DOI 10.1007/s10958-022-05867-z
Journal of Mathematical Sciences, Vol. 262, No. 6, April, 2022

PIECEWISE LINEAR PRICE FUNCTION OF A DIFFERENTIAL GAME WITH SIMPLE DYNAMICS AND INTEGRAL TERMINAL PRICE FUNCTIONAL

L. G. Shagalova UDC 517.977

Abstract. In this paper, we consider an antagonistic differential game of two persons with dynamics described by a differential equation with simple motions and an integral terminal payment functional. In this game, there exists a price function, which is a generalized (minimax or viscous) solution of the corresponding Hamilton–Jacobi equation. For the case where the terminal function and the Hamiltonian are piecewise linear and the dimension of the phase space is equal to 2, we propose a finite algorithm for the exact construction of the price function. The algorithm consists of the sequential solution of elementary problems arising in a certain order. The piecewise linear price function of a differential game is constructed by gluing piecewise linear solutions of elementary problems. Structural matrices are a convenient tool of representing such functions.

***Keywords and phrases*:** differential game, simple motion, price function, Hamilton–Jacobi equation, generalized solution, minimax solution, algorithm.

***AMS Subject Classification*:** 49N70, 49N75, 91A05, 91A24

1. Introduction. Antagonistic differential games form a branch of the mathematical theory of optimal control in which control problems are investigated in the case of opposite interests of subjects controlling the system (players). The dynamics of the system is described by a differential equation containing vectors controlled by the players. There are various formalizations of differential games.

In this paper, the game problem is considered within the framework of the positional formalization introduced in the works of N. N. Krasovsky and A. I. Subbotin (see [5, 6]). To choose a control, each player can use only the current information about the state of the system. The first player seeks to minimize the given payment functional on motions of the system, whereas the goal of the second player is opposite, i.e., to maximize this functional. In the case considered, the approach to solving a differential game consists of the search for an appropriate price function. The price function assigns the same optimal guaranteed result for all players to a given initial state of the system, i.e., the best guaranteed value of the payment functional. If the price function is known, one can construct the optimal controls of the players by the feedback principle.

In the study of differential games, as a rule, nonlinear first-order partial differential equations arise. It was noted in [12, 13] that the price function is a minimax solution of the Hamilton–Jacobi equation corresponding to the differential game considered, where the concept of a generalized minimax solution is equivalent to the concept of a viscosity solution introduced by M. G. Crandall and P.-L. Lions (see [2]).

Thus, the problem of constructing the price function of a differential game can be reduced to solving a first-order differential equation. However, there is no universal methods of solving the nonlinear

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 168, Proceedings of the International Conference “Geometric Methods in the Control Theory and Mathematical Physics” Dedicated to the 70th Anniversary of Prof. S. L. Atanasyan, 70th Anniversary of Prof. I. S. Krasil’shchik, 70th Anniversary of Prof. A. V. Samokhin, and 80th Anniversary of Prof. V. T. Fomenko. Ryazan State University named for S. Yesenin, Ryazan, September 25–28, 2018. Part I, 2019.

Hamilton–Jacobi equations, and a solution an analytical form cannot be obtained in all cases. Therefore, the development of computational methods and analysis of the structure of piecewise smooth minimax solutions are topical directions of research. Despite the simplicity of the dynamics, the solutions of such games are known only in some special cases, and in the general case, finding solutions is also not an easy task.

For local approximation of a piecewise smooth prove function of a differential game with general dynamics, one can use solutions of a differential game with simple dynamics, which depends only on the controls of the players and does not depend on the phase vector. Despite the simplicity of the dynamics, the solutions of such games are known only in some special cases, whereas the search for solutions in the general case is a difficult problem. Therefore, the study of differential games with simple motions is also of independent interest. Geometric methods and convex and nonsmooth analysis are used for the study of such games.

Differential games with simple motions were studied by many authors; in particular, we mention the works [4, 7, 8, 15].

In this paper, we present a finite algorithm for constructing an exact minimax solution of the Hamilton–Jacobi equation corresponding to a differential game with simple motions and an integral terminal payment functional in the case of a two-dimensional phase space and piecewise linear input data. The results presented summarize the results obtained in [10, 11, 14].

2. Statement of the problem. We consider the following position differential game of two persons governed by the dynamical equation

$$\dot{x} = u(t) + v(t), \quad t \in [0, \vartheta], \quad x \in \mathbb{R}^n, \quad u(t) \in P \subset \mathbb{R}^n, \quad v(t) \in Q \subset \mathbb{R}^n; \tag{1}$$

where t is time, ϑ is the given moment of the end of the game, x is the phase vector, and $u(\cdot)$ and $v(\cdot)$ are the controls of the first and second players, respectively, constructed by the feedback principle. We assume that the sets P and Q are compact.

Let $(t_0, x_0) \in [0, \vartheta] \times \mathbb{R}^n$ be an initial state. On trajectories of the control system (1) corresponding to the given initial state, we consider the integral terminal payment functional

$$I = I(t_0, x_0, u(\cdot), v(\cdot)) = \sigma(x(\vartheta)) + \int_{t_0}^{\vartheta} g(u(\tau), v(\tau))d\tau, \tag{2}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function and the function $g : P \times Q \to \mathbb{R}$ is continuous. The first player intends to minimize the pay by choosing his control, whereas the second player intends to maximize it.

Assume that for the differential game considered, the following condition is fulfilled:

$$\min_{u \in P} \max_{v \in Q} [\langle s, u + v \rangle + g(u, v)] = \max_{v \in Q} \min_{u \in P} [\langle s, u + v \rangle + g(u, v)] = H(s), \quad s \in \mathbb{R}^n, \tag{3}$$

where $\langle s, f \rangle$ is the scalar product of the vectors s and f. The function $H(\cdot)$ defined by Eq. (3) is called the *Hamiltonian* of the differential game (1), (2).

It is well known (see [5, 6]) that if the condition (3) is fulfilled, then for any initial state $(t_0, x_0) \in [0, \vartheta] \times \mathbb{R}^n$, there exists the price $\omega(t_0, x_0)$ of the game. Thus, there exists the price function $\omega : [0, \vartheta] \times \mathbb{R}^n \to \mathbb{R}$. However, the search for the price function is a difficult problem and there no universal methods to solve it. We note that in the case where the integrand function $g(\cdot)$ is identical zero, i.e., the payment functional is terminal, the problem is greatly simplified. In particular, if one of the functions $H(\cdot)$ or $\sigma(\cdot)$ is convex or concave, one can obtain explicit formulas for the price function using the well-known Pshenichny–Sagaidak (see [9]) and Hopf–Lax (see [1, 3]) formulas. Also, in the case where the piecewise linear functions $H(\cdot)$ and $\sigma(\cdot)$ are not necessarily convex, the dimension n of the phase space is equal to 2, and the terminal price if positively homogeneous, i.e., the function $\sigma(\cdot)$ satisfies the condition

$$\sigma(\lambda x) = \lambda\sigma(x), \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}, \quad \lambda > 0, \tag{4}$$

there exists a finite algorithm for constructing the price function (see [10, 11, 14]). The aim of this work is to generalize this algorithm to the case of a monotonic nonzero integrand $g(\cdot)$.

3. Price of the game as a minimax solution of the Hamilton–Jacobi equation. In this section, we recall present some facts from [12, 13] needed in the sequel for constructing the price function of the differential game (1)–(3) and some of their consequences.

The price function $\omega : [0, \vartheta] \times \mathbb{R}^n \to \mathbb{R}$ coincides with the minimax solution of the following Cauchy problem:

$$\frac{\partial\omega(t,x)}{\partial t} + H\left(\frac{\partial\omega(t,x)}{\partial x}\right) = 0, \quad t \leq \vartheta, \quad x \in \times\mathbb{R}^n, \tag{5}$$

$$\omega(\vartheta, x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{6}$$

The minimax solution of the problem (5), (6) exists and is unique. We assume that the terminal function $\sigma(\cdot)$ is positively homogeneous, i.e., satisfies the condition (4). The Hamiltonian $H(\cdot)$ is defined by Eq. (3) and is not positively homogeneous.

We introduce the functions

$$H^*(s,r) = \begin{cases} |r|H\left(\dfrac{s}{|r|}\right) & \text{for } r \neq 0, \\ \lim\limits_{r\downarrow 0} rH\left(\dfrac{s}{r}\right) & \text{for } r = 0, \end{cases} \qquad (s,r) \in \mathbb{R}^n \times \mathbb{R}, \tag{7}$$

$$\sigma^\sharp(x,y) = \sigma(x) + y, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}. \tag{8}$$

We assume that the limit in (7) exists.

Consider the following Cauchy problem for the Hamilton–Jacobi equation with the Hamiltonian $H^*(\cdot)$, which is positively homogeneous with respect to the variable $\overline{s} = (s, r)$:

$$\frac{\partial u(t,x,y)}{\partial t} + H^*\left(\frac{\partial u(t,x,y)}{\partial x}, \frac{\partial u(t,x,y)}{\partial y}\right) = 0, \quad t \leq \vartheta, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}, \tag{9}$$

$$u(\vartheta, x, y) = \sigma^\sharp(x,y), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}. \tag{10}$$

The following assertion holds.

Theorem 1. *A function $\omega(t,x)$ is a minimax solution of the problem* (5), (6) *if and only if the function $u(t,x,y) = \omega(t,x) + y$ is a miinimax solution of the problem* (9), (10).

Thus, the problem of the search for the price function with an integral terminal payment functional is reduced to the solution of the Hamilton–Jacobi equation with a positively homogeneous Hamiltonian. In this case, the dimension of the phase space in increased by 1.

If the Hamiltonian $H^*(\cdot)$ satisfies the Lipschitz condition, then the minimax solution $u(t,x,y)$ satisfies the relation

$$u(t,x,y) = (\vartheta - t)u\left(0, \frac{x}{\vartheta - t}, \frac{y}{\vartheta - t}\right), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}. \tag{11}$$

Using the relation (11), one can reduce the problem (9), (10) to a problem for the function

$$\varphi(x,y) = u(0,x,y) \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}. \tag{12}$$

The function $\varphi(\cdot)$ is a minimax solution of the following first-order partial differential equation:

$$H^*\left(\frac{\partial\varphi(x,y)}{\partial x}, \frac{\partial\varphi(x,y)}{\partial y}\right) + \left\langle \frac{\partial\varphi(x,y)}{\partial x}, x\right\rangle + \frac{\partial\varphi(x,y)}{\partial y}\cdot y - \varphi(x,y) = 0, \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}, \tag{13}$$

considered together with the limit relation

$$\lim_{\alpha\downarrow 0}\alpha\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) = \sigma^\sharp(x,y), \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}. \tag{14}$$

A minimax solution of Eq. (13) is a continuous function satisfying a pair of differential inequalities. These inequalities can be written in various forms, which however are equivalent. We write these inequalities in the following form:

$$H^*(l,m) + \langle l,x\rangle + m\cdot y \le \varphi(x,y), \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}, \quad (l,m)\in D^-\varphi(x,y), \tag{15}$$

$$H^*(l,m) + \langle l,x\rangle + m\cdot y \ge \varphi(x,y), \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}, \quad (l,m)\in D^+\varphi(x,y), \tag{16}$$

where the sets $D^-\varphi(x,y)$ and $D^+\varphi(x,y)$ are the subdifferential and the superdifferential of the function $\varphi(\cdot)$ at the point (x,y), respectively.

4. Algorithm of the construction of the price function. In the case where the dimension of the phase space is equal to 2 and the terminal function $\sigma(\cdot)$ and the integrand $g(\cdot)$ are piecewise linear, the price function $\omega(\cdot)$ of the differential game (1)–(3) is piecewise linear and can be constructed in the exact form. We describe an algorithm of constructing the function $\varphi(\cdot)$, which then allows one to obtain the function $\omega(\cdot)$ by the relation (11) and Theorem 1.

4.1. Representation of the limit function. Let

$$y_+ = \max\{0; y\}, \quad y_- = \min\{0; y\}, \quad y\in\mathbb{R},$$
$$\sigma_+(x) = \max\{0;\sigma(x)\}, \quad \sigma_-(x) = \min\{0;\sigma(x)\}, \quad x\in\mathbb{R}^n,$$
$$\sigma^\sharp_+(x,y) = \sigma_+(x) + y_+, \quad \sigma^\sharp_-(x,y) = \sigma_-(x) + y_-, \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}.$$

It is eqsy to see that the limit function $\sigma^\sharp(\cdot)$ in (8) can be represented in the form

$$\sigma^\sharp(x,y) = \sigma^\sharp_+(x,y) + \sigma^\sharp_-(x,y), \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}. \tag{17}$$

Moreover, for the solution $\varphi(\cdot)$ of the problem (13), (14) we have the representation

$$\varphi(x,y) = \varphi_+(x,y) + \varphi_-(x,y), \quad x\in\mathbb{R}^n, \quad y\in\mathbb{R}, \tag{18}$$

where $\varphi_+(\cdot)$ and $\varphi_-(\cdot)$ are solutions of the problem (13), (14) corresponding to the limit functions $\sigma^\sharp_+(\cdot)$ and $\sigma^\sharp_-(\cdot)$, respectively. Under the conditions imposed below, the functions $\varphi_+(\cdot)$ and $\varphi_-(\cdot)$ are constructed similarly.

5. Assumptions. The algorithm is developed under the following assumptions.

A1. The integrand $g(\cdot)$ has the form

$$g(u,v) = g_1(u) + g_2(v), \quad u\in\mathbb{R}^2, \quad v\in\mathbb{R}^2, \tag{19}$$

where $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous, piecewise linear functions glued from a finite number of linear functions. Thus, their sum $g(\cdot)$ is also a continuous, piecewise linear function.

A2. The sets P and Q are polyhedra. It follows from (3) that the Hamiltonian $H(\cdot)$ of the differential game (1)–(3) is also piecewise linear and can be glued from a finite number of linear functions

$$H^i(s) = \langle h^i, s\rangle + p^i, \quad i\in\overline{1,n_H}, \quad h^i\in\mathbb{R}^2, \quad p^i\in\mathbb{R}, \quad s\in\mathbb{R}^2. \tag{20}$$

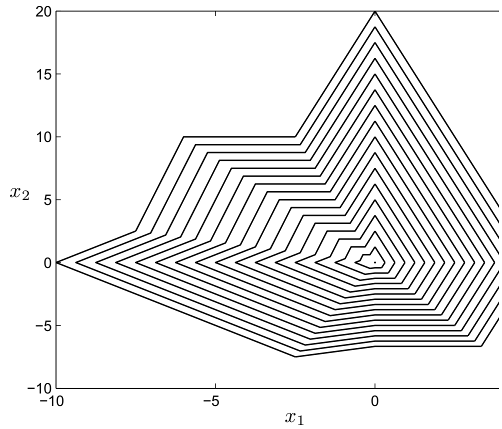

Fig. 1.

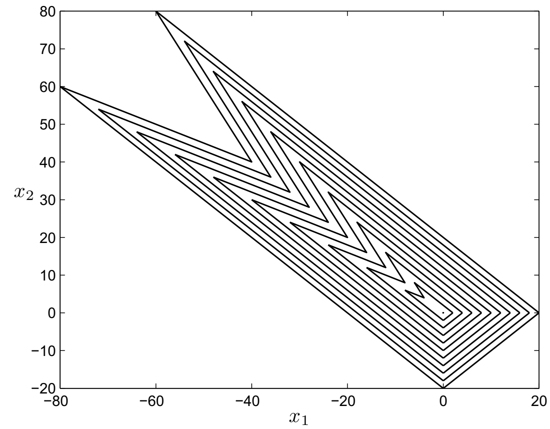

Fig. 2.

A3. The function $\sigma(\cdot)$ is positively homogeneous (i.e., satisfies the condition (4)) and piecewise linear, i.e., is glued from a finite number of linear functions:

$$\sigma^i(x) = \left\langle s^i, x \right\rangle, \quad i \in \overline{1, n_\sigma}, \quad s^i \in \mathbb{R}^2, \quad x \in \mathbb{R}^2.$$

Introduce the notation

$$Z = \left\{ s^i \mid i \in \overline{1, n_\sigma} \right\}. \tag{21}$$

Moreover, without loss of generality, due to the representations (17) and (18), we can assume that the function $\sigma(\cdot)$ is nonnegative,

$$\sigma(x) \geq 0, \quad x \in \mathbb{R}^2, \tag{22}$$

and consider an algorithm of constructing the function $\varphi(\cdot)$ corresponding to the limit function

$$\sigma^\sharp(x, y) = \sigma(x) + y_+, \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R} \tag{23}$$

6. Simple piecewise linear functions. The notion of a simple piecewise linear function (SPLF) used in [10, 14] is useful for developing an algorithm. The main property of SPLFs is as follows: if a function $\psi : \mathbb{R}^2 \supset D \to \mathbb{R}$ is an SPLF, then for any point $x_* \subset D$, there exists a neighborhood $O_\varepsilon(x_*)$ in which $\psi(\cdot)$ has one of the following three representations:

$$\psi(x) = \langle s_i, x \rangle + h_i,$$
$$\psi(x) = \max \Big\{ \langle s_i, x \rangle + h_i, \ \langle s_j, x \rangle + h_j \Big\},$$
$$\psi(x) = \min \Big\{ \langle s_i, x \rangle + h_i, \ \langle s_j, x \rangle + h_j \Big\},$$

where s_i and s_j are vectors from $\mathbb{R}^2$ and h_i and h_j are numbers. Thus, the domain of an SPLF does not contain points in a small neighborhood of which three or more linear functions are glued.

For a formal definition of SPLFs, matrices are used. We do not give a rigorous definition here; we only note that the structure matrix contains information about all linear functions that form the corresponding SPLF. If the structure matrix is known, then one can calculate the value of the SPLF at each points of its domain.

We also note that under the condition **A3**, a nonnegative function $\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an SPLF in the domain $\mathbb{R}^2 \setminus 0$, where 0 is the null vector. In Figs. 1 and 2, examples of the level lines of the function σ are presented.

7. Elementary problems. The algorithm for constructing the function $\varphi(\cdot)$ is essentially a sequential solution of elementary problems that arise in a certain order.

Let

$$\varsigma^{\sharp+}(x,y) = \max\{\langle a,x\rangle + y, \langle b,x\rangle + y\}, \quad \varsigma^{\sharp-}(x,y) = \min\{\langle a,x\rangle + y, \langle b,x\rangle + y\},$$

where a, b, and x are vectors from $\mathbb{R}^2$ and $y \in \mathbb{R}$.

Problems 1 and 2. Let a and b be given linearly independent vectors. Problem 1 (respectively, Problem 2) consists of the construction of a minimax solution of the problem (13), (14), (20) for $\sigma^\sharp = \varsigma^{\sharp+}$ (respectively, for $\sigma^\sharp = \varsigma^{\sharp-}$).

Since the function $\varsigma^{\sharp+}$ is convex and the function $\varsigma^{\sharp-}$ is concave, we can obtain explicit formulas for solutions of the Problems 1 and 2. One can prove that the functions

$$\phi^+(x,y) = \max_{l\in[a,b]} \phi_l(x,y), \quad \phi^-(x,y) = \min_{l\in[a,b]} \phi_l(x,y),$$

are solutions of these problems; here

$$[a,b] = \big\{\lambda a + (1-\lambda)b \mid \lambda \in [0,1]\big\}, \quad \phi_l(x,y) = \langle l,x\rangle + y + H(l).$$

The first step of the algorithm of constructing a solution $\varphi(\cdot)$ of the problem (13), (14), (23) consists of the consecutive solution of the problems 1 and 2 and gluing these solutions. Specific problems are determined by the function $\sigma(\cdot)$.

Further construction of the solution consists of solving elementary problems of another type.

Let $\bar{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$. Introduce the notation

$$\varphi_{\bar{s}}(x,y) = \langle s,x\rangle + s_3 \cdot y + H^*(\bar{s}), \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R},$$

where the vector $s \in \mathbb{R}^2$ is formed by the first two components of the vector $\bar{s}$, $s = (s_1, s_2)$. Note that if $s_3 = 1$, then $H^*(\bar{s}) = H(s)$ and $\varphi_{\bar{s}}(x,y) = \phi_s(x,y)$.

For a given set M, we denote its closure by $\operatorname{cl} M$ and its boundary by ∂M.

Problems 3 and 4. Consider linearly independent vectors $\bar{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $\bar{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ and a number $r > 0$. Let

$$\varphi^*(x,y) = \max\big\{\varphi_{\bar{a}}(x,y),\ \varphi_{\bar{b}}(x,y)\big\}, \quad \varphi_*(x,y) = \min\big\{\varphi_{\bar{a}}(x,y),\ \varphi_{\bar{b}}(x,y)\big\},$$
$$G^* = \big\{(x,y) \in \mathbb{R}^3 \mid \varphi^*(x,y) < r\big\}, \quad G_* = \big\{(x,y) \in \mathbb{R}^3 \mid \varphi_*(x,y) < r\big\}.$$

Problem 3: Construct a continuous function $\varphi^0 : \operatorname{cl} G^* \to \mathbb{R}$, which is a minimax solution of the first-order partial differential equation (13) in the domain G^* satisfying the relations

$$\varphi^0(x,y) < r \ \forall (x,y) \in G^*; \quad \varphi^0(x,y) = r \ \forall (x,y) \in \partial G^*.$$

Problem 4: Construct a continuous function $\varphi_0 : \operatorname{cl} G_* \to \mathbb{R}$, which is a minimax solution of Eq. (13) in the domain G_* satisfying the relations

$$\varphi_0(x,y) < r \ \forall (x,y) \in G_*; \quad \varphi_0(x,y) = r \ \forall (x,y) \in \partial G_*.$$

The function

$$\varphi^0(x,y) = \max_{\bar{s}} \varphi_{\bar{s}}(x,y) \quad \text{for} \quad \bar{s} \in S_r(\bar{a},\bar{b}),$$

where

$$S_r(\bar{a},\bar{b}) = \big\{\bar{s} \in \operatorname{con}(\bar{a},\bar{b}) \mid \langle s, w_0\rangle + H^*(\bar{s}) = r\big\},$$
$$\operatorname{con}(\bar{a},\bar{b}) = \big\{\lambda\bar{a} + \mu\bar{b} \mid \lambda \geq 0,\ \mu \geq 0\big\},$$

is a solution of the Problem 3. The point $w_0 \in \mathbb{R}^2$ is a solution of the system of two linear equations

$$\langle a, w_0\rangle + H^*(\bar{a}) = r, \quad \langle b, w_0\rangle + H^*(\bar{b}) = r.$$

The components of the vectors $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$ coincide with the first two components of the vectors $\bar{a}$ and $\bar{b}$, respectively.

The function

$$\varphi_0(x, y) = \min_{\bar{s}} \varphi_{\bar{s}}(x, y) \quad \text{for} \quad \bar{s} \in S_r(\bar{a}, \bar{b})$$

is a solution of the Problem 4 in the cases that appear in the construction of a solution $\varphi(\cdot)$ of the problem (13), (14), (23),

8. Main result. We denote by Ω the set of points of the space $\mathbb{R}^3$ at which the Hamiltonian $H^* : \mathbb{R}^3 \to \mathbb{R}$ is not differentiable. Let 0 be the null vector in $\mathbb{R}^3$. By a set $Z \subset \mathbb{R}^2$ (see (21)) of vectors that form the function σ, we define the set

$$Z^{\natural} = \big\{\bar{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 \mid s = (s_1, s_2) \in Z,\ s_3 = 1\big\} \subset \mathbb{R}^3.$$

Now we formulate the main result of this paper.

Theorem 2. *Let the conditions* **A2** *and* **A3** *be fulfilled. Then the following assertions hold.*

- **A.** *A solution $\varphi(\cdot)$ of the problem* (13), (14), (23) *is a nonnegative, piecewise linear function formed by gluing the linear functions*

$$\varphi_{\bar{s}}(x, y) = \langle s, x\rangle + s_3 \cdot y + H^*(\bar{s}), \quad \bar{s} \in L, \tag{24}$$

where the set L consists of a finite number of elements and

$$Z^{\natural} \subset L, \quad (L \setminus Z^{\natural}) \subset (\Omega \cup 0).$$

- **B.** *For any $y_* \in \mathbb{R}$, the function $\varphi(x, y_*)$ in the domain $\{x \in \mathbb{R}^2 \mid \varphi(x, y_*) > 0\}$ is formed by gluing a finite number of simple piecewise linear functions.*

The proof of Theorem 2 is based on the algorithm described above. The nonnegativity of the function $\varphi(\cdot)$ follows from the nonnegativity of the function $\sigma(\cdot)$ and the form (24) of linear functions that form the solution follows from specific elementary problems that arise in the course of its construction.

To prove that the function $\varphi(\cdot)$ constructed by the above algorithm is a minimax solution of Eq. (13), we must verify the inequalities (15) and (16). At points where $\varphi(\cdot)$ coincides with the solution of one of elementary problems considered above, these inequalities are fulfilled. Thus, we must verify the inequalities (15) and (16) on the surfaces of gluing solutions of different elementary problems that form the function $\varphi(\cdot)$. Note that if a gluing surface Γ does not belong to the domain of any solution of an elementary problem, then there exists a number $r \geq 0$ such that $\Gamma \subset \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid \varphi(x, y) = r\}$.

Let $(x_*, y_*) \in \Gamma$. If there exists a neighborhood $O_\varepsilon(x_*, y_*)$ in which the function φ is linear, then the inequalities (15) and (16) are fulfilled. Indeed, in this case there exists a vector $\bar{s}^* = \{s^*, s_3^*\}$, $s^* \in \mathbb{R}^2$, $s_3^* \in \mathbb{R}$, such that

$$\varphi(x, y) = \varphi_{\bar{s}^*} = \langle s^*, x\rangle + s_3^* \cdot y + H^*(\bar{s}^*), \quad (x_*, y_*) \in O_\varepsilon(x_*, y_*).$$

The function φ is differentiable in $O_\varepsilon(x_*, y_*)$ and $D^-\varphi(x_*, y_*) = D^+\varphi(x_*, y_*) = \bar{s}^*$ so that the inequalities (15) and (16) are fulfilled at the point (x_*, y_*) .

Consider the case where the point $(x_*, y_*) \in \Gamma$ is a gluing point of two linear functions. It follows from the algorithm that the following two situations are possible. In the first situation, for any $\varepsilon > 0$, there exists a point $(x_\varepsilon, y_\varepsilon)$ in the neighborhood $O_\varepsilon(x_*, y_*)$ at which the function φ is glued from the same linear functions and is a solution of a certain elementary problem; moreover, $\varphi(x_\varepsilon, y_\varepsilon) > r$. Obviously, in this situation

$$D^-\varphi(x_*, y_*) = D^-\varphi(x_\varepsilon, y_\varepsilon), \quad D^+\varphi(x_*, y_*) = D^+\varphi(x_\varepsilon, y_\varepsilon).$$

Since the inequalities (15) and (16) are fulfilled at the point $(x_\varepsilon, y_\varepsilon)$, we conclude due to the continuity that they also hold at the point (x_*, y_*).

In the second situation, in a neighborhood $O_\varepsilon(x_*, y_*)$ of the point (x_*, y_*), the function φ has the form

$$\varphi(x, y) = \max\left\{\varphi_{\bar a}(x, y),\ \varphi_{\bar b}(x, y)\right\},$$

where the vectors $\bar a$ and $\bar b$ from $\mathbb{R}^3$ are related by the formula $\bar b = \mu\bar a$, where $\mu > 0$ and the vector $\bar a$ is nonzero. We have $D^+\varphi(x_*, y_*)$,

$$D^-\varphi(x_*, y_*) = \left\{\bar l \in \mathbb{R}^3 \mid \bar l = \lambda\bar a,\ \lambda \in [0, 1]\right\} \cup \left\{\bar l \in \mathbb{R}^3 \mid \bar l = \lambda\bar b,\ \lambda \in [0, 1]\right\}.$$

Since the functions $\varphi_{\bar a}$ and $\varphi_{\bar b}$ are positively homogeneous and

$$\varphi(x_*, y_*) = \varphi_{\bar a}(x_*, y_*) = \varphi_{\bar b}(x_*, y_*) = r \geq 0,$$

we conclude that in this situation the inequalities (15) and (16) also holds.

Finally, we consider the case where (x_*, y_*) is a node point, i.e., a point at which n ($n \geq 3$) linear functions are glued:

$$\varphi_{\bar a^p}(x, y) = \langle a^p, x\rangle + a_3^p \cdot y + H^*(\bar a^p), \quad p = 1, \ldots, n. \tag{25}$$

The function φ is not differentiable at the point (x_*, y_*) and hence at least one of the sets $D^-\varphi(x_*, y_*)$ or $D^+\varphi(x_*, y_*)$ is empty. For definiteness, we assume that $D^+\varphi(x_*, y_*) = \varnothing$. If, in addition, $D^-\varphi(x_*, y_*) = \varnothing$, then the inequalities (15) and (16) are obviously fulfilled.

Assume that $D^-\varphi(x_*, y_*) \neq \varnothing$. We denote by $\varphi_\varepsilon(\cdot)$ the restriction of the function φ to a sufficiently small convex neighborhood $O_\varepsilon(x_*, y_*)$ such that in this neighborhood the function φ is glued only from the function (25). One can prove that

$$D^-\varphi(x_*, y_*) = D^-\tilde\varphi_\varepsilon(x_*, y_*),$$

where $\tilde\varphi_\varepsilon(\cdot)$ is the convex hull of the function φ_ε. Thus, $D^-\varphi(x_*, y_*)$ is a bounded, closed, convex set. Moreover, if $\bar l^* \in D^-\varphi(x_*, y_*)$, then there exist vectors $\bar a \in \mathbb{R}^3$ and $\bar b \in \mathbb{R}^3$ such that

$$\bar l^* \in [\bar a, \bar b] = \left\{(1 - \lambda)\bar a + \lambda\bar b \mid \lambda \in [0, 1]\right\},$$

and for any $n = 1, 2, \ldots,$ there exists a point $(x_n, y_n) \in O_{\varepsilon_n}(x_*, y_*)$ such that $D^-\varphi(x_n, y_n) = [a, b]$; here $\varepsilon_n = \varepsilon/n$. The points (x_n, y_n) are not nodal points; therefore, the inequality (15) does not hold at them. In particular,

$$H^*(\bar l^*) + \langle l, x_n\rangle + l_3 \cdot y_n \leq \varphi(x_n, y_n). \tag{26}$$

Passing in (26) to the limit as $n \to \infty$, we obtain

$$H^*(\bar l^*) + \langle l, x_*\rangle + l_3 \cdot y_* \leq \varphi(x_*, y_*);$$

this proves the inequality (15). The inequality (16) holds due to the fact that the superdifferential is empty.

The case where $D^+\varphi(x_*, y_*) \neq \varnothing$ and $D^+\varphi(x_*, y_*) = \varnothing$ is considered similarly. Thus, the function $\varphi(\cdot)$ constructed by the algorithm described above is a minimax solution of Eq. (13).

Acknowledgment. This work was supported by the Russian Foundation for Basic Research (project No. 17-01-00074) and the Ministry of Education and Science of the Russian Federation (project No. 02.A03.21.0006).

REFERENCES

1. M. Bardi and L. Evans, "On Hopf's formulas for solutions of Hamilton–Jacobi equations," *Nonlin. Anal. Th. Meth. Appl.*, **8**, No. 11, 1373–1381 (1984).
2. M. G. Crandall and P. L. Lions, "Viscosity solutions of Hamilton–Jacobi equations," *Trans. Am. Math. Soc.*, **377**, No. 1, 1–42 (1983).
3. E. Hopf, "Generalized solutions of nonlinear equations of first order," *J. Math. Mech.*, **14**, No. 6, 951–973 (1965).
4. L. V. Kamneva and V. S. Patsko, "Construction of the maximal stable bridge in games with simple motions on the plane," *Tr. Inst. Mat. Mekh.*, **20**, No. 4, 128–142 (2014).
5. N. N. Krasovsky and A. I. Subbotin, *Positional Differential Games*, Nauka, Moscow (1974).
6. N. N. Krasovsky and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, New York (1988).
7. M. Pachter and Y. Yavin, "Simple-motion pursuit-evasion differential games. Part 1. Stroboscopic strategies in collision-course guidance and proportional navigation," *J. Optim. Theory Appl.*, **51**, No. 1, 95–127 (1986).
8. L. A. Petrosjan, *Differential Games of Pursuit*, World Scientific, Singapore (1993).
9. B. N. Pshenichny and M. I. Sagaidak, "On differential games with fixed time," *Kibernetika*, No. 2, 54–63 (1970).
10. L. G. Shagalova, "A piecewise linear minimax solution of the Hamilton–Jacobi equation," *IFAC Proc. Vols.*, **31**, No. 13, 193–197 (1998).
11. L. G. Shagalova, "Piecewise linear price function of a differential game with simple motions," *Vestn. Tambov. Univ. Ser. Estestv. Tekhn. Nauki*, **12**, No. 4, 564–565 (2007).
12. A. I. Subbotin, *Minimax Inequalities and Hamilton–Jacobi Equations*, Nauka, Moscow (1991).
13. A. I. Subbotin, *Generalized Solutions of First Order PDEs. The Dynamical Optimization Perspective*, Birkhäuser, Boston (1995).
14. A. I. Subbotin and L. G. Shagalova, "Piecewise linear solution of the Cauchy problem for the Hamilton–Jacobi equation," *Dokl. Ross. Akad. Nauk*, **325**, No. 5, 144–148 (1992).
15. V. I. Ukhobotov and I. V. Tseunova, "Differential games with simple motions," *Vestn. Chelyabinsk. Univ.*, **11**, 84–96 (2009).

L. G. Shagalova
N. N. Krasovsky Institute of Mathematics and Mechanics
of the Ural Branch of the Russian Academy of Sciences;
Ural Federal University named after the First President of Russia B. N. Yeltsin
E-mail: `shag@imm.uran.ru`