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Phase Portraits of Stabilized Hamiltonian Systems in Growth Models

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Abstract. The paper investigates a qualitative behavior of solutions of Hamiltonian systems generated by the Pontryagin maximum principle for the optimal control problems at the infinite time horizon. Based on the stabilization procedure, authors show that the only possible portraits are stable focus or saddle. As an example, the resource consumption model is considered. By varying the discount factor in the model, authors demonstrate phase portraits in the corresponding optimal control problem.
Keywords: optimal control problems, Growth Models, Hamiltonian systems, Pontryagin maximum principle.

INTRODUCTION

The paper analyzes a qualitative behavior of the Hamiltonian systems arising in the Pontryagin maximum principle used for an optimal control problem at the infinite time interval. Many applied areas such as financial, demographic, economic, ecological etc., face with this type of problems. Stabilizability property of the Hamiltonian systems becomes very important sine it allows for constructing sub-optimal (stabilized) solutions of an optimal control problem and provide a good approximation of an optimal trajectories. As a result, these approximate solutions one can use for restoring the optimal strategies, at least for the problems of small dimensions (see [1, 6, 7]).

In [7], authors established conditions for the existence of the nonlinear stabilizer. This paper demonstrates possible phase portraits of the Hamiltonian systems based on the analysis of the steady state's character. The paper is organized as follows. Next section poses the problem, then we construct the Hamiltonian function and discuss its properties. The third section analyzes the corresponding Hamiltonian system and its Jacobian. Using properties of the Jacobi matrix, we investigate characters of a steady state. As an example, we refer on the previous work and consider the particular problem, for which we show that under different values of the model parameters, optimal solutions behave only in two possible ways depending on steady state character.

CONTROL PROBLEM

Many optimal control problems based on the growth models that describe development of the processes in many applied areas deal with the following dynamics

$$\dot{x}(t) = F(x(t))u(t) + G(x(t)) = \Phi(x(t),u(t)), \;\; x(0) = x_0 \, , \tag{1}$$

where $x\in\mathbb{R}^n$ is the vector of investigated factors, the functional matrix $F(x)=\{f_{ij}(x)\}^{n,m}_{i,j=1}$, and the vector–function $G(x)=\{g_i(x)\}^n_{i=1}$ are twice continuously differentiable functions. The symbol $u=(u_1, \ldots, u_m)$ stands for the control parameter. The quality of the control process is estimated by the functional of the form

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$$J(\cdot)=\int_0^{+\infty} e^{-\rho t}\ln c(x(t),u(t))dt\,, \tag{2}$$

where $c(x, u)$ is approximated by the equality

$$c(t)=\prod_{i=1}^{m}(1-u_i(t)-w_i(x(t)))f(x(t))\,, \tag{3}$$

Functions $f(x)$ and $w_i(x)$ ($i = 1, \ldots, m$) are supposed to be twice continuously differentiable. In the economic growth models, function $f(x)$ is called production function. In (3), each multiplier is strictly positive. This restriction is not critical, because otherwise, one can introduce an additional parameter $d(x)=f(x)\sum_{i=1}^{m}k_i(x)$ determining the debt, i.e. it simply means that the economy spend more goods than produces. In this case, the right-hand part is changed as follows $c(t)=\prod_{i=1}^{m}(1-u_i(t)-w_i(x(t))+k_i(x(t)))f(x(t))$.

The structure of the quality function (2) imposes an additional constraint on the controls u_i in (3), i.e.

$$0<\sum_{i=1}^{m}u_i(x)<1\Rightarrow\exists\bar{u}_i\in(0,1):u_i(t)\in[0,\bar{u}_i],\ i=1,2,\ldots,m\,. \tag{4}$$

In the growth models, these restrictions follow from the assumption on closedness of the designed system. Based on the introduced dynamics (1) and the quality functional (2), one can formulate the following control problem.

Problem. The problem is to construct such a control process ($x(t)$, $u(t)$) that maximizes the quality functional (2) along trajectories of the system (1) under the control restriction (4). The problem analysis is provided within the Pontryagin maximum principle for the control problems with infinite time interval (see [4], [1]).

CONTROL PROBLEM ANALYSIS

For Problem, the stationary Hamiltonian function that is independent explicitly on time has the following form

$$H(\cdot)=\sum_{i=1}^{m}\ln(1-u_i-w_i(x))+\ln f(x)+\psi^T\Phi(x,u)\,, \tag{5}$$

where $\psi\in\mathbb{R}^n$ is a vector of adjoint parameters. The Hamiltonian function is strictly concave with respect to the control parameters u (see [7]). Therefore, one can guarantee existence of such controls u that satisfy constraints (4) and provide maximum to the Hamiltonian function (5).

$$u_j(x,\psi)=\begin{cases}0, & (x,\psi)\in\Delta_j^1=\left\{(x,\psi):w_j(x)+\Gamma_j(x,\psi)\ge 1\right\},\ \Gamma_j(x,\psi)=\left(\psi^T F_j(x)\right)^{-1}\\ 1-w_j-\Gamma_j(x,\psi), & (x,\psi)\in\Delta_j^2=\left\{(x,\psi):1-u_j(x)\le w_j(x)+\Gamma_j(x,\psi)\le 1\right\}\\ \bar{u}_j, & (x,\psi)\in\Delta_j^3=\left\{(x,\psi):w_j(x)+\Gamma_j(x,\psi)\le 1-\bar{u}_j\right\},\quad j\in\{1,\ldots,m\}\end{cases}. \tag{6}$$

The derived structure of the control vector $u=u(x, \psi)$ allows for splitting up the $2n$-dimensional space of phase x and conjugate ψ variables in 3^m domains with different control regimes. The maximized Hamiltonian function constructed by the rule:

$$H(x,\psi)=H(x,\psi,u)\,, \tag{7}$$

is continuous and smooth in variables x and ψ in all domains $\Delta_j^k\left(j\in\{1,\ldots,m\},k\in\{1,2,3\}\right)$.

Once the Hamiltonian function is derived, we generate the Hamiltonian system according the formulae:

$$\begin{cases}\dot{x}(t)=\dfrac{\partial H(x,\psi)}{\partial\psi}, & x(0)=x_0\\ \dot{\psi}(t)=\rho\psi-\dfrac{\partial H(x,\psi)}{\partial x}, & \lim\limits_{t\to+\infty}e^{-\rho t}(x(t))^T\psi(t)=0\end{cases}. \tag{8}$$

Assumption. Let the Hamiltonian system (8) have a unique steady state $P^*=(x^*, \psi^*)$ with the strictly positive phase components $x\in\mathbb{R}^n_{>0}$ and non-zero conjugate elements $\psi_i^*\neq 0$, $i\in\{1,\ldots,n\}$. Using Assumption, the system (8) is linearized at the vicinity O_δ^* of the steady state P^*, and we derive the following linear system:

$$\begin{cases}\dot{\tilde{x}}=A\tilde{x}+B\tilde{\psi}, & \tilde{x}(t)=x(t)-x^*\\ \dot{\tilde{\psi}}=C\tilde{x}+\left(\rho E_n-A^T\right)\tilde{\psi}, & \tilde{\psi}(t)=\psi(t)-\psi^*\end{cases}. \tag{9}$$

where matrices A, B, and C are Jacobian blocks calculated as follows:

$$J^*=\begin{pmatrix} A & B \\ C & \rho E_n - A^T \end{pmatrix} := \begin{pmatrix} \dfrac{\partial^2 H(x^*,\psi^*)}{\partial\psi\partial x} & \dfrac{\partial^2 H(x^*,\psi^*)}{\partial\psi^2} \\ -\dfrac{\partial^2 H(x^*,\psi^*)}{\partial x^2} & \rho E_n - \dfrac{\partial^2 H(x^*,\psi^*)}{\partial x\partial\psi} \end{pmatrix}. \tag{10}$$

Next section studies the eigenvalues of the Jacobian (10).

EIGENVALUES OF THE JACOBIAN

Eigenvalues of a Jacobian determine qualitative behavior of the system (8) around the steady state P^*. It plays significant role for stabilization problem. Here, we show that the eigenvalues of the Jacobian (10) can be real or complex conjugate, and exactly n roots of the characteristics polynomial have positive real parts, while n other roots have negative real parts. Based on the results derived by Yu.S. Ledyaev in [2], the stabilization problem of the linearized system (9) requires for constructing a such matrix X that linearly connects phase variables and conjugate components $\tilde{\psi} = X\tilde{x}$ at the vicinity O_δ^* of the steady state P^* and asymptotically stabilizes the following system:

$$\dot{\tilde{x}} = (A + BX)\tilde{x}, \quad \tilde{\psi} = X\tilde{x}. \tag{11}$$

Consider an auxiliary linear system:

$$\begin{cases} \dot{\xi} = (A - \dfrac{\rho}{2}E_n)\xi + Bz, \quad \xi = \tilde{x}e^{-\rho/2t} \\ \dot{z} = C\xi - \left(A^T - \dfrac{\rho}{2}E_n\right)z, \quad z = \tilde{\psi}e^{-\rho/2t} \end{cases}. \tag{12}$$

and suppose that eigenvalues of the Jacobi matrix M of the system (12) does not have pure imaginary eigenvalues:

$$\mathrm{Re}(\lambda(M)) \neq 0, \quad where \ M = \begin{pmatrix} A - \dfrac{\rho}{2}E_n & B \\ C & \dfrac{\rho}{2}E_n - A^T \end{pmatrix}. \tag{13}$$

Remark. Matrix M is the Hamiltonian matrix (see [3, 7]). Therefore, its eigenvalues are symmetric with respect to the imaginary axes. In [7], authors prove that the Jacobian J^* and matrix M are related by the equality.

$$J^* = M + \frac{\rho}{2}E_{2n}. \tag{14}$$

The following theorem establishes the existence of the matrix X (see [11]).

Lemma 1. Matrix X stabilizing dynamics (12) does exist and can be found as a solution of the following matrix Riccati equation if the condition (13) holds.

$$C - X\left(A - \frac{\rho}{2}E_n\right) + \left(\frac{\rho}{2}E_n - A^T\right)X - XBX = 0. \tag{15}$$

Lemma 2. Matrix X constructed in Lemma 1 stabilizes the original linearized system (9) in the stable subspace Ω, where $\tilde{\psi} = X\tilde{x}$.

Lemma 2 shows that the Hamiltonian system is asymptotically stable by the first approximation around the steady state in the linear subspace Ω. In other words, the stabilized dynamics, derived by substituting $\psi = \psi^* + X(x - x^*)$ into the first n equations of Hamiltonian system (8), provides asymptotically stable system of the form:

$$\dot{x}(t) = \left.\frac{\partial H(x,\psi)}{\partial\psi}\right|_{\psi=\psi^*+X(x-x^*)}. \tag{16}$$

Around the steady state, the stabilized Hamiltonian system has stable focal or stable node character.

Remark. In the paper [9], for the resource productivity optimization problem, the phase portrait of the stabilized solution demonstrates the cyclic behavior (see Fig. 1, left). Another example can be found in [10], where for other values of the model parameters the model, mentioned in the paper [9], behaves in a different way and stabilizes at the steady state of the node type (see Fig. 1, right).

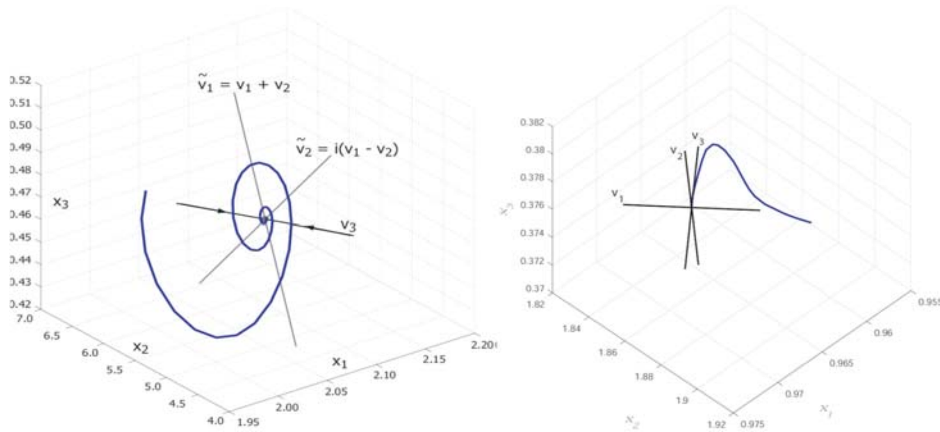

FIGURE 1. The stable focal behavior (left), the stable node behavior (right).

CONCLUSIONS

The paper discusses phase portraits of the stabilized Hamiltonian dynamics in the optimal control problems arising in growth models of different applied areas. It is shown that the existence of a steady state guarantees the stabilizability of the Hamiltonian dynamics at least locally in a vicinity of the equilibrium. This fact plays an important role for numerical algorithms synthesizing optimal solutions of the control problems.

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REFERENCES

1. Aseev, S.M., Kryazhimskiy, A.V. (2007). The Pontryagin maximum principle and optimal economic growth problems. Proc. of the Steklov Inst. of Math., volume 257
2. Ledyaev, Yu. (2011). On analytical solutions of matrix Riccati equations. Proc. of the Steklov Inst. of Math., 273(1), 214-228.
3. Paige, C., Loan, C.V. (1981). A Schur decomposition for Hamiltonian matrices. Linear Algebra and its Applications, 41, 11-32.
4. Pontryagin, L., Boltyanskii, V., and et al. (1962). The Mathematical Theory of Optimal Processes. Interscience, New York.
5. Tarasyev, A., Zhu, B. (2012). Optimal proportions in growth trends of resource productivity. In Proc. of the 15th IFAC Workshop CAO 12, 45(25), 182-187. Italy.
6. Tarasyev, A., Usova, A. (2010). Construction of a regulator for the Hamiltonian system in a two-sector economic growth model. Proc. of the Steklov Inst. of Math., 271, 1-21.
7. Tarasyev, A., Usova, A. (2015). Structure of the Jacobian in economic growth models. In Proc. of the 16th IFAC Workshop CAO 2015, 48(25), 191-196. Germany.
8. Tarasyev, A., Usova, A. (2016). Cyclic behavior of optimal trajectories in growth models. In Proc. of the 10th IFAC Symposium NOLCOS 2016, 49(18), 1048-1053. CA, USA.
9. Alexander Tarasyev, Anastasiia Usova, Wei Wang. (2015). Hamiltonian Trajectories in a Heterogeneous Economic Growth Model for Optimization Resource Productivity, In: IFAC-PapersOnLine, Volume 48, Issue 25, pp. 74–79.
10. Alexander Tarasyev, Anastasiia Usova. (2018). Robust methods for stabilization of Hamiltonian systems in economic growth models. In: IFAC-PapersOnLine, Volume 51, Issue 32, pp. 7-12.
11. Usova, A.A., Tarasyev, A.M. (2020) Structure of a Stabilizer for the Hamiltonian Systems. In proc.: Tarasyev, A., Maksimov, V., Filippova, T. (eds) Stability, Control and Differential Games. Lecture Notes in Control and Information Sciences - Proceedings. Springer, Cham.