Various types of completeness in topologized semilattices

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Abstract

A topologized semilattice X is called *complete* if each non-empty chain $C \subset X$ has $\inf C$ and $\sup C$ that belong to the closure $\overline{C}$ of the chain C in X. In this paper, we introduce various concepts of completeness of topologized semilattices in the context of operators that generalize the closure operator, and study their basic properties. In addition, examples of specific topologized semilattices are given, showing that these classes do not coincide with each other. Also in this paper, we prove theorems that allow us to generalize the available results on complete semilattices endowed with topology.

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1. Introduction

The use of various algebraic structures, additionally endowed with the structure of a topological space, has long established itself as a convenient and powerful tool in various fields of modern mathematics. This phenomenon motivates the fundamental study of the properties of these objects.

When studying spaces that have some additional structure consistent with the topology, quite often the concept of completeness naturally arises, as some internal property of these objects. In most cases, completeness can also be described as an external property, and often it is associated with the concept of absolute closedness, understood in a suitable sense. For example, a metric space X is complete if and only if it is closed in every metric space Y containing X as a metric subspace. A uniform space X is complete if and only if it is closed in every uniform space Y containing X as a uniform subspace. A topological group X is complete if and only if X, together with its two-sided uniform structure, is a complete uniform space, and so on. The completeness of semilattices is a well-studied algebraic property, which generalizes quite naturally (using the closure operator in a topological space) to semilattices endowed with a topological structure.

It should be noted that one of the first mathematicians who studied the absolute closedness of various topologized algebraic objects (including semilattices) was O. V. Gutik (see for example [16–18])

The question of the closedness of the images of complete topologized semilattices under continuous homomorphisms in Hausdorff semitopological semilattices was first raised by T. Banakh and S. Bardyla in [3] and is currently solved positively for some special cases. Historically, the

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first results in this direction belong to J. W. Stepp; in particular, he proved that semilattices having only finite chains are always closed in every topological semilattice containing it as a subsemilattice [15]. The above circumstances motivate the study of the notion of completeness of topologized semilattices in the context of operators that generalize the closure operator in a topological space.

The objectives of this paper are to determine the corresponding classes of semilattices endowed with topology, to study their basic properties, to construct examples showing that these classes do not coincide, and to generalize the well-known theorems on the closure of images of complete semilattices under continuous homomorphisms.

2. The completeness of topologized semilattices

A *semilattice* is any commutative semigroup of idempotents (an element x of a semigroup is called an *idempotent* if $xx = x$).

A semilattice endowed with a topology is called a *topologized semilattice*. A topologized semilattice X is called a *(semi)topological semilattice* if the semigroup operation $X \times X \to X$, $(x, y) \mapsto xy$, is (separately) continuous.

It is well known that semilattices can be viewed as partially ordered sets, namely: in every semilattice X we can consider the following order relation $\leq$: $x \leq y \leftrightarrow xy = x = yx$. Endowed with this partial order, the semilattice is a *poset*, i.e., partially ordered set. It is easy to see that the element xy is, in the sense of a given order, an infimum (a greatest lower bound) of the elements x and y. Conversely, if in a partially ordered set $(X, \leq)$ each pair of elements has a greatest lower bound, then X together with the operation of taking the infimum is a semilattice.

A subset D of a poset $(X, \leq)$ is called

• a *chain* if any elements $x, y \in D$ are comparable in the sense that $x \leq y$ or $y \leq x$. This can be written as $y \in \updownarrow x$ where $\uparrow x := \{y \in D : x \leq y\}$, $\downarrow x := \{y \in D : y \leq x\}$, and $\updownarrow x := (\uparrow x) \cup (\downarrow x)$;

• *up-directed* if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$;

• *down-directed* if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$.

It is clear that each chain in a poset is both up-directed and down-directed.

A semilattice X is called *chain-finite* if each chain in X is finite. A semilattice is called *linear* if it is a chain in itself.

In [15] Stepp proved that for any homomorphism $h : X \to Y$ from a chain-finite semilattice to a Hausdorff semitopological semilattice Y, the image $h(X)$ is closed in Y.

In [3] Banakh and Bardyla improved result of Stepp by proving that for any homomorphism $h : X \to Y$ from a chain-finite semilattice to a Hausdorff semitopological semilattice Y, the image $h(X)$ is closed in Y.

The notion of completeness of semilattices is a well-known algebraic property and is naturally transferred to topologized semilattices: a topologized semilattice X is called *complete* if each non-empty chain $C \subset X$ has $\inf C$ and $\sup C$ that belong to the closure $\overline{C}$ of the chain C in X.

A Hausdorff space X is said to be *H-closed* if it is closed in every Hausdorff space in which it can be embedded.

Complete topologized semilattices play an important role in the theory of (absolutely) H-closed semilattices, see [1-8]. By [1], a *Hausdorff semitopological semilattice X is complete if and only if each closed chain in X is compact if and only if for any continuous homomorphism $h : S \to Y$ from a closed subsemilattice $S \subset X$ to a Hausdorff topological semigroup Y the image $h(S)$ is closed in Y.*

A topologized semilattice X is called
• $\uparrow\downarrow$-*closed* if for each $x \in X$ the sets $\uparrow x$ and $\downarrow x$ are closed;
• *chain-compact* if each closed chain in X is compact.

On each topologized semilattice we shall consider weaker topologies:
- the *weak chain$^\bullet$-topology* generated by the subbase consisting of complements to closed chains in X,
- the *weak$^\bullet$ topology* $\mathcal{W}^\bullet_X$, generated by the subbase consisting of complements to closed subsemilattices in X.

A topologized semilattice X is called
• *chain-compact* if each closed chain in X is compact,
• *weak chain$^\bullet$ compact* if its weak chain$^\bullet$ topology is compact,
• $\mathcal{W}^\bullet_X$*-compact* if its weak$^\bullet$ topology $\mathcal{W}^\bullet_X$ is compact.

The weak$^\bullet$-topology $\mathcal{W}^\bullet_X$ was introduced and studied in [6]. According to Lemmas 5.4, 5.5 of [6], for any topologized semilattice we have the implications:
complete $\Rightarrow$ $\mathcal{W}^\bullet_X$-compact $\Rightarrow$ chain-compact.

By Theorem 5.4 in [7], a chain-compact $\uparrow\downarrow$-closed topologized semilattice is complete.

Note that the weak chain$^\bullet$ topology of a topologized semilattice X is obviously contained in the weak$^\bullet$- topology, which immediately implies:

Lemma 2.1. *A complete topologized semilattice X is weak chain$^\bullet$ compact.*

It is also easy to prove the following statement.

Lemma 2.2. *A weak chain$^\bullet$-compact topologized semilattice X is chain-compact.*

Proof. Let C be a closed chain in X. Consider a centered family $\{F_\alpha\}$ of closed subsets of C. Then every set F_α is also a chain and is closed in X; this means that F_α is also closed in the weak chain$^\bullet$-topology on X. Since X is a weak chain$^\bullet$ compact semilattice, $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \neq \emptyset$. □

From this we obtain the following theorem, which is necessary for the study of other types of completeness of semilattices, discussed in the following chapters.

Theorem 2.3. *For a $\uparrow\downarrow$-closed topologized semilattice X, the following statements are equivalent:*
1) X is complete;
2) X is weak chain$^\bullet$ compact;
3) X is chain compact.

A *multi-valued map* $\Phi : X \multimap Y$ between sets X, Y is a function assigning to each point $x \in X$ a subset $\Phi(x)$ of Y. The *image* of any set $A \subset X$ under a multi-valued map Φ is called the set $\Phi(A) = \bigcup\limits_{x\in A} \Phi(x)$, the *preimage* of any set $B \subset Y$ is the set $\Phi^{-1}(B) = \{x \in X : \Phi(x) \cap B \neq \emptyset\}$. A multi-valued map $\Phi : X \multimap Y$ between semigroups is called a *multimorphism* if $\Phi(x)\Phi(y) \subset \Phi(xy)$ for any elements $x, y \in X$. Here $\Phi(x)\Phi(y) := \{ab : a \in \Phi(x), b \in \Phi(y)\}$.

A multi-valued map $\Phi : X \multimap Y$ between topological spaces is called *upper semicontinuous* if for any closed subset $F \subset Y$ the preimage $\Phi^{-1}(F)$ is closed in X.

A subset F of a topological space X is called T_1-*closed* (resp. T_2-closed) in X if each point $x \in X \setminus F$ has a (closed) neighborhood, disjoint with F.

A multimorphism $\Phi : X \multimap Y$ is called a *T_i-multimorphism* for $i \in \{1, 2\}$ if for any $x \in X$ the set $\Phi(x)$ is T_i-closed in Y.

It was shown in [2] that the completeness of semilattices is preserved by images under continuous homomorphisms.

Note that the map $\Phi : X \multimap Y$ where $\Phi(x) = Y$ (between semilattices X and Y) always is an upper semicontinuous T_1 multimorphism; it shows the images of semilattices under maps of this type do not preserve completeness. However, a positive result was achieved by imposing additional algebraic constraints, which will be shown in Theorem 2.5. To prove it, we need the following simple proposition.

Proposition 2.4. *A $\uparrow\downarrow$-closed topologized semilattice X is complete if and only if each non-empty closed chain C contains* $\inf C$ *and* $\sup C$.

Proof. Necessity obviously follows from the definition of completeness.

Let $C \subset X$ be a non-empty chain. By Lemma 4.2, proved in [7], the set $\overline{C}$ is also a chain. Let a be a the smallest element of $\overline{C}$; the inclusion $C \subset \overline{C}$ implies that a is the lower bound of the set C. If we assume that there is a lower bound c of chain C such that $c \not\leq a$, then $a \notin \uparrow c \supset \overline{C}$, a contradiction. Hence, $a = \inf C$. Similarly, we can show that the largest element b of the chain $\overline{C}$ is $\sup C$, which completes the proof. □

Theorem 2.5. *Let X be a complete topologized semilattice, Y be a $\uparrow\downarrow$-closed topologized semilattice and $\Phi : X \multimap Y$ be an upper semicontinuous T_1 multimorphism such that for any two points $x, y \in X$ inequality $x \leq y$ implies $\Phi(x) \cap \uparrow\Phi(y) \subset \Phi(y)$. Then the semilattice $\Phi(X)$ is complete if and only if the semilattice $\Phi(x)$ is complete for each $x \in X$.*

Proof. Let us first prove sufficiency. Since Φ is a T_1 multi-valued map, $\Phi(x)$ is closed for every $x \in X$ and, hence, it is complete.

Let $\Phi(x)$ be a complete semilattice for each $x \in X$ and let $C \subset Y$ be a closed chain. Since the semilattice Y is $\uparrow\downarrow$-closed, it is sufficient to show that C contains the largest and smallest elements. Note that the semilattice $\Phi^{-1}(C)$ is closed in X and, hence, it is complete. Consider the map $\Phi_C : \Phi^{-1}(C) \multimap C$ such that $\Phi_C(x) = \Phi(x) \cap C$. We show that Φ_C is an upper semicontinuous T_1 multimorphism. Since $\Phi_C(x)\Phi_C(y) := (\Phi(x) \cap C)(\Phi(y) \cap C) \subset \Phi(xy) \cap C = \Phi_C(xy)$ (inclusion holds due to the fact that Φ is a multimorphism and C is a semilattice), then Φ_C is a multimorphism.

Let $F \subset C$ be a closed subset of C. Then F is closed in Y and $\Phi_C^{-1}(F) = \{x \in \Phi^{-1}(C) : \Phi_C(x) \cap F \neq \emptyset\} = \{x \in \Phi^{-1}(C) : \Phi(x) \cap C \cap F \neq \emptyset\} = \Phi^{-1}(C)$, that is $\Phi(C)$ is upper semicontinuous. Finally, $\Phi_C(x) = \Phi(x) \cap C$ is closed in $\Phi(x)$ and therefore is complete as closed subsemilattice of semilattice $\Phi(x)$.

For each $c \in C$ consider a closed semilattice $S_c = \Phi_C^{-1}(\uparrow c)$. Since X is complete, $\inf S_c \in \overline{S_c} = S_c$. Take $y \in C$ such that $c \in \Phi_C(y)$. Since $\Phi_C(\inf S_c) \cap \uparrow c \neq \emptyset$, then $c \in \Phi_C(\inf S_c)\Phi_C(y) \subset \Phi_C(y \inf S_c) = \Phi(\inf S_c)$. Note that for $c_1, c_2 \in C$ the inequality $c_1 \leq c_2$ implies the inclusion $S_{c_2} \subset S_{c_1}$, whence it follows that $\inf S_{c_1} \leq \inf S_{c_2}$,i.e. the set $S = \{S_c\}_{c \in C}$ is a chain. By the completeness of X, $\inf S \in \overline{S} \subset \Phi^{-1}(C)$. Since the chains $\Phi_C(\inf S)$ and $\Phi_C(\sup S)$ are complete and closed, they contain the largest and the smallest elements. Let $a = \min \Phi_C(\inf S)$, $b = \max \Phi_C(\sup S)$. We show that a and b are the smallest and largest elements of C, respectively. Indeed, suppose that the chain C contains the element $c > b$. Then $\inf S_c \leq \sup S$ and, by the condition, $c \in \Phi_C(\inf S_c) \cap \uparrow\Phi_C(\sup S) \subset \Phi_C(\sup S)$. Since b is a largest element of $\Phi_C(\sup S)$, $c \leq b$.

Now suppose that there is an element $c < a$. Then $c \in \Phi_C(\inf S_c)\Phi_C(\inf S) \subset \Phi_C(\inf S)$ and $c \geq a$ because of a is the smallest element of $\Phi_C(\inf S)$. The resulting contradictions complete the proof. □

Note important special case of Theorem 2.5.

Corollary 2.6. Let X be a complete topologized semilattice, let Y be a $\uparrow\downarrow$-closed topologized semilattice, $\Phi : X \multimap Y$ be a upper semicontinuous T_1 multimorphism such that $\Phi(x) \cap \Phi(y) = \emptyset$ for $x \neq y$. Then the semilattice $\Phi(X)$ is complete if and only if the semilattice $\Phi(x)$ is complete for each $x \in X$.

Proof. We show that the inequality $x \leq y$ for $x, y \in X$ implies $\Phi(x) \cap \uparrow\Phi(y) = \emptyset$. Suppose the opposite. Let $a \in \Phi(x) \cap \uparrow\Phi(y)$. This means that there is $b \in \Phi(y)$ such that $b \leq a$. But then $b = ab \in \Phi(x)\Phi(y) \subset \Phi(xy) = \Phi(x)$ and $b \in \Phi(x) \cap \Phi(y)$, which is a contradiction. □

Theorem 2.5 allows to generalize the results on the closure of semilattices.

The *Lawson number* $\overline{\Lambda(X)}$ of a Hausdorff topologized semilattice X is defined as the smallest cardinal κ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a subsemilattice of X that does not contain y. A topologized semilattice X is ω-Lawson if and only if it is Hausdorff and has at most countable Lawson number $\overline{\Lambda(X)}$.

A topological space X is called *functionally Hausdorff* if for any two points $x, y \in X$ there is a continuous real-valued function f such that $f(x) \neq f(y)$.

A space X is *sequential* if for non-closed set A there is a sequence of elements A converging to some point $x \in \overline{A} \setminus A$.

In [3–5] it was proved that a complete subsemilattices of semitopological functionally Hausdorff (sequential Hausdorff, ω-Lawson) semilattice are closed. It is also known that a semitopological semilattice is $\uparrow\downarrow$-closed [12]. These results, together with Theorem 2.5, allow us to formulate:

Corollary 2.7. Let X be a complete topologized semilattice, and let Y be an ω-Lawson (functionally Hausdorff, sequential) semitopological semilattice, $\Phi : X \multimap Y$ be an upper semicontinuous T_1 multimorphism such that for any two points $x, y \in X$ inequality $x \leq y$ implies $\Phi(x) \cap \uparrow\Phi(y) \subset \Phi(y)$ and the semilattice $\Phi(x)$ is complete for each $x \in X$. Then the set $\Phi(X)$ is closed in Y.

We can also reformulate the question of the closure of semilattices in terms of preimages under certain maps.

A point x of a topological space X is called *θ-adherent point* of the set $A \subset X$ if $A \cap \overline{U} \neq \emptyset$ for any neighborhood U of x.

The following concepts was introduced by N.V. Velichko in [13].

• The θ-closure of a subset A of a topological space X is called the set $\overline{A}^{\theta} = \{x \in X : x$ is a θ-adherent point of $A\}$.

• A subset A of a topological space X is called θ-closed if $\overline{A}^{\theta} = A$.

Theorem 2.8. *Let X be a subsemilattice of a topological semilattice Y. If there exists a closed homomorphism $h : X \to E$ from X to a complete topologized semilattice E such that for each $e \in E$ the set $h^{-1}(e)$ is θ-closed in Y, then X is closed in Y.*

Proof. Since h is a closed map, the image $h(X)$ is closed in E. Given that completeness is inherited by closed subsemilattices, we can assume, without loss of generality, that h is a surjective map. We define a multi-valued map $\Phi : E \multimap Y$ as follows: $\Phi(e) = h^{-1}(e)$. We show that Φ is an upper semicontinuous T_2 multimorphism.

Claim that $\Phi^{-1}(F) = h(F)$ for each $F \subset X$.

Take an element $x \in \Phi^{-1}(F)$. By definition of the preimage of a set under a multivalued map $\Phi(x) \cap F = h^{-1}(x) \cap F \neq \emptyset$ and, hence, there is $z \in F$ such that $h(z) = x$ and $x \in h(F)$. Now take $e \in h(F)$ and find $y \in F$ such that $h(y) = e$. This means that $y \in F \cap h^{-1}(e) = F \cap \Phi(e)$, so $e \in \Phi^{-1}(F)$; the resulting inclusions prove the equality $\Phi^{-1}(F) = h(F)$.

Since h is a closed map, $\Phi^{-1}(F) = \Phi^{-1}(F \cap X) = h(F \cap X)$ is closed in E for every closed set $F \subset Y$, so Φ is upper semicontinuous. Since $\Phi(e) = h^{-1}(e)$ is θ-closed in Y, the multi-valued map Φ has the property T_2.

Now we check that Φ is a multimorphism. Take $e_1, e_2 \in E$ and $x_1 \in \Phi(e_1) = h^{-1}(e_1), x_2 \in \Phi(e_2) = h^{-1}(e_2)$. Since h is a homomorphism, $e_1e_2 = h(x_1)h(x_2) = h(x_1x_2)$ and $x_1x_2 \in h^{-1}(e_1e_2)$. Then we have that $\Phi(e_1)\Phi(e_2) = h^{-1}(e_1)h^{-1}(e_2) \subset h^{-1}(e_1e_2) = \Phi(e_1e_2)$, that is, Φ is a multimorphism.

To complete the proof, it remains only to note that the closedness of $\Phi(E) = X$ in Y now follows from Theorem 2.1 in [1]. □

Corollary 2.9. Let X be a subsemilattice of a regular semitopological semilattice Y. If there exists a closed homomorphism $h : X \to E$ from X into a complete topologizing semilattice E such that for each $e \in E$ the set $h^{-1}(e)$ is closed in Y, then X is closed in Y.

Proof. It immediately follows from the fact that in regular spaces closure and θ-closure operators coincide. □

3. The δ-completeness of topologized semilattices

A point x of topological space X is called *δ-adherent point* of a set $A \subset X$ if $A \cap Int\overline{U} \neq \emptyset$ for any neighborhood U of x.

The *δ-closure* of a subset A of a topological space X is called the set $\overline{A}^{\delta} = \{x \in X : x$ is a δ-adherent point of $A\}$.

A subset A of a topological space X is called δ-closed, if $\overline{A}^{\delta} = A$.

The concept of δ-closure was introduced by N.V. Velichko in [13]. It is also proved that the intersection and finite union of δ-closed sets is δ-closed. Obviously, the empty set and the entire space are δ-closed sets. It follows that for any topological space (X, τ) there exists a topology τ_δ such that closed (in (X, τ_δ)) sets are exactly δ-closed sets of the space (X, τ). It is easy to check that the δ-closure of a set is a δ-closed set. It follows that $\overline{A}^{\delta}$ is the intersection of all δ-closed sets containing A. Now it is easy to see that the closure operator in (X, τ_δ) is the same as the δ-closure operator in (X, τ).

Complements to δ-closed sets are called δ-open sets.

Definition 3.1. A topologized semilattice X is called *δ-complete* if each non-empty chain $C \subset X$ has $\inf C$ and $\sup C$ that belong to the δ-closure $\overline{C}^{\delta}$ of the chain C in X.

It follows that a topologized semilattice (X, τ) is δ-complete if and only if the semilattice (X, τ_δ) is complete.

Definition 3.2. A topologized semilattice X is called *δ-$\uparrow\downarrow$-closed* if for each $x \in X$ sets $\uparrow x$ and $\downarrow x$ are δ-closed in X.

Definition 3.3. A *weak δ-chain$^\bullet$-topology* on a topologized semilattice X is called a topology generated by a subbase consisting of complements to δ-closed chains in X.

Definition 3.4. The topologized semilattice X is called *weak δ-chain$^\bullet$ compact*, if X is compact in its weak *δ-chain$^\bullet$* topology.

We now formulate an analog of Theorem 1.6 for δ-complete semilattices.

Theorem 3.5. *For a δ-$\uparrow\downarrow$-closed topologized semilattice (X, τ) the following conditions are equivalent:*

1) X is δ-complete;

2) X is weak δ-chain$^\bullet$ compact;

3) any δ-open cover of δ-closed (in X) chain C contains finite subcover.

Proof. Since the closure operator in the semilattice (X, τ_δ) coincides with the δ-closure operator in (X, τ), the semilattice (X, τ_δ) is complete. Recall that closed sets in (X, τ_δ) are exactly δ-closed sets in (X, τ). The statement of this theorem now follows from Theorem 2.3. □

4. The θ-completeness of topologized semilattices

Definition 4.1. A topologized semilattice X is called *θ-complete*, if for each non-empty chain $C \subset X$ $\inf C \in \overline{C}^\theta$ and $\sup C \in \overline{C}^\theta$.

Definition 4.2. A topologized semilattice X is called *θ-$\uparrow\downarrow$-closed* (*θ-$\updownarrow$-closed*), if for any element $x \in X$ sets $\uparrow x$ and $\downarrow x$ (set $\updownarrow x$) are θ-closed.

Note that unlike the closure and δ-closure operators, the θ-closure operator is not necessarily idempotent, which makes the class of θ-complete semilattices a bit more interesting.

Definition 4.3. Let X be a topologized semilattice and $D \subset X$ is up (down)-directed. We say that D *up-θ-converges* (*down-θ-converges*) to the point $x \in X$, if for any neighborhood U of x there is $d \in D$ such that $D \cap \uparrow d \subset \overline{U}$ ($D \cap \downarrow d \subset \overline{U}$).

Lemma 4.4. *Let X be a θ-complete topologized semilattice. Then any up-directed set $D \subset X$ up-θ-converges to* $\sup D$.

Proof. Suppose the opposite. Let $D \subset X$ be a up-directed set does not up-θ-converge to $\sup D$. Then there exists a neighborhood U of the point $\sup D$ that the set $(D \cap \uparrow d) \setminus \overline{U} \neq \emptyset$ for each $d \in D$.

We claim that the set $E = D \setminus \overline{U}$ is up-directed. Indeed, let $e_1, e_2 \in E$; then there is $d \in D$ such that $d \geq e_1$ and $d \geq e_2$. Since $(D \cap \uparrow d) \setminus \overline{U} \subset E$, there is $e' \in E$ such that $e' \geq d \geq e_1$ and $e' \geq d \geq e_2$, as required. Note that $\sup D = \sup E$.

Since X is a θ-complete semilattice, $\sup E = \sup D \in \overline{E}^\theta$. But, on the other hand, $E \cap \overline{U} = \emptyset$, hence, $\sup E \notin \overline{E}^\theta$, because $\sup E$ is an inner point of $\overline{U}$. □

The following statement is proved in exactly the same way.

Lemma 4.5. *Let X be a θ-complete topologized semilattice. Then any down-directed set $D \subset X$ down-θ-converges to* $\inf D$.

Lemma 4.6. *Let X be a θ-$\updownarrow$-closed semilattice. Then for any chain $C \subset X$, the set $\overline{C}^{\theta}$ is a chain.*

Proof. Suppose the opposite, let the set $\overline{C}^{\theta}$ contain incomparable elements x and y. Since $x \notin \updownarrow y$ and $\updownarrow y$ is θ-closed, there is a neighborhood U of the point x such that $\overline{U} \cap \updownarrow y = \emptyset$. Since $x \in \overline{C}^{\theta}$, there is $z \in \overline{U} \cap C$. Since $z \notin \updownarrow y$, we have $y \notin \updownarrow z$. Then there is a neighborhood V of the point y such that $\overline{V} \cap \updownarrow z = \emptyset$, which impossible, since $C \subset \updownarrow z$ and $y \in \overline{C}^{\theta}$. □

Theorem 4.7. *Let X be a θ-complete, θ-$\uparrow\downarrow$-closed topologized semilattice, and $C \subset X$ be a chain. Then $\overline{C}^{\theta}$ is a θ-closed set.*

Proof. Assume that $\overline{C}^{\theta}$ is not θ-closed and there exists $x \in \overline{C}^{\theta^2} \setminus \overline{C}^{\theta}$ (where $\overline{C}^{\theta^2} = \overline{\overline{C}^{\theta}}^{\theta}$). Note that, by Lemma 4.6, the sets $\overline{C}^{\theta}$ and $\overline{C}^{\theta^2}$ are chains. By θ-completeness, $\inf C \in \overline{C}^{\theta}$ and $\sup C \in \overline{C}^{\theta}$. Since X is a θ-$\uparrow\downarrow$-closed semilattice, $\overline{C}^{\theta^2} \subset \uparrow \inf C \cap \downarrow \sup C$, hence, $\inf C \leq x$ and $\sup C \geq x$. By choice x, $\inf C \neq x$ and $\sup C \neq x$, hence, $\inf C < x < \sup C$. Note that there are elements $c_1, c_2 \in C$ such that $c_1 < x < c_2$ and, hence, sets $\downarrow x \cap C$ and $\uparrow x \cap C$ non empty. Since X is a θ-complete semilattice, $a := \sup(\downarrow x \cap C) \in \overline{\downarrow x \cap C}^{\theta} \subset \overline{C}^{\theta}$ è $b := \inf(\uparrow x \cap C) \in \overline{\uparrow x \cap C}^{\theta} \subset \overline{C}^{\theta}$. By the choice of x, the double inequality $a < x < b$ is satisfied. Then x does not belong to θ-closed set $\downarrow a \cup \uparrow b$; this means that there is a neighborhood U of the point x such that $\overline{U} \cap \downarrow a \cup \uparrow b = \emptyset$. Since $x \in \overline{C}^{\theta^2}$, there is $y \in \overline{U} \cap \overline{C}^{\theta}$. Since $\overline{C}^{\theta}$ is a chain and $y \notin \downarrow a \cup \uparrow b$, we again the double inequality $a < y < b$. Let us now find a neighborhood V of y such that $\overline{V} \cap \downarrow a \cup \uparrow b = \emptyset$ and element $c \in \overline{V} \cap C$. Since $c \notin \downarrow a \cup \uparrow b$, $a < c < b$. This inequality leads us to a contradiction: since $c \in C$, if $c > a = \sup(\downarrow x \cap C)$ then $c > x$ (otherwise $c \leq a$); similarly, if $c < b = \inf(\uparrow x \cap C)$ then $c < x$. □

A subset M of a topological space X is an *H-set* if every cover of it by open sets of X has a finite subfamily which covers M with the closures of its members.

The concept of an H-set, which generalizes the concept of a compact subset of a space, was introduced by N.V. Velichko in [13].

Recall that a topological space X is called *Urysohn space*, if for any two distinct points $x, y \in X$ there are neighborhoods U_x, U_y of points x, y such that $\overline{U_x} \cap \overline{U_y} = \emptyset$.

The following results are well known:

- Every θ-closed subset of an H-closed space is an H-set [14].
- If X is H-closed and Urysohn, then $M \subset X$ is θ-closed if and only if it is an H-set [9].
- M is an H-set of a space X if and only if for every filter $\mathcal{F}$ on X, which meets M, $M \cap ad_{\theta}\mathcal{F} \neq \emptyset$, where $ad_{\theta}\mathcal{F} = \bigcap\{\overline{F}^{\theta} : F \in \mathcal{F}\}$ [10].

Lemma 4.8. *Let X be a θ-complete topologized semilattice. Then every θ-closed chain C is an H-set.*

Proof. Let C be a non-empty θ-closed chain in X. Consider a family of open sets $\mathcal{U}$, covering C. Let A be a set of points $a \in C$ such that the set $\downarrow x \cap C$ can be covered by a finite subfamily $\overline{\mathcal{U}}$. $A \neq \emptyset$, since it obviously contains the smallest element c of the chain C. By θ-completeness of X, $\sup A \in \overline{A}^{\theta} \subset \overline{C}^{\theta} = C$.

We show that $b := \sup A \in A$. Suppose that $b \notin A$. Choose $U_b \in \mathcal{U}$ such that $b \in U_b$. By Lemma 4.4, there exists a point $a \in A$ such that $A \cap \uparrow a \subset \overline{U_b}$. By definition of A, $x \in A$ implies $\downarrow x \cap C \subset A$. By assumption, $a < \sup A = b$, so for each $a \leq x \leq b$ there is $y \in A$ such that

$x \le y \le b$, i.e., $A \cap \uparrow a = (C \cap \uparrow a \cap \downarrow b) \setminus \{b\}$. By definition of A, there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \overline{\mathcal{V}} \supset \downarrow a \cap C$. Let $\mathcal{U}' = \mathcal{V} \cup \{U_b\}$. Then $\overline{\mathcal{U}'}$ is the finite cover of $\downarrow b \cap C$, and $b \in A$.

Now we claim that $C = \downarrow b \cap C$. Suppose that $E := C \setminus \downarrow b \neq \emptyset$. Then $e := \inf E \in \overline{E}^{\theta} \subset \overline{C}^{\theta} = C$. Note that $a \in \downarrow b \cap C = A$ for every $a \in C$ such that $a < e$.

Choose $U_e \in \mathcal{U}$ such that $e \in U_e$. Obviously that $\mathcal{U}' \cup \{\overline{U_e}\}$ is the finite cover of $\downarrow e \cap C$, and $e \in A$. Note that b is a lower bounded of $C \setminus \downarrow b = E$, and $b \le e$. Consider two cases:

1) $b < e$. Since $e \in A$, this contradicts the equality $b = \sup A$.

2) $b = e$. By Lemma 4.5, there exists $d \in E$ such that $E \cap \downarrow d \subset \overline{U_b}$. Note that $d > e = b$ ($e \notin E$). Then $\downarrow d \cap C \subset \bigcup \overline{\mathcal{U}'}$ and $d \in A$, which contradicts the equality $b = \sup A$. □

Combining this Lemma with Theorem 4.7, we obtain

Corollary 4.9. Let X be a θ-complete θ-$\uparrow\downarrow$-closed topologized semilattice. Then for each chain $C \subset X$ the chain $\overline{C}^{\theta}$ is an H-set.

Lemma 4.10. *Let X be a θ-$\uparrow\downarrow$-closed semilattice in which $\overline{C}^{\theta}$ is an H-set for any chain $C \subset X$. Then X is a θ-complete semilattice.*

Proof. Let $C \subset X$ be a chain. By Lemma 4.6, $\overline{C}^{\theta}$ is also chain. Consider a centered family $\mathcal{F}_< = \{\overline{C}^{\theta} \cap \downarrow x : x \in C\}$ subsets $\overline{C}^{\theta}$. Since the chain $\overline{C}^{\theta}$ is an H-set and $\overline{\overline{C}^{\theta} \cap \downarrow x}^{\theta} \subset \overline{\downarrow x}^{\theta} = \downarrow x$, we have $\bigcap \mathcal{F}_< = \bigcap\limits_{x \in C} \overline{C}^{\theta} \cap \downarrow x \supset \bigcap\limits_{x \in C} \overline{\overline{C}^{\theta} \cap \downarrow x}^{\theta} \cap \overline{C}^{\theta} \neq \emptyset$ by the criterion of the H-set mentioned above and proved in [10]. Obviously, $\bigcap \mathcal{F}_<$ contains the only element c that is the smallest in $\overline{C}^{\theta}$. Clearly, $c \le x$ for each $x \in C$; suppose that there is $a < c$, and $a = inf C$. Then $c \notin \uparrow a \supset \overline{C}^{\theta}$, which contradicts the choice c. Thus, $c = \inf C \in \overline{C}^{\theta}$; similarly, it can be shown that the intersection of a centered family $\mathcal{F}_> = \{\overline{C}^{\theta} \cap \uparrow x : x \in C\}$ contains $\sup C$. □

By Corollary 4.9 and Lemma 4.10 we have the following theorem.

Theorem 4.11. *A θ-$\uparrow\downarrow$-closed topologized semilattice X is θ-complete if and only if for any chain $C \subset X$ the chain $\overline{C}^{\theta}$ is an H-set.*

We show that a slightly more general result can be obtained from other arguments. We give the necessary definitions.

Definition 4.12. Let X be a topologized semilattice. The *weak θ-chain$^{\bullet}$-topology* on X is called a topology generated by a subbase consisting of complements to θ-closures of chains in X.

Definition 4.13. A topologized semilattice X is called *weak θ-chain$^{\bullet}$ compact* if X is compact in its weak θ-*chain*$^{\bullet}$ topology.

Lemma 4.14. *Let (X, τ) be a θ-complete θ-$\uparrow\downarrow$-closed topologized semilattice. Then X is weak θ-chain$^{\bullet}$ compact.*

Proof. In [13], it was proved that the intersection and finite union of θ-closed sets is θ-closed. Obviously, the empty set and the entire space are θ-closed sets. It follows that for any topological space (X, τ) there exists a topology τ_θ such that closed (in (X, τ_θ)) sets are exactly θ-closed sets of the space (X, τ).

Clearly, the θ-closure of a set $A \subset X$ in (X, τ) is contained in the closure of A in (X, τ_θ). It follows that a semilattice (X, τ_θ) is complete and $\uparrow\downarrow$-closed. By Lemmas 4.6 and 4.7, for each chain C the chain $\overline{C}^\theta$ is θ-closed. It follows that the weak θ-chain$^\bullet$-topology on X is contained in the weak$^\bullet$ $\mathcal{W}^\bullet$-topology of the space (X, τ). By Lemma 5.4 of [6], the semilattice (X, τ_θ) is compact in its weak$^\bullet$ $\mathcal{W}^\bullet$-topology. Moreover, (X, τ) is weak θ-chain$^\bullet$ compact. □

Lemma 4.15. *Let X be a weak θ-chain$^\bullet$ compact, θ-$\updownarrow$-closed topologized semilattice. Then $\overline{C}^\theta$ is an H-set for any chain $C \subset X$.*

Proof. Let $C \subset X$ be a chain. We show that for any centered family $\{F_\alpha\}_{\alpha\in\Lambda}$ of subsets $\overline{C}^\theta$, $\bigcap\limits_{\alpha\in\Lambda} \overline{F_\alpha}^\theta \cap \overline{C}^\theta \neq \emptyset$. By Lemma 4.6, the set $\overline{C}^\theta$ is a chain; then so is F_α. This means that the sets $\overline{F_\alpha}^\theta$ closed in weak θ-*chain*$^\bullet$ topology on X. □

Theorem 4.16. *For a θ-$\uparrow\downarrow$-closed semilattice X the following statements are equivalent:*

1) X is θ-complete;

2) X is θ-chain$^\bullet$ compact;

3) $\overline{C}^\theta$ is an H-set for any chain $C \subset X$.

Lemma 4.17. *Let X be a Urysohn space, $r : X \to X$ be a retraction. Then the set $r(X)$ is θ-closed in X.*

Proof. Suppose that $\overline{r(X)}^\theta \setminus r(X) \neq \emptyset$ and let $x \in \overline{r(X)}^\theta \setminus r(X)$. Note that $r(x) \neq x$. Find neighborhoods $U_x, U_{r(x)}$ of x and $r(x)$, respectively, such that $\overline{U_x} \cap \overline{U_{r(x)}} = \emptyset$. By the continuity of r, there is a neighborhood $V_x \subset U_x$ of x such that $r(V_x) \subset U_{r(x)}$. Clearly, then $\overline{V_x} \cap \overline{U_{r(x)}} = \emptyset$. Since $x \in \overline{r(X)}^\theta$, $\overline{V_x} \cap r(X) \neq \emptyset$. Let $z \in \overline{V_x} \cap r(X)$. Since $r(z) = z$ and r is continuous, $z \in r(\overline{V_x}) \subset \overline{r(V_x)}^{r(X)} \subset \overline{U_{r(x)}}^{r(X)} \subset \overline{U_{r(x)}}$, that is $z \in \overline{V_x} \cap \overline{U_r(x)}$, contradiction. □

Proposition 4.18. *An Urysohn semitopological semilattice X is θ-$\uparrow\downarrow$-closed.*

Proof. Consider an element $x \in X$ and the mapping $s_x : X \to X, s_x : y \mapsto xy$. Since X is Hausdorff, $\{x\}$ is θ-closed. Is easy to check, that the preimage of a θ-closed set under continuous mapping is θ-closed. It is easy to see that $\uparrow x = s_x^{-1}(x)$ and s_x is a retraction X on $\downarrow x$; θ-closedness $\uparrow x$ and $\downarrow x$ follows from the continuity of s_x. □

5. The Θ-completeness of topologized semilattices

Unlike the δ-closure, the operation of taking the θ-closure of a set is not necessarily idempotent, that is, the θ-closure of set may not be θ-closed. This fact motivates the following definition.

Definition 5.1. A Θ-closure of a subset A of a topological space X is called the set $\overline{A}^\Theta$, equal to the intersection of all θ-closed subsets of X, containing A.

In [13], it was proved that the intersection and finite union of θ-closed sets is θ-closed. Obviously, the empty set and the entire space are θ-closed sets. It follows that for any topological space (X, τ) there exists a topology τ_θ, such that the closed sets in space (X, τ_θ) exactly the θ-closed sets in (X, τ). Obviously, the closure operator in (X, τ_θ) is the same as the Θ-closure operator in (X, τ).

Complement to a θ-closed set is called a θ-open set.

Definition 5.2. A topologized semilattice X is called Θ-*complete* if for every non-empty chain $C \subset X$ $\inf C \in \overline{C}^{\Theta}$ and $\sup C \in \overline{C}^{\Theta}$.

Definition 5.3. Let X be a topologized semilattice. A *weak* Θ-*chain*$^{\bullet}$ *topology* on X is called a topology generated by subbase consisting of the complements of θ-closed chains in X.

Definition 5.4. The topologized semilattice X is called *weak* Θ-*chain*$^{\bullet}$ *compact* if X is compact in its weak Θ-*chain*$^{\bullet}$ topology.

Theorem 5.5. *For a θ-$\uparrow\downarrow$-closed topologized semilattice X the following conditions are equivalent:*

1) X is Θ-complete;

2) X is weak Θ-chain$^{\bullet}$ compact;

3) any θ-open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of a θ-closed (in X) chain C contains a finite subcover.

6. The relationship of different types of completeness

In this chapter, we will look at the relationships of the classes of topologized semilattices that we introduced earlier.

In [13], it was proved that for an arbitrary topological space X and its subset A the following inclusions hold: $\overline{A} \subset \overline{A}^{\delta} \subset \overline{A}^{\theta}$. From the definitions, it is clear that the same is always true for $\overline{A}^{\theta} \subset \overline{A}^{\Theta}$. In addition, the same paper shows that the operators δ-closures and θ-closures coincide in the class of regular spaces.

Proposition 6.1. *Let X be a topologized semilattice. Then*

1) if X is complete, then it is δ-complete;

2) if X is δ-complete, then it is θ-complete;

3) if X is θ-complete, then it is Θ-complete.

Proposition 6.2. *For a regular topologized semilattice X the following conditions are equivalent:*

1) X is complete;

2) X is δ-complete;

3) X is θ-complete;

4) X is Θ-complete.

Proposition 6.3. *Let X be a topologized semilattice.*

1). If X is weak chain$^{\bullet}$ compact, it is also weak δ-chain$^{\bullet}$ compact.

2). If X is a weak δ-chain$^{\bullet}$ compact or weak θ-chain$^{\bullet}$-compact, it is weak Θ-chain$^{\bullet}$-compact. If, in addition, X is θ-$\updownarrow$-closed, then weak δ-chain$^{\bullet}$-compactness implies weak θ-chain$^{\bullet}$-compactness.

Proof. The first part of the statement follows from the fact that θ-closed chain is δ-closed, and, hence, it is closed; thus weak Θ-chain$^{\bullet}$-topology is contained in weak δ-chain$^{\bullet}$-topology and weak θ-chain$^{\bullet}$-topology. Note that weak δ-chain$^{\bullet}$-topology weaker than weak chain$^{\bullet}$-topology. It remains to note that compactness in some topology implies compactness in any weaker topology.

The second part of the statement follows from the fact that in a θ-$\updownarrow$-closed semilattice the set $\overline{C}^{\theta}$ is a δ-closed chain. □

Finally, we show how the various types of compactness of chains are related.

Proposition 6.4. *Let (X,τ) be a topologized semilattice. Then*

1) if every closed in (X,τ) chain is compact (i.e. X is chain-compact) then every δ-closed chain is compact in (X,τ_δ);

2) if every δ-closed chain is compact in (X,τ_δ) then every θ-closed chain is compact in (X,τ_θ);

3) if $\overline{C}^\theta$ is an H-set for any chain $C\subset X$ then every θ-closed chain is compact in (X,τ_θ).

Moreover, if X is a θ-$\updownarrow$-closed subsemilattice then

4) if every δ-closed chain is compact in (X,τ_δ) then $\overline{C}^\theta$ is an H-set for any chain $C\subset X$.

Proof. Points 1) and 2) are obvious. We prove point 3).

Let $C\subset X$ be a θ-closed (in X) chain and $\{U_\alpha\}_{\alpha\in\Lambda}$ be a family of θ-open (in X) sets covering C. By definition of θ-open set, for each point $x\in C\cap U_\alpha$ there exists a neighborhood U_x such that $\overline{U_x}\cap(X\setminus U_\alpha)=\emptyset$ that is $\overline{U_x}\subset U_\alpha$. Since $\overline{C}^\theta=C$, the set C is a H-set. Since $\{U_x\}_{x\in C}$ is an open cover of C, there is a finite set $x_1,...,x_n$ such that $C\subset\bigcup\limits_{i=1}^{n}\overline{U_{x_i}}$. For every x_i there is $\alpha_i\in\Lambda$ such that $\overline{U}_{x_i}\subset U_{\alpha_i}$. It follows that $\{U_{\alpha_i}\}$ is a finite cover of C.

Now we prove 4). Consider a chain $C\subset X$. Since X is θ-$\updownarrow$-closed, the set $\overline{C}^\theta$ is a δ-closed chain. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ is a cover of C. It is known [1], that the sets $Int\overline{U}$ are δ-open. Then there is a finite set $\alpha_1,...,\alpha_n$ such that $C\subset\bigcup\limits_{i=1}^{n}Int\overline{U_{\alpha_i}}$. Then $C\subset\bigcup\limits_{i=1}^{n}\overline{U_{\alpha_i}}$. □

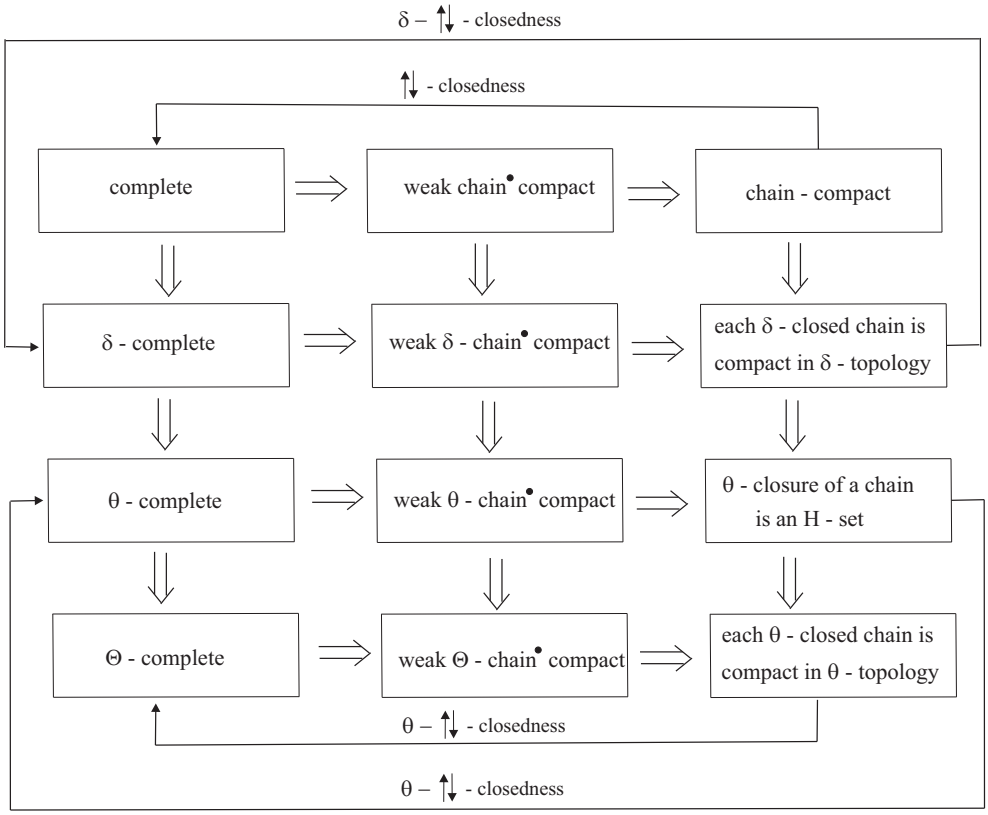

Diagram 1. Implications between various types of completeness of semitopological semilattices

7. Examples

In conclusion, we show examples that separate the classes we have introduced.

Example 7.1. *There exists a δ-complete topologized semilattice such that it is not complete.*

Consider the segment $I = [0, 1]$, ordered by the natural order. We define on I the topology τ in terms of fundamental systems of neighborhoods $\mathcal{B}(x)$:

$$\mathcal{B}(x) = \begin{cases} (x-\varepsilon, x+\varepsilon) \cap I : \varepsilon > 0, & x \neq 0. \\ ([0, \varepsilon) \cap I) \setminus \{\frac{1}{n} : n \in \mathbb{N}\} : \varepsilon > 0, & x = 0. \end{cases}$$

Note that $Int(\overline{(x-\varepsilon, x+\varepsilon) \cap I}) = (x-\varepsilon, x+\varepsilon) \cap I$ and $Int(\overline{([0, \varepsilon) \cap I) \setminus \{\frac{1}{n} : n \in \mathbb{N}\}}) = [0, \varepsilon) \cap I$, i.e. δ-topology on (I, τ) coincide with the Euclidean topology on I, hence, (I, τ) is δ-complete.

Since $0 \notin \overline{\{\frac{1}{n} : n \in \mathbb{N}\}}$ then (I, τ) is not complete.

Example 7.2. *There exists a θ-complete topologized semilattice such that it is not δ-complete.*

Let $X = [0, 1] \times [0, 1]$. Denote by $B(x, \varepsilon)$ — an open ε-ball of x in $\mathbb{R}^2$, $B[x, \varepsilon]$ – a closed ε-ball of x. We define the topology on X in terms of fundamental systems of neighborhoods $\mathcal{B}(x)$:

$$\mathcal{B}(x) = \begin{cases} \{B(x, \varepsilon) \cap X : \varepsilon > 0\}, x = (a, b), b \neq 0. \\ \{(B(x, \varepsilon)) \cap X) \setminus \{0\} \times (a-\varepsilon, a+\varepsilon) \cup \{x\} : \varepsilon > 0, x = (a, 0). \end{cases}$$

Define the operation on X:

$$xy = \begin{cases} x, x = y \vee x = (a_1, 0), y = (a_2, 0), a_1 < a_2. \\ (0, 0), \textit{otherwise}. \end{cases}$$

Note that X is a semilattice in which the only maximum infinite chain is the set $C = [0, 1] \times \{0\}$, order isomorphic to segment $[0, 1]$. Consider a chain $A \subset C$. By order isomorphism, there are $\sup A \in C$ and $\inf A \in C$. Clearly, the closure of the neighborhood $U = (B((0, a), \varepsilon)) \cap X) \setminus \{0\} \times (a-\varepsilon, a+\varepsilon) \cup \{x\}$ of the point $(0, a)$ is the set $B[a, \varepsilon] \cap X$, i.e., $\overline{U} \cap C = \{0\} \times [a-\varepsilon, a+\varepsilon]$. By definitions sup *and* inf*, we have that $\sup A \in \overline{A}^{\theta}$ and $\inf A \in \overline{A}^{\theta}$. But, X is not δ-complete semilattice: for the chain $A = (\frac{1}{3}, \frac{2}{3}) \times \{0\}$ we have $\sup A = \frac{1}{3} \times \{0\}, \inf A = \frac{2}{3} \times \{0\}$, but $U = Int\overline{U}$ for a base neighborhood U of the point $(\frac{1}{3}, 0)$ and $U \cap A = \emptyset$.*

Example 7.3. *There exists a Θ-complete topologized semilattice such that it is not θ-complete.*

Let $A_0 = \{0\} \times \omega_1, A_1 = \{1\} \times \omega_1, A_2 = \{2\} \times (\omega_1 + 1)$. For an ordinal number α define the sets $\lim^2(\alpha) := \{\beta \in \lim(\alpha) : \beta = \sup(\lim(\beta))\}$ and $\lim^1(\alpha) := \lim(\alpha) \setminus \lim^2(\alpha)$. Clearly, the sets $\lim(\omega_1), \lim^1(\omega_1)$ and $\lim^2(\omega_1)$ are unbounded in ω_1, and because they has the same order type; fix isomorphisms $f_1 : \lim(\omega_1) \to \lim^1(\omega_1)$ and $f_2 : \lim^2(\omega_1) \to \lim^2(\omega_1)$ of well-ordered sets.

Let $A = A_0 \sqcup A_1 \sqcup A_2$. Define on A the following equivalence relation: $x \sim y$ if and only if $x = y$ or $x = (0, \alpha), y = (1, f_1(\alpha))$ where $\alpha \in \lim(\omega_1)$ or $x = (1, \alpha), y = (2, f_2(\alpha))$ where $\alpha \in \lim^1(\omega_1)$.

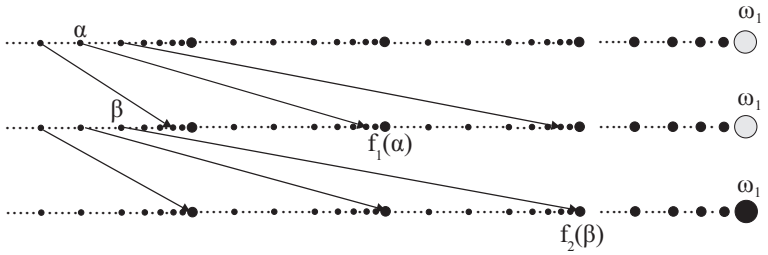

Diagram 2. Example 7.3

Define on the $Y = A/\sim$ *the following operation:*

$$[x]\cdot[y] = \begin{cases} [x], \ \ [x]=[y]. \\ [(0,\min\{\alpha,\beta\})], \ \ (0,\alpha)\in[x],(0,\beta)\in[y]. \\ [x], \ \ [y]=[(2,\omega_1)]. \\ [y], \ \ [x]=[(2,\omega_1)]. \\ [(0,0)], \ \ \textit{otherwise.} \end{cases}$$

Note that $(Y,\cdot)$ *is semilattice in which only maximum infinite chain is the set* $Z = \{[(0,\alpha)] : \alpha\in\omega_1\}\cup\{[(2,\omega_1)]\}$.

The set $\{[(i,\beta)] : \beta>\gamma$ *and* $\beta\le\alpha\}$ *and* $\{[(i,\beta) : \beta>\gamma$ *and* $\beta\le\alpha$ *and* $\beta\in I(\omega_1)]$, *where* $i=\overline{0,2}$ *denote by* $(\gamma,\alpha]^i$ *and* $(\gamma,\alpha]^i_I$, *respectively. We define the topology on* Y *in terms of fundamental systems of neighborhoods* $\mathcal{B}(x)$*:*

$$\mathcal{B}(x) = \begin{cases} \{\{x\}\}, x=[(i,\alpha)], i=\overline{0,2}, \alpha\in I(\omega_1). \\ \{(\beta,\alpha]^0\cup(\gamma,f_1(\alpha)]^1_I : \beta<\alpha,\gamma<f_1(\alpha)\}, x=[(0,\alpha)], \alpha\in\lim(\omega_1). \\ \{(\beta,\alpha]^1\cup(\gamma,f_2(\alpha)]^2_I : \beta<\alpha,\gamma<f_2(\alpha)\}, x=[(1,\alpha)], \alpha\in\lim^2(\omega_1). \\ \{(\beta,\alpha]^2_I : \beta<\omega_1\}, x=[(2,\alpha)], \alpha=\omega_1\vee\alpha\in\lim^1(\omega_1). \end{cases}$$

We show that the semilattice Y *is* Θ*-complete. Let* $C\subset Y$ *be a chain. If* C *is finite, then it contains infimum and supremum. Otherwise* $C\subset Z$*. Since the chain* Z *well-ordered by the natural order,* C *contains infimum. If the set* $E := \{\alpha\in\omega_1 : [(0,\alpha)]\in C\}$ *is bounded in* ω_1, *then* $\sup C = [(0,\sup E)]\in\overline{C}\subset\overline{C}^{\Theta}$*. If* E *is unbounded then* $\sup C = [(2,\omega_1)]$*. Note that* $B = \{[(0,\alpha)] : \alpha\in\omega_1\}$ *(in subspace topology) is homeomorphic to* ω_1*. It follows that the set* $\overline{C}^B\cap\{[(0,\alpha)] : \alpha\in\lim(\omega_1)\}$ *is of power* ω_1 *(since it is the intersection of closed unbounded sets). Then* $\overline{C}\cap\{[(1,\alpha)] : \alpha\in\lim^1(\omega_1)\}$ *is also of power* ω_1 *(since* $\overline{C}^B\subset\overline{C}$, $[(0,\alpha)] = [1,f_1(\alpha)]$ *for limit ordinals* $\alpha<\omega_1$ *and* f_1 *is bijection). Then the set* $\overline{\overline{C}}^{\theta}\cap\{[(1,\alpha)] : \alpha\in\lim^2(\omega_1)\}\subset$ $\subset\overline{C}^{\theta^2}\cap\{[(1,\alpha)] : \alpha\in\lim^2(\omega_1)\}$ *is of power* ω_1*, since the closure of a standard neighborhood of the point* $[(1,\alpha)]$*, where* $\alpha\in\lim^2(\omega_1)$*, contains an infinite number of points* $[(1,\beta)]$*, where* $\beta\in\lim^1(\omega_1)$*. Note that* $\overline{(\beta,\omega_1]^2_I}\supset(\beta,\omega_1]^2$*, so it is easy to see that* $[(2,\omega_1)]\in\overline{C}^{\theta^2}\subset\overline{C}^{\Theta}$.

Note that if $C = \{[(0,\alpha] : \alpha\in I(\omega_1)\}$*, then* $[(2,\omega_1)]\in\overline{C}^{\theta^2}$*, but* $[(2,\omega_1)]\notin\overline{C}^{\theta}$*, i.e.,* Y *is not* θ*-complete semilattice.*

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