G$ has a unique nonabelian composition factor which
is isomorphic to $\operatorname{PSL}(2, p)$ and if $p=13$, then $G$ has a unique nonabelian composition factor which is isomorphic to $\operatorname{PSL}(2,13)$ or $\operatorname{PSL}(2,27)$. In [3], it is proved that if $q=p^{\alpha}$, where $p$ is a prime and $\alpha>1$, then $\operatorname{PGL}(2, q)$ is uniquely determined by its element orders. Also in [1], it is proved that if $q=p^{\alpha}$, where $p$ is an odd prime and $\alpha$ is an odd natural number, then $\operatorname{PGL}(2, q)$ is uniquely determined by its prime graph. However, in this paper as the main result we prove that the almost simple group $\operatorname{PGL}(2,49)$ is unrecognizable by prime graph. Also, finally we put a question about the existence of Frobenius groups with the same prime graph as the almost simple groups PGL $(2, q)$.

## 2 Preliminary Results

Lemma 2.1. ([17]) Let $G$ be a finite group and $N \unlhd G$ such that $G / N$ is a Frobenius group with kernel $F$ and cyclic complement $C$. If $(|F|,|N|)=1$ and $F$ is not contained in $N C_{G}(N) / N$, then $p|C| \in \pi_{e}(G)$ for some prime divisor $p$ of $|N|$.

Lemma 2.2. ([8]) Let $G$ be a finite group and $|\pi(G)| \geq 3$. If there exist prime numbers $r$, $s$, $t \in \pi(G)$, such that $\{t r, t s, r s\} \cap \pi_{e}(G)=\emptyset$, then $G$ is non-solvable.

Lemma 2.3. ([19, Theorem 18.6]) Let $G$ be a nonsolvable Frobenius complement. Then $G$ has a normal subgroup $G_{0}$ with $\left|G: G_{0}\right|=1$ or 2 such that $G_{0}=\mathrm{SL}(2,5) \times M$ with $M$ a $Z$-group of order prime to 2, 3 and 5.

Using [14, Theorem A], we have the following result:

Lemma 2.4. Let $G$ be a finite group with $t(G) \geq 2$. Then one of the following holds:
(a) $G$ is a Frobenius or 2-Frobenius group;
(b) there exists a nonabelian simple group $S$ such that $S \leq \bar{G}:=G / N \leq A u t(S)$ for some nilpotent normal subgroup $N$ of $G$.

Lemma 2.5. ([20]) Let $G=L_{n}^{\varepsilon}(q), q=p^{m}$, be a simple group which acts absolutely irreducibly on a vector space $W$ over a field of characteristic $p$. Denote $H=W \lambda G$. If $n=2$ and $q$ is odd then $2 p \in \pi_{e}(H)$.

## 3 Main Results

Lemma 3.1. There are infinitely many finite Frobenius group $G$ such that $\Gamma(G)=\Gamma(\operatorname{PGL}(2,49))$.

Proof. Let $F$ be a finite field of characteristic 7. Also let there are some elements $\alpha$ and $\beta$ included in $F$ such that $\alpha^{2}=-1$ and $\beta^{2}=5$. We know that such a finite filed exists and moreover there are infinitely many filed with these properties.

Now we construct some linear groups as follow:

$$
\begin{gathered}
C:=\left\langle\left(\begin{array}{ccc}
-1 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{ccc}
0 & \alpha & 0 \\
\alpha & \frac{\beta+1}{2} & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right)\right\rangle, \\
K:=\left\langle\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{array}\right)\right\rangle .
\end{gathered}
$$

By the above definition, $C \cong\left\langle x, y, z \mid x^{3}=y^{5}=z^{2}=1, x^{z}=z, y^{z}=y,(x y)^{2}=z\right\rangle$. This implies that $C \cong \mathrm{SL}(2,5)$. Also we have $K \cong F \oplus F$, is a direct sum of additive group $F$ by itself. This means $K$ is isomorphic to a vector space of dimension 2 over $F$ and so $|K|=|F|^{2}$. It is obvious that $C$ belongs to the normalizer of $K$ in $\mathrm{GL}(3, F)$.

Now we define $G:=K \rtimes C$. Since $K$ is an elementary abelian 7-group, it is easy to prove that $C$ acts fixed point freely on $K$ by conjugation. Hence $G$ is a Frobenius group with kernel $K$ and complement $C$. This implies that in the prime graph of $G, 7$ is an isolated vertex. Also by $\Gamma(\operatorname{SL}(2,5))$, we get that 2 is adjacent to 3 and 5 and there is no edge between 3 and 5 in $\Gamma(G)$. Therefore, $\Gamma(G)$ coincides to $\Gamma(\operatorname{PGL}(2,49))$, which completes the proof.

Lemma 3.2. Each following group $G$ is an almost simple group related to the simple group $S$. Moreover, $G$ has a prime graph which coincide with the prime graph of the almost simple group $\operatorname{PGL}(2,49):$
(1) $G=S_{7}$ and $S=A_{7}$.
(2) $G=U_{4}(3) \cdot 2$ and $S=U_{4}(3)$.
(3) $G=U_{3}(5)$ or $G=U_{3}(5) \cdot 2$ and $S=U_{3}(5)$.

Proof. Using [4], it is straightforward.
Theorem 3.3. Let $G$ be a finite group with the prime graph as same as the prime graph of PGL $(2,49)$. Then $G$ is isomorphic to one of the following groups:
(1) A Frobenius group $K \rtimes C$, such that $K$ is a 7 -group and $C$ contains a subgroup $C_{0}$ whose index in $C$ is at most 2 and $C_{0}$ is isomorphic to $\operatorname{SL}(2,5)$.
(2) On of the almost simple group: $S_{7}, U_{4}(3) \cdot 2, U_{3}(5) \cdot 2$ or $\operatorname{PGL}(2,49)$.
(3) The simple group: $U_{3}(5)$.

In particular, $\mathrm{PGL}(2,49)$ is unrecognizable by prime graph.
Proof. By [18, Lemma 7], it follows that $\mu(\operatorname{PGL}(2,49))=\{7,48,50\}$. Hence, the connected components of the prime graph of $\operatorname{PGL}(2,49)$ are exactly $\{7\}$ and $\{2,3,5\}$. Also by $\mu(\operatorname{PGL}(2,49))$, there is no edge between 3 and 5 in $\Gamma(\operatorname{PGL}(2,49))$. Now since $\Gamma(G)=\Gamma(\operatorname{PGL}(2,49))$, we deduce that these relations hold in the prime graph of $G$.

First we claim that $G$ is not solvable. On the contrary, let $G$ be a solvable group. So there is a Hall $\{3,5,7\}$-subgroup in $G$, say $H$. On the other hand $\{3,5,7\}$ is an independent subset of $\Gamma(G)$, which is a contradiction by Lemma 2.2. Therefore, $G$ is not solvable and so by Lemma [2.4, either $G$ is a Frobenius group or there is a nonabelian simple group $S$ such that $S \leq \bar{G}:=G / \operatorname{Fit}(G) \leq \operatorname{Aut}(S)$.

Let $G$ be a Frobenius group with kernel $K$ and complement $C$. By Lemma 2.3, we know that $K$ is nilpotent and $\pi(C)$ is a connected component of the prime graph of $G$. Hence we conclude that $\pi(K)=\{7\}$ and $\pi(C)=\{2,3,5\}$, since 7 is an isolated vertex in $\Gamma(G)$. Hence if $C$ is solvable, then $G$ is a solvable which is a contradiction by the above argument.

Thus we suppose that $C$ is non-solvable. Then by Lemma [2.3, the complement $C$ has a normal subgroup $C_{0}$ with index at most 2 which is isomorphic to $\operatorname{SL}(2,5) \times M$, where $\pi(M) \cap$ $\{2,3,5\}=\emptyset$. On the other hand, by the previous argument, we know that $\pi(C)=\{2,3,5\}$. This implies that $M=1$ and so $C_{0} \cong \mathrm{SL}(2,5)$. Also by Lemma 3.1, we know that this such Frobenius complement exists. Hence $G$ can be isomorphic to a Frobenius group $K$ : $C$, where $K$ is a 7 -subgroup and $C$ contains a subgroup isomorphic to $\mathrm{SL}(2,5)$ whose index is at most 2 , Therefore if $G$ is a Frobenius group, then we get Case (1).

Now we assume that $G$ is neither Frobenius nor 2-Frobenius group. Hence by Lemma 2.4, there exists a nonabelian simple group $S$ such that:

$$
S \leq \bar{G}:=G / K \leq \operatorname{Aut}(S)
$$

in which $K$ is the Fitting subgroup of $G$. Since $\{2,7\}$ is an independent subset of $\Gamma(G)$, by Lemma 2.4, we conclude that $7 \in \pi(S)$ and $7 \notin \pi(K) \cup \pi(\bar{G} / S)$. Also we know that $\pi(S) \subseteq \pi(G)$. Since $\pi(G)=\{2,3,5,7\}$, so by [13, Table 8], we get that $S$ is isomorphic to $A_{7}, A_{8}, A_{9}, A_{10}$, $S_{6}(2), O_{8}^{+}(2), L_{3}\left(2^{2}\right), L_{2}\left(2^{3}\right), U_{3}(3), U_{4}(3), U_{3}(5), L_{2}(7), S_{4}(7), L_{2}\left(7^{2}\right)$ or $J_{2}$. Now we consider each possibility for the simple group $S$.

Let $S \cong L_{2}(7)$. Then $5 \in \pi(K)$, since $5 \notin(\pi(S) \cup \pi(\bar{G} / S))$. On the other hand $S$ contains a $\{3,7\}$-subgroup $H$. Hence $G$ has a subgroup isomorphic to $K_{5}: H$ where $K_{5}$ is 5 -group. On the other hand $K_{5}: H$ is solvable and so there is an edge between to prime numbers in $\Gamma\left(K_{5}: H\right)$, which is impossible since $\Gamma\left(K_{5}: H\right)$ is a subgraph of $\Gamma(G)$. Thus $S \neq L_{2}(7)$.

Let $S \cong L_{2}\left(2^{3}\right)$. In this case, $5 \in \pi(K)$. Also we know that $S$ contains a Frobenius group isomorphic to $8: 7$. Hence by Lemma 2.1, we get that $G$ has an element order $5 \cdot 7$, which is a contradiction.

Let $S \cong A_{8}, A_{9}$ or $A_{10}$. Thus $S$ consists an element of order $3 \cdot 5$, which contradicts to the prime graph of $G$.

Let $S \cong J_{2}, O_{8}^{+}(2)$ or $S_{6}(2)$. In this case $S$ contains an element of order 15 , which is a contradiction.

By Lemma 3.2, the finite group $S$ can be isomorphic to each simple group $A_{7}, U_{3}(3), U_{4}(3)$ and $U_{3}(5)$.

Let $S$ be isomorphic to $\mathrm{PSL}_{2}(49)$. Hence $\mathrm{PSL}_{2}(49) \leq \bar{G} \leq \operatorname{Aut}\left(\mathrm{PSL}_{2}(49)\right)$.
Let $\pi(K)$ contains a prime $r$ such that $r \neq 7$. Since $K$ is nilpotent, we may assume that $K$ is a vector space over a field with $r$ elements (analogous to the proof of Lemma ??). Hence the prime graph of the semidirect product $K \rtimes \mathrm{PSL}_{2}(49)$ is a subgraph of $\Gamma(G)$. Let $B$ be a Sylow 7 -subgroup of $\mathrm{PSL}_{2}$ (49). We know that $B$ is not cyclic. On the other hand $K \rtimes B$ is a Frobenius group such that $\pi(K \rtimes B)=\{r, 7\}$. Hence $B$ should be cyclic which is a contradiction. This implies that $K=1$, since $7 \notin \pi(K)$.

We know that $\operatorname{Aut}\left(\mathrm{PSL}_{2}(49)\right) \cong Z_{2} \times Z_{2}$. Since in the prime graph of $\mathrm{PSL}_{2}(49)$ there is not any edge between 7 and 2 , we get that $G \neq \mathrm{PSL}_{2}(49)$. Also if $G=\mathrm{PSL}_{2}(49):\langle\theta\rangle$, where $\theta$ is a field automorphism, then we get that 2 and 7 are adjacent in $G$, which is a contradiction. If $G=\mathrm{PSL}_{2}(49):\langle\gamma\rangle$, where $\gamma$ is a diagonal-field automorphism, then we get that $G$ does not contain any element with order $2 \cdot 7$ (see [3, Lemm 12]), which is contradiction, since in $\Gamma(G)$, $2 \sim 7$. This argument shows that $G \cong \mathrm{PSL}_{2}(49)$, which completes the proof.

Problem 3.4. Let $G=P G L(2, q)$ be an almost simple group related to the simple group $\operatorname{PSL}(2, q)$. Find all Frobenius group $H$ such that $\Gamma(H)=\Gamma(G)$.

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