

NUMERICAL ALGORITHMS FOR SEMILINEAR PARABOLIC EQUATIONS WITH SMALL PARAMETER BASED ON APPROXIMATION OF STOCHASTIC EQUATIONS

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ABSTRACT. The probabilistic approach is used for constructing special layer methods to solve the Cauchy problem for semilinear parabolic equations with small parameter. Despite their probabilistic nature these methods are nevertheless deterministic. The algorithms are tested by simulating the Burgers equation with small viscosity and the generalized KPP-equation with a small parameter.

1. INTRODUCTION

Nonlinear partial differential equations (nonlinear PDE) usually are not susceptible of analytic solution, and are mostly investigated by numerical methods. Numerical methods used for solving PDE are traditionally based on deterministic approaches (see, e.g., [23, 24, 27] and references therein). A class of layer methods intended to solve semilinear parabolic equations is introduced in [18], where the well-known probabilistic representations of solutions to linear parabolic equations and the ideas of weak sense numerical integration of stochastic differential equations (SDE) are used to construct numerical algorithms. Despite their probabilistic nature these methods are nevertheless deterministic.

Nonlinear parabolic equations with small parameter arise in a variety of applications (see, e.g., [2, 4, 8, 11, 26] and references therein). For instance, they are used in gas dynamics, where one has to take into account small viscosity and small heat conductivity. Some problems of combustion are described by PDE with small parameter. They also arise as the result of introducing artificial viscosity in systems of first-order hyperbolic equations, which is one of the popular approaches to the numerical solution of inviscid problems of gas dynamics [21, 25, 30].

Here we construct some layer methods for solving the Cauchy problem for semilinear parabolic equations with small parameter of the form

$$(1.1) \quad \begin{aligned} \frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2} \sum_{i,j=1}^d a^{ij}(t, x, u) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^d (b^i(t, x, u) \\ + \varepsilon^2 c^i(t, x, u)) \frac{\partial u}{\partial x^i} + g(t, x, u) = 0, \quad t \in [t_0, T), \quad x \in R^d, \end{aligned}$$

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$$(1.2) \quad u(T, x) = \varphi(x).$$

The probabilistic representations of the solution to the problem (1.1)-(1.2) are connected with systems of SDE with small noise. Just for such systems, special weak approximations are proposed in [19]. Applying these special approximations, we get new layer methods intended to solve the Cauchy problem (1.1)-(1.2).

When the solution of (1.1)-(1.2) is regular, it turns out that errors of the proposed methods have the form of $O(h^p + \varepsilon^l h^q)$, where $p > q$, $l > 0$, and h is a step of time discretization. Thanks to the fact that the accuracy order of such methods is equal to a comparatively small q , they are not too complicated, while due to the large p and the small factor ε^l at h^q , their errors are fairly low and therefore these methods are highly efficient. The singular case, when derivatives of the solution go to infinity as $\varepsilon \rightarrow 0$, requires a special theoretical investigation. For equations of a particular type, we obtain the corresponding theoretical results. As to the equation of the general type (1.1), we restrict ourselves to the analysis of the one-step error. Further theoretical investigations should rest on a stability analysis and on particular properties of the solution. However, we test the methods constructed here on model problems for which shock waves are observed. The tests give quite good results not only in simulations of wave formation—that corresponds to the regular case—but also in simulations of wave propagation, i.e., in the singular case. The reasons for these experimental facts and possible ways to get realistic estimates of errors of the methods in the singular case are discussed.

Section 2 gives some results from [18, 19] used in this paper. In Section 3, new implicit and explicit layer methods for the problem (1.1)-(1.2) are proposed. Some theorems on their rates of convergence both in the regular and in the singular cases are proved. For implementation of the layer methods, we need a space discretization. The numerical algorithms based on the proposed layer methods and on the linear interpolation are constructed in Section 4. For simplicity, Sections 3 and 4 deal with the one-dimensional case of the problem (1.1)-(1.2). In Section 5, extensions to the multi-dimensional case and systems of reaction-diffusion equations are given. In Section 6, we propose both two-layer and three-layer methods in a particular case of the problem (1.1)-(1.2).

All the numerical algorithms presented in the paper are tested through computer experiments. Some results of numerical tests on the Burgers equation with small viscosity and on the generalized KPP-equation with a small parameter are given in Section 7.

This paper is devoted to initial value problems. Boundary value problems for nonlinear parabolic equations with small parameter will be considered in a separate work. The probability approach to linear boundary value problems is treated in [15, 16, 17].

2. PRELIMINARIES

Here we give, in the required form, some results from [18, 19] which are used in the next sections.

2.1. Probabilistic approach to constructing numerical methods for semi-linear PDE. Let the Cauchy problem (1.1)-(1.2) have the unique solution $u = u(t, x)$ which is sufficiently smooth and satisfies some needed conditions of boundedness (see the corresponding theoretical results, e.g., in [13, 29]). If we substitute

$u = u(t, x)$ in the coefficients of (1.1), we obtain a linear parabolic equation with small parameter. The solution of this linear equation has the following probabilistic representation:

$$(2.1) \quad u(t, x) = E(\varphi(X_{t,x}(T)) + Z_{t,x,0}(T)), \quad t \leq T, x \in R^d,$$

where $X_{t,x}(s), Z_{t,x,z}(s), s \geq t$, is the solution of the Cauchy problem for the system of stochastic differential equations

$$(2.2) \quad dX = (b(s, X, u(s, X)) + \varepsilon^2 c(s, X, u(s, X)))ds + \varepsilon \sigma(s, X, u(s, X))dw, \quad X(t) = x,$$

$$(2.3) \quad dZ = g(s, X, u(s, X))ds, \quad Z(t) = z.$$

Here $w(s) = (w^1(s), \dots, w^d(s))^T$ is a d -dimensional standard Wiener process, $b(s, x, u)$ and $c(s, x, u)$ are d -dimensional column-vectors compounded from the coefficients $b^i(s, x, u)$ and $c^i(s, x, u)$ of (1.1), $\sigma(s, x, u)$ is a $d \times d$ matrix obtained from the equation $a(s, x, u) = \sigma(s, x, u)\sigma^T(s, x, u)$, where $a = \{a^{ij}\}$; the equation is solvable with respect to σ (for instance, by a lower triangular matrix) at least in the case of a positive definite a .

To simplify the notation, we write $u(t, x)$ and $X_{t,x}(s)$ instead of $u(t, x; \varepsilon)$ and $X_{t,x}^\varepsilon(s)$ throughout the paper.

Introduce a discretization, for definiteness the equidistant one:

$$T = t_N > t_{N-1} > \dots > t_0 = t, \quad h := \frac{T - t_0}{N}.$$

Note that all the methods given in the paper can easily be adapted for a nonequidistant discretization. For instance, we use a variable discretization step h in some of our numerical tests (see Section 7.1).

We have

$$\begin{aligned} u(t_k, x) &= E(\varphi(X_{t_k,x}(T)) + Z_{t_k,x,0}(T)) \\ &= E(\varphi(X_{t_{k+1},X_{t_k,x}(t_{k+1})}(T)) + Z_{t_{k+1},X_{t_k,x}(t_{k+1}),Z_{t_k,x,0}(t_{k+1})}(T)) \\ &= EE(\varphi(X_{t_{k+1},X_{t_k,x}(t_{k+1})}(T)) \\ &\quad + Z_{t_{k+1},X_{t_k,x}(t_{k+1}),Z_{t_k,x,0}(t_{k+1})}(T) / X_{t_k,x}(t_{k+1}), Z_{t_k,x,0}(t_{k+1})). \end{aligned}$$

Because $Z_{t,x,z}(s) = Z_{t,x,0}(s) + z, t \leq s$, from this we get

$$(2.4) \quad \begin{aligned} u(t_k, x) &= EE(\varphi(X_{t_{k+1},X_{t_k,x}(t_{k+1})}(T)) + Z_{t_{k+1},X_{t_k,x}(t_{k+1}),0}(T) / X_{t_k,x}(t_{k+1}), Z_{t_k,x,0}(t_{k+1})) \\ &\quad + EE(Z_{t_k,x,0}(t_{k+1}) / X_{t_k,x}(t_{k+1}), Z_{t_k,x,0}(t_{k+1})) \\ &= E(u(t_{k+1}, X_{t_k,x}(t_{k+1})) + Z_{t_k,x,0}(t_{k+1})). \end{aligned}$$

In accordance with the probabilistic approach to constructing numerical methods for semilinear PDE from [18], the ideas of weak sense numerical integration of SDE [12, 14, 22] are employed to obtain some approximate relations from (2.2)-(2.4). The relations allow us to express approximations $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$ in terms of $\bar{u}(t_{k+1}, x)$ recurrently, i.e., to construct layer methods which are discrete in the variable t only. To clarify the approach, it is relevant to derive one of the methods from [18] which is used in this paper in a broad fashion. For simplicity, we restrict ourselves to the case $d = 1$.

Applying the explicit weak Euler scheme with the simplest noise simulation [12, 14, 22] to the system (2.2)-(2.3), we get

$$(2.5) \quad \begin{aligned} X_{t_k, x}(t_{k+1}) &\simeq \bar{X}_{t_k, x}(t_{k+1}) = x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) \\ &\quad + \varepsilon h^{1/2} \sigma(t_k, x, u(t_k, x)) \xi_k, \\ Z_{t_k, x, z}(t_{k+1}) &\simeq \bar{Z}_{t_k, x, z}(t_{k+1}) = z + hg(t_k, x, u(t_k, x)), \end{aligned}$$

where $\xi_{N-1}, \xi_{N-2}, \dots, \xi_0$ are i.i.d. random variables with the law $P(\xi = \pm 1) = 1/2$.

Using (2.4)-(2.5), we obtain

$$(2.6) \quad \begin{aligned} u(t_k, x) &\simeq E(u(t_{k+1}, \bar{X}_{t_k, x}(t_{k+1})) + \bar{Z}_{t_k, x, 0}(t_{k+1})) \\ &= \frac{1}{2} u(t_{k+1}, x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) + \varepsilon h^{1/2} \sigma(t_k, x, u(t_k, x))) \\ &\quad + \frac{1}{2} u(t_{k+1}, x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) - \varepsilon h^{1/2} \sigma(t_k, x, u(t_k, x))) \\ &\quad + hg(t_k, x, u(t_k, x)). \end{aligned}$$

Thus, we can calculate the approximations $\bar{u}(t_k, x)$ layerwise:

$$(2.7) \quad \bar{u}(t_N, x) = \varphi(x),$$

$$\begin{aligned} \bar{u}(t_k, x) &= \frac{1}{2} \bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_k, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_k, x)) + \varepsilon h^{1/2} \sigma(t_k, x, \bar{u}(t_k, x))) \\ &\quad + \frac{1}{2} \bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_k, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_k, x)) - \varepsilon h^{1/2} \sigma(t_k, x, \bar{u}(t_k, x))) \\ &\quad + hg(t_k, x, \bar{u}(t_k, x)), \quad k = N-1, \dots, 1, 0. \end{aligned}$$

The method (2.7) is an implicit layer method for solving the Cauchy problem (1.1)-(1.2). This method is deterministic, even though the probabilistic approach is used for its construction.

Applying the method of simple iteration to (2.7) with $\bar{u}(t_{k+1}, x)$ as a null iteration, we get the first iteration (we denote it as $\bar{u}(t_k, x)$ again, since this does not cause any confusion):

$$(2.8) \quad \bar{u}(t_N, x) = \varphi(x),$$

$$\begin{aligned} \bar{u}(t_k, x) &= \frac{1}{2} \bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_{k+1}, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_{k+1}, x)) \\ &\quad + \varepsilon h^{1/2} \sigma(t_k, x, \bar{u}(t_{k+1}, x))) \\ &\quad + \frac{1}{2} \bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_{k+1}, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_{k+1}, x)) \\ &\quad - \varepsilon h^{1/2} \sigma(t_k, x, \bar{u}(t_{k+1}, x))) \\ &\quad + hg(t_k, x, \bar{u}(t_{k+1}, x)), \quad k = N-1, \dots, 1, 0. \end{aligned}$$

According to the propositions proved in [18], this explicit layer method has one-step error estimated by $O(h^2)$ and global error $O(h)$.

Of course, all the results from [18] can be applied to solving the problem with small parameter (1.1)-(1.2). But since the probabilistic representation of the solution to (1.1)-(1.2) is connected with the system of differential equations with small noise (2.2)-(2.3), one can expect that use of weak approximations for SDE with small noise [19] leads to new effective methods for solving (1.1)-(1.2).

2.2. Some weak approximations for SDE with small noise. Here we recall some weak approximations for SDE with small noise:

$$(2.9) \quad dX = b(t, X)dt + \varepsilon^2 c(t, X)dt + \varepsilon \sigma(t, X)dw(t), \quad X(t_0) = x, \\ t \in [t_0, T], \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

where X , $b(t, x)$, and $c(t, x)$ are d -dimensional column-vectors, $\sigma(t, x)$ is a $d \times m$ matrix, $w(s) = (w^1(s), \dots, w^m(s))^\top$ is an m -dimensional standard Wiener process, ε_0 is a positive number. The coefficients are assumed to satisfy the corresponding conditions of smoothness and boundedness (the details are given, e.g., in [14, 19]).

The one-step error ρ and the global error R of a weak approximation $X_k = \bar{X}_{t,x}(t_k)$ at the point (t, x) are defined as

$$\rho = |Ef(X_{t,x}(t+h)) - Ef(\bar{X}_{t,x}(t+h))|, \\ R = |Ef(X_{t,x}(T)) - Ef(\bar{X}_{t,x}(T))|,$$

where $f(x)$ is a function belonging to a sufficiently wide class (the details are given, e.g., in [12, 14, 22]).

Below we write down a number of weak schemes from [14] and [19] which are used in the next sections.

The Euler scheme:

$$(2.10) \quad X_{k+1} = X_k + hb(t_k, X_k) + \varepsilon^2 hc(t_k, X_k) + \varepsilon h^{1/2} \sigma(t_k, X_k) \xi_k, \\ \rho = O(h^2), \quad R = O(h),$$

where the $\xi_k = (\xi_k^1, \dots, \xi_k^m)$ are i.i.d. m -dimensional vectors with i.i.d. components and each component is distributed by the law $P(\xi = \pm 1) = 1/2$.

The Runge-Kutta scheme with error $O(h^2 + \varepsilon^2 h)$:

$$(2.11) \quad X_{k+1} = X_k + \frac{1}{2}hb(t_k, X_k) + \frac{1}{2}hb(t_{k+1}, X_k + hb(t_k, X_k)) \\ + \varepsilon^2 hc(t_k, X_k) + \varepsilon h^{1/2} \sigma(t_k, X_k) \xi_k, \\ \rho = O(h^3 + \varepsilon^2 h^2), \quad R = O(h^2 + \varepsilon^2 h),$$

where the ξ_k are the same as in (2.10).

The special second-order Runge-Kutta scheme (for SDE with small additive noise, i.e., σ is a constant matrix; $c \equiv 0$):

$$(2.12) \quad X_{k+1} = X_k + \frac{1}{2}hb(t_k, X_k) + \frac{1}{2}hb(t_{k+1}, X_k + hb(t_k, X_k)) + \varepsilon h^{1/2} \sigma \xi_k + \varepsilon h^{1/2} \sigma \xi_k, \\ \rho = O(h^3), \quad R = O(h^2),$$

where the $\xi_k = (\xi_k^1, \dots, \xi_k^m)$ are i.i.d. m -dimensional vectors with i.i.d. components and each component is distributed by the law $P(\xi = 0) = 2/3$, $P(\xi = \pm\sqrt{3}) = 1/6$.

The special Runge-Kutta scheme with error $O(h^4 + \varepsilon^2 h^2)$ (for SDE with small additive noise, i.e., σ is a constant matrix; $c \equiv 0$):

$$(2.13) \quad \begin{aligned} X_{k+1} &= X_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + \varepsilon h^{1/2} \sigma \xi_k, \\ k_1 &= hb(t_k, X_k), \quad k_2 = hb(t_{k+1/2}, X_k + k_1/2), \\ k_3 &= hb(t_{k+1/2}, X_k + k_2/2 + \varepsilon h^{1/2} \sigma \xi_k), \quad k_4 = hb(t_{k+1}, X_k + k_3 + \varepsilon h^{1/2} \sigma \xi_k), \\ \rho &= O(h^5 + \varepsilon^2 h^3), \quad R = O(h^4 + \varepsilon^2 h^2), \end{aligned}$$

where the ξ_k are the same as in (2.12).

Note that in Sections 6 and 7 we also handle some other special weak approximations.

3. METHODS FOR A GENERAL SEMILINEAR PARABOLIC EQUATION WITH SMALL PARAMETER

Here we construct new layer methods for semilinear parabolic equations with small parameter (1.1)-(1.2), using the probabilistic approach from [18] and applying specific weak approximations from [19] to the system with small noise (2.2)-(2.3).

3.1. Implicit layer method. For simplicity, let us consider the Cauchy problem (1.1)-(1.2) for $d = 1$:

$$(3.1) \quad \frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2} \sigma^2(t, x, u) \frac{\partial^2 u}{\partial x^2} + (b(t, x, u) + \varepsilon^2 c(t, x, u)) \frac{\partial u}{\partial x} + g(t, x, u) = 0,$$

$$t \in [t_0, T], \quad x \in R^1,$$

$$(3.2) \quad u(T, x) = \varphi(x).$$

The probabilistic representation of the solution $u(t, x)$ to this problem has the form (2.1)-(2.3) with $d = 1$.

Applying the Runge-Kutta scheme (2.11) to the system (2.2)-(2.3), we get

$$(3.3) \quad \begin{aligned} X_{t_k, x}(t_{k+1}) &\simeq \bar{X}_{t_k, x}(t_{k+1}) \\ &= x + \frac{1}{2}hb_k + \frac{1}{2}hb(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) \\ &\quad + \varepsilon^2 hc_k + \varepsilon h^{1/2} \sigma_k \xi_k, \end{aligned}$$

$$Z_{t_k, x, z}(t_{k+1}) \simeq \bar{Z}_{t_k, x, z}(t_{k+1}) = z + \frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)),$$

where $b_k, c_k, \sigma_k,$ and g_k are the coefficients $b, c, \sigma,$ and g calculated at the point $(t_k, x, u(t_k, x))$.

Using the probabilistic representation (2.4), we obtain

$$\begin{aligned} u(t_k, x) &\simeq E(u(t_{k+1}, \bar{X}_{t_k, x}(t_{k+1})) + \bar{Z}_{t_k, x, 0}(t_{k+1})) \\ &= \frac{1}{2}u(t_{k+1}, x + h[b_k + b(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k))]/2 + \varepsilon^2 hc_k + \varepsilon h^{1/2} \sigma_k) \\ &\quad + \frac{1}{2}u(t_{k+1}, x + h[b_k + b(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k))]/2 + \varepsilon^2 hc_k - \varepsilon h^{1/2} \sigma_k) \\ &\quad + \frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)). \end{aligned}$$

We can approximate $u(t_k, x)$ by $v(t_k, x)$ found from

$$\begin{aligned}
 (3.4) \quad & v(t_k, x) \\
 &= \frac{1}{2}u(t_{k+1}, x + h[\tilde{b}_k + b(t_{k+1}, x + h\tilde{b}_k), u(t_{k+1}, x + h\tilde{b}_k)]) / 2 + \varepsilon^2 h \tilde{c}_k + \varepsilon h^{1/2} \tilde{\sigma}_k \\
 &+ \frac{1}{2}u(t_{k+1}, x + h[\tilde{b}_k + b(t_{k+1}, x + h\tilde{b}_k), u(t_{k+1}, x + h\tilde{b}_k)]) / 2 + \varepsilon^2 h \tilde{c}_k - \varepsilon h^{1/2} \tilde{\sigma}_k \\
 &\quad + \frac{1}{2}h\tilde{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\tilde{b}_k, u(t_{k+1}, x + h\tilde{b}_k)),
 \end{aligned}$$

where $\tilde{b}_k, \tilde{c}_k, \tilde{\sigma}_k,$ and \tilde{g}_k are the coefficients $b, c, \sigma,$ and g calculated at the point $(t_k, x, v(t_k, x))$.

The corresponding implicit layer method has the form

$$\begin{aligned}
 (3.5) \quad & \bar{u}(t_N, x) = \varphi(x), \\
 & \bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h[\bar{b}_k + b(t_{k+1}, x + h\bar{b}_k), \bar{u}(t_{k+1}, x + h\bar{b}_k)]) / 2 \\
 & \quad + \varepsilon^2 h \bar{c}_k + \varepsilon h^{1/2} \bar{\sigma}_k \\
 & + \frac{1}{2}\bar{u}(t_{k+1}, x + h[\bar{b}_k + b(t_{k+1}, x + h\bar{b}_k), \bar{u}(t_{k+1}, x + h\bar{b}_k)]) / 2 \\
 & \quad + \varepsilon^2 h \bar{c}_k - \varepsilon h^{1/2} \bar{\sigma}_k \\
 & + \frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k)), \quad k = N - 1, \dots, 0,
 \end{aligned}$$

where $\bar{b}_k, \bar{c}_k, \bar{\sigma}_k,$ and \bar{g}_k are the coefficients $b, c, \sigma,$ and g calculated at the point $(t_k, x, \bar{u}(t_k, x))$.

3.2. Convergence theorem in the regular case. Let us make the following two assumptions.

(i) The coefficients $b(t, x, u)$ and $g(t, x, u)$ and their first and second derivatives are continuous and uniformly bounded, and so are the coefficients $c(t, x, u)$ and $\sigma(t, x, u)$ and their first derivatives:

$$\begin{aligned}
 (3.6) \quad & \left| \frac{\partial^{i+j+l} b}{\partial t^i \partial x^j \partial u^l} \right| \leq K, \quad \left| \frac{\partial^{i+j+l} g}{\partial t^i \partial x^j \partial u^l} \right| \leq K, \quad 0 \leq i + j + l \leq 2, \\
 & \left| \frac{\partial^{i+j+l} c}{\partial t^i \partial x^j \partial u^l} \right| \leq K, \quad \left| \frac{\partial^{i+j+l} \sigma}{\partial t^i \partial x^j \partial u^l} \right| \leq K, \quad 0 \leq i + j + l \leq 1, \\
 & t_0 \leq t \leq T, \quad x \in R^1, \quad u_o < u < u^\circ,
 \end{aligned}$$

where $-\infty \leq u_o, u^\circ \leq \infty$ are constants.

(ii) There exists a unique bounded solution $u(t, x)$ of problem (3.1)-(3.2) such that

$$(3.7) \quad u_o < u_* \leq u(t, x) \leq u^* < u^\circ,$$

where u_*, u^* are constants, and there exist the uniformly bounded derivatives

$$\begin{aligned}
 (3.8) \quad & \left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j} \right| \leq K, \quad i = 0, j = 1, 2, 3, 4; \quad i = 1, j = 0, 1, 2; \quad i = 2, j = 0, 1; \\
 & \quad i = 3, j = 0; \quad t_0 \leq t \leq T, \quad x \in R^1, \quad 0 < \varepsilon \leq \varepsilon^*.
 \end{aligned}$$

Below, in Section 3.4, we consider the singular case, when the condition (3.8) is not fulfilled.

In Lemma 3.1 and Theorem 3.1 we use the letters K and C without any index for various constants which do not depend on h, k, x, ε .

Lemma 3.1. *Under assumptions (i) and (ii), the one-step error of the implicit layer method (3.5) is estimated by $O(h^3 + \varepsilon^2 h^2)$, i.e.,*

$$|v(t_k, x) - u(t_k, x)| \leq C \cdot (h^3 + \varepsilon^2 h^2),$$

where $v(t_k, x)$ is found from (3.4), and C does not depend on h, k, x, ε .

Proof. Introduce the function

$$\begin{aligned} U_{t_k, x}(v) &:= \frac{1}{2} u(t_{k+1}, x + h[\check{b}_k + b(t_{k+1}, x + \check{h}\check{b}_k, u(t_{k+1}, x + \check{h}\check{b}_k))]/2 + \varepsilon^2 h\check{c}_k + \varepsilon h^{1/2}\check{\sigma}_k) \\ &\quad + \frac{1}{2} u(t_{k+1}, x + h[\check{b}_k + b(t_{k+1}, x + \check{h}\check{b}_k, u(t_{k+1}, x + \check{h}\check{b}_k))]/2 + \varepsilon^2 h\check{c}_k - \varepsilon h^{1/2}\check{\sigma}_k) \\ &\quad + \frac{1}{2} h\check{g}_k + \frac{1}{2} hg(t_{k+1}, x + \check{h}\check{b}_k, u(t_{k+1}, x + \check{h}\check{b}_k)), \end{aligned}$$

where $\check{b}_k, \check{c}_k, \check{\sigma}_k$, and \check{g}_k are the coefficients b, c, σ , and g calculated at (t_k, x, v) .

To prove the lemma, we make use of the method of simple iteration. Define the sequence

$$v^{(i)}(t_k, x) := U_{t_k, x}(v^{(i-1)}(t_k, x)), \quad i = 1, 2, \dots,$$

and take $u(t_k, x)$ as a null iteration:

$$v^{(0)}(t_k, x) = u(t_k, x).$$

First we prove that

$$(3.9) \quad |v^{(1)}(t_k, x) - v^{(0)}(t_k, x)| = |v^{(1)}(t_k, x) - u(t_k, x)| \leq C \cdot (h^3 + \varepsilon^2 h^2).$$

We have

$$\begin{aligned} (3.10) \quad v^{(1)}(t_k, x) &= \frac{1}{2} u(t_{k+1}, x + h[b_k + b(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k))]/2 + \varepsilon^2 hc_k + \varepsilon h^{1/2}\sigma_k) \\ &\quad + \frac{1}{2} u(t_{k+1}, x + h[b_k + b(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k))]/2 + \varepsilon^2 hc_k - \varepsilon h^{1/2}\sigma_k) \\ &\quad + \frac{1}{2} hg_k + \frac{1}{2} hg(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)), \end{aligned}$$

where b_k, c_k, σ_k , and g_k are the coefficients b, c, σ , and g calculated at the point $(t_k, x, u(t_k, x))$.

Using assumptions (i) and (ii), we expand the functions u and g :

$$\begin{aligned}
 & u(t_{k+1}, x + hb_k/2 + hb(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)))/2 + \varepsilon^2 hc_k \pm \varepsilon h^{1/2} \sigma_k \\
 &= u(t_k, x) + \frac{\partial u}{\partial t} h + \frac{\partial u}{\partial x} \cdot (b_k h + \frac{1}{2} \frac{\partial b}{\partial t} h^2 + \frac{1}{2} \frac{\partial b}{\partial x} b_k h^2 + \frac{1}{2} \frac{\partial b}{\partial u} \frac{\partial u}{\partial t} h^2 \\
 & \quad + \frac{1}{2} \frac{\partial b}{\partial u} \frac{\partial u}{\partial x} b_k h^2 + \varepsilon^2 c_k h \pm \varepsilon \sigma_k h^{1/2}) \\
 & + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} h^2 + \frac{\partial^2 u}{\partial t \partial x} \cdot (b_k h^2 \pm \varepsilon \sigma_k h^{3/2}) \\
 & + \frac{\partial^2 u}{\partial x^2} \cdot (\frac{1}{2} \varepsilon^2 \sigma_k^2 h + \frac{1}{2} b_k^2 h^2 \pm \varepsilon h^{1/2} \sigma_k \cdot (b_k h + \varepsilon^2 c_k h)) \\
 & \pm \frac{1}{6} \frac{\partial^3 u}{\partial x^3} \varepsilon^3 \sigma_k^3 h^{3/2} + O(h^3 + \varepsilon^2 h^2), \\
 & g(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) \\
 &= g_k + \frac{\partial g}{\partial t} h + \frac{\partial g}{\partial x} b_k h + \frac{\partial g}{\partial u} \frac{\partial u}{\partial t} h + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} b_k h + O(h^2),
 \end{aligned}$$

where the derivatives of u are calculated at the point (t_k, x) , the derivatives of the coefficients b and g are calculated at the point $(t_k, x, u(t_k, x))$, and

$$(3.11) \quad |O(h^3 + \varepsilon^2 h^2)| \leq C \cdot (h^3 + \varepsilon^2 h^2), \quad |O(h^2)| \leq C \cdot h^2.$$

Substituting these expansions in (3.10), we get

$$\begin{aligned}
 v^{(1)}(t_k, x) &= u(t_k, x) + h \cdot (\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot (b_k + \varepsilon^2 c_k) + \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} \sigma_k^2 + g_k) \\
 & + \frac{1}{2} h^2 \cdot (\frac{\partial^2 u}{\partial t^2} + \frac{\partial b}{\partial t} \frac{\partial u}{\partial x} + \frac{\partial b}{\partial u} \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} + b_k \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial g}{\partial t} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial t}) \\
 & + \frac{1}{2} b_k h^2 \cdot (\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial b}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial b}{\partial u} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + b_k \frac{\partial^2 u}{\partial x^2} + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x}) \\
 & + O(h^3 + \varepsilon^2 h^2).
 \end{aligned}$$

Then, adding and subtracting the appropriate terms of order $O(\varepsilon^2 h^2)$ in the above expression, we obtain

$$\begin{aligned}
 (3.12) \quad v^{(1)}(t_k, x) &= u(t_k, x) + h \cdot (\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot (b_k + \varepsilon^2 c_k) + \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} \sigma_k^2 + g_k) \\
 & + \frac{1}{2} h^2 \frac{\partial}{\partial t} (\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot (b + \varepsilon^2 c) + \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 + g) \\
 & + \frac{1}{2} b_k h^2 \frac{\partial}{\partial x} (\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot (b + \varepsilon^2 c) + \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 + g) + O(h^3 + \varepsilon^2 h^2),
 \end{aligned}$$

where, after differentiation, all the expressions are calculated at the point (t_k, x) .

Since $u(t, x)$ is the solution of problem (3.1)-(3.2), the relation (3.12) implies

$$v^{(1)}(t_k, x) = u(t_k, x) + O(h^3 + \varepsilon^2 h^2).$$

The estimate (3.9) is proved.

Clearly, for a sufficiently small h , we obtain

$$u_\circ < -Ch^2(h + \varepsilon) + u_* \leq v^{(1)}(t_k, x) \leq u^* + Ch^2(h + \varepsilon) < u^\circ, \quad x \in R^1.$$

Using assumptions (i) and (ii), we get

$$\begin{aligned} |v^{(2)}(t_k, x) - v^{(1)}(t_k, x)| &= |U_{t_k, x}(v^{(1)}(t_k, x)) - U_{t_k, x}(v^{(0)}(t_k, x))| \\ &\leq Kh|v^{(1)}(t_k, x) - v^{(0)}(t_k, x)|, \end{aligned}$$

whence it follows that

$$|v^{(2)}(t_k, x) - v^{(0)}(t_k, x)| \leq (1 + Kh) |v^{(1)}(t_k, x) - v^{(0)}(t_k, x)|.$$

It is not difficult to show that there exists a sufficiently small h such that the procedure can be continued infinitely (i.e., $u_\circ < v^{(i)}(t_k, x) < u^\circ$, $i = 2, 3, \dots$) and

$$\begin{aligned} |v^{(n)}(t_k, x) - v^{(n-1)}(t_k, x)| &\leq (Kh)^{n-1} |v^{(1)}(t_k, x) - v^{(0)}(t_k, x)|, \\ |v^{(n)}(t_k, x) - v^{(0)}(t_k, x)| &\leq \frac{1 - (Kh)^n}{1 - Kh} |v^{(1)}(t_k, x) - v^{(0)}(t_k, x)|, \quad n = 1, 2, 3, \dots \end{aligned}$$

Further, we prove by the usual arguments that there is a unique root of the equation

$$v(t_k, x) = U_{t_k, x}(v(t_k, x))$$

such that

$$(3.13) \quad |v(t_k, x) - v^{(0)}(t_k, x)| \leq \frac{1}{1 - Kh} |v^{(1)}(t_k, x) - v^{(0)}(t_k, x)|.$$

Substituting (3.9) in (3.13), we come to the statement of the lemma. Lemma 3.1 is proved.

Let us prove the following theorem on global convergence.

Theorem 3.1. *Under assumptions (i) and (ii), the global error of the implicit layer method (3.5) is estimated by $O(h^2 + \varepsilon^2 h)$:*

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K \cdot (h^2 + \varepsilon^2 h),$$

where the constant K does not depend on h, k, x, ε .

Proof. We follow the proof of the corresponding theorem in [18].

Denote the error of the method (3.5) accumulated at the k -th step ($(N - k)$ -th layer) as

$$(3.14) \quad R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x).$$

Introduce the notation

$$X_{k+1}^{(\pm)} := x + \frac{1}{2} h \bar{b}_k + \frac{1}{2} h b(t_{k+1}, x + h \bar{b}_k, \bar{u}(t_{k+1}, x + h \bar{b}_k)) + \varepsilon^2 h \bar{c}_k \pm \varepsilon h^{1/2} \bar{\sigma}_k,$$

where (we remember) \bar{b}_k, \bar{c}_k , and $\bar{\sigma}_k$ are the coefficients $b(t, x, u), c(t, x, u)$, and $\sigma(t, x, u)$ calculated at $t = t_k, x = x, u = \bar{u}(t_k, x) = u(t_k, x) + R(t_k, x)$.

Using this notation, (3.5), and (3.14), we get

$$\begin{aligned} (3.15) \quad &u(t_k, x) + R(t_k, x) = \bar{u}(t_k, x) \\ &= \frac{1}{2} \bar{u}(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2} \bar{u}(t_{k+1}, X_{k+1}^{(-)}) + \frac{1}{2} h \bar{g}_k + \frac{1}{2} h g(t_{k+1}, x + h \bar{b}_k, \bar{u}(t_{k+1}, x + h \bar{b}_k)) \\ &= \frac{1}{2} u(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2} u(t_{k+1}, X_{k+1}^{(-)}) + \frac{1}{2} R(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2} R(t_{k+1}, X_{k+1}^{(-)}) \\ &\quad + \frac{1}{2} h \bar{g}_k + \frac{1}{2} h g(t_{k+1}, x + h \bar{b}_k, \bar{u}(t_{k+1}, x + h \bar{b}_k)). \end{aligned}$$

Clearly, $R(t_N, x) = 0$. Below we prove recurrently that $R(t_k, x)$, $k = N-1, \dots, 0$, is sufficiently small under a small h . Using the assumption (3.7), we shall be able to justify the following suggestion (we need it now): the value $u(t_k, x) + R(t_k, x)$ remains in the interval (u_\circ, u°) for h small enough.

We have

$$\begin{aligned}\bar{b}_k &= b(t_k, x, \bar{u}(t_k, x)) = b(t_k, x, u(t_k, x) + R(t_k, x)) \\ &= b(t_k, x, u(t_k, x)) + \Delta b = b_k + \Delta b,\end{aligned}$$

where $b_k := b(t_k, x, u(t_k, x))$ and due to assumption (i) Δb satisfies the inequality

$$(3.16) \quad |\Delta b| \leq K|R(t_k, x)|.$$

Analogously,

$$(3.17) \quad \begin{aligned}\bar{c}_k &= c_k + \Delta c, \quad |\Delta c| \leq K|R(t_k, x)|, \quad \bar{\sigma}_k = \sigma_k + \Delta\sigma, \quad |\Delta\sigma| \leq K|R(t_k, x)|, \\ \bar{g}_k &= g_k + \Delta g, \quad |\Delta g| \leq K|R(t_k, x)|.\end{aligned}$$

Using (3.16)-(3.17) and (3.14), we get

$$(3.18) \quad \begin{aligned}X_{k+1}^{(\pm)} &= x + \frac{1}{2}h(b_k + \Delta b) \\ &+ \frac{1}{2}hb(t_{k+1}, x + h[b_k + \Delta b], u(t_{k+1}, x + h[b_k + \Delta b]) + R(t_{k+1}, x + h\bar{b}_k)) \\ &\quad + \varepsilon^2 h[c_k + \Delta c] \pm \varepsilon h^{1/2}[\sigma_k + \Delta\sigma] \\ &= x + \frac{1}{2}hb_k + \frac{1}{2}hb(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) + \varepsilon^2 hc_k \pm \varepsilon h^{1/2}\sigma_k \\ &\quad + h[\Delta b/2 + \varepsilon^2 \Delta c] \pm \varepsilon h^{1/2}\Delta\sigma + h^2\Delta_1 + h\Delta_2\end{aligned}$$

and

$$(3.19) \quad \begin{aligned}\frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k)) \\ = \frac{1}{2}hg_k + \frac{1}{2}h\Delta g \\ + \frac{1}{2}hg(t_{k+1}, x + h[b_k + \Delta b], u(t_{k+1}, x + h[b_k + \Delta b]) + R(t_{k+1}, x + h\bar{b}_k)) \\ = \frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) + \frac{1}{2}h\Delta g + h^2\Delta_3 + h\Delta_4,\end{aligned}$$

where $|\Delta_1|, |\Delta_3| \leq K|R(t_k, x)|$ and $|\Delta_2|, |\Delta_4| \leq K|R(t_{k+1}, x + h\bar{b}_k)|$.

Substituting (3.18) and (3.19) in (3.15) and expanding $\frac{1}{2}u(t_{k+1}, X_{k+1}^{(\pm)})$, it is not difficult to obtain

$$\begin{aligned}u(t_k, x) + R(t_k, x) &= v^{(1)}(t_k, x) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(+)}) \\ &\quad + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(-)}) + r(t_k, x) + \tilde{r}(t_k, x),\end{aligned}$$

where $v^{(1)}(t_k, x)$ is defined in (3.10) and

$$|r(t_k, x)| \leq Kh|R(t_{k+1}, x + h\bar{b}_k)|, \quad |\tilde{r}(t_k, x)| \leq Kh|R(t_k, x)|.$$

Then by (3.9) we get

$$(3.20) \quad R(t_k, x) = \frac{1}{2}R(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(-)}) + r(t_k, x) + \tilde{r}(t_k, x) + O(h^3 + \varepsilon^2 h^2).$$

Introduce the notation

$$R_k := \max_{-\infty < x < +\infty} |R(t_k, x)|.$$

We have, from (3.20),

$$R_N = 0, \quad R_k \leq R_{k+1} + KhR_k + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2), \quad k = N - 1, \dots, 0.$$

Hence

$$R_k \leq \frac{C}{2K} \left(\left(\frac{1 + Kh}{1 - Kh} \right)^N - 1 \right) \cdot (h^2 + \varepsilon^2 h),$$

and therefore (remember that $N = (T - t_0)/h$)

$$R_k \leq \frac{C}{2K} (e^{4K(T-t_0)} - 1) \cdot (h^2 + \varepsilon^2 h).$$

Theorem 3.1 is proved.

Remark 3.1. For linear parabolic equations, i.e., when the coefficients of (3.1) do not depend on u , the method (3.5) becomes the explicit one with the global error $O(h^2 + \varepsilon^2 h)$, and can be applied to linear parabolic equations with small parameter. Note also that if the dimension d of the linear problem is high ($d > 3$ in practice) and it is enough to find the solution at a few points only, the Monte-Carlo technique is preferable.

3.3. Explicit layer methods. For implementation of the implicit method (3.5), one can use the method of simple iteration. If we take $u(t_{k+1}, x)$ as a null iteration, in the case of $b(t, x, u) \neq b(t, x)$ or $g(t, x, u) \neq g(t, x)$ the first iteration provides the one-step error $O(h^2)$ only. One can show that by applying the second iteration we get $O(h^3 + \varepsilon^2 h^2)$ as the one-step error. However it is possible to reach the same one-step accuracy by some modification of the first iteration that reduces the number of recalculations. The explicit layer method obtained on this way has the form (we use the same notation $\bar{u}(t_k, x)$ again)

$$\begin{aligned} (3.21) \quad & \bar{u}(t_N, x) = \varphi(x), \\ & \hat{b}_k = b(t_k, x, \bar{u}(t_{k+1}, x)), \quad \hat{c}_k = c(t_k, x, \bar{u}(t_{k+1}, x)), \quad \hat{\sigma}_k = \sigma(t_k, x, \bar{u}(t_{k+1}, x)), \\ & \bar{u}^{(1)}(t_k, x) = \bar{u}(t_{k+1}, x + h\hat{b}_k) + hg(t_k, x, \bar{u}(t_{k+1}, x)), \\ \bar{u}(t_k, x) = & \frac{1}{2} \bar{u}(t_{k+1}, x + h[b(t_k, x, \bar{u}^{(1)}(t_k, x)) + b(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k))]) / 2 \\ & + \varepsilon^2 h \hat{c}_k + \varepsilon h^{1/2} \hat{\sigma}_k \\ & + \frac{1}{2} \bar{u}(t_{k+1}, x + h[b(t_k, x, \bar{u}^{(1)}(t_k, x)) + b(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k))]) / 2 \\ & + \varepsilon^2 h \hat{c}_k - \varepsilon h^{1/2} \hat{\sigma}_k \\ & + \frac{1}{2} hg(t_k, x, \bar{u}^{(1)}(t_k, x)) + \frac{1}{2} hg(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k)), \\ & k = N - 1, \dots, 0. \end{aligned}$$

The following theorem can be proved by arguments like those used for Lemma 3.1 and Theorem 3.1.

Theorem 3.2. *Under assumptions (i) and (ii), the global error of the explicit layer method (3.21) is estimated by $O(h^2 + \varepsilon^2 h)$:*

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K \cdot (h^2 + \varepsilon^2 h),$$

where the constant K does not depend on h, k, x, ε .

Remark 3.2. Naturally, we can take other weak approximations (more accurate than the ones we use above) of SDE with small noise [19] to construct the corresponding high-order (with respect to h and ε) methods for the problem (3.1)-(3.2). In Section 6 we give high-order methods in some particular cases of the equation (3.1).

3.4. Singular case. The estimates of errors for the methods proposed above (Theorems 3.1 and 3.2) are obtained provided the bounds of derivatives of the solution to the problems considered are uniform with respect to $x \in R^d, t \in [t_0, T]$, and $0 < \varepsilon \leq \varepsilon^*$ (see (3.8)). This assumption is ensured, e.g., in the following case. Consider the first-order partial differential equation obtained from (3.1) with $\varepsilon = 0$

$$(3.22) \quad \frac{\partial u^0}{\partial t} + b(t, x, u^0) \frac{\partial u^0}{\partial x} + g(t, x, u^0) = 0, \quad t \in [t_0, T], \quad x \in R^1,$$

$$(3.23) \quad u^0(T, x) = \varphi(x).$$

If the coefficients of the equation (3.22) and the initial condition (3.23) are such that the solution $u^0(t, x), x \in R^1$, is sufficiently smooth for $t_0 \leq t \leq T$, then the derivatives of the solution $u(t, x)$ to (3.1)-(3.2) can be uniformly bounded with respect to $0 \leq \varepsilon \leq \varepsilon^*$ for $t \in [t_0, T]$ (see [7, 28]). Note that generally the assumption that the coefficients of (3.22) and the initial condition $\varphi(x)$ are bounded and smooth functions is not enough to ensure the regular behavior of $u^0(t, x)$ at any $t < T$ [7].

A lot of physical phenomena (e.g., formation and propagation of shock waves) having singular behavior is described by equations with small parameter. The derivatives of their solutions go to infinity as $\varepsilon \rightarrow 0$ and, rigorously speaking, Theorems 3.1 and 3.2 become inapplicable.

After the known change of variables $t = \varepsilon^2 t', x = \varepsilon^2 x'$, the problem (3.1)-(3.2) is rewritten for $v(t', x') := u(\varepsilon^2 t', \varepsilon^2 x')$ in the form

$$(3.24) \quad \frac{\partial v}{\partial t'} + \frac{1}{2} \sigma^2(\varepsilon^2 t', \varepsilon^2 x', v) \frac{\partial^2 v}{\partial x'^2} + (b(\varepsilon^2 t', \varepsilon^2 x', v) + \varepsilon^2 c(\varepsilon^2 t', \varepsilon^2 x', v)) \frac{\partial v}{\partial x'} + \varepsilon^2 g(\varepsilon^2 t', \varepsilon^2 x', v) = 0, \quad t' \in [t_0/\varepsilon^2, T/\varepsilon^2], \quad x' \in R^1, \quad 0 < \varepsilon \leq \varepsilon^*,$$

$$(3.25) \quad v(T/\varepsilon^2, x') = \varphi(\varepsilon^2 x').$$

If assumptions like (ii) hold for the solution $v(t', x')$ of (3.24)-(3.25) (we observe that the problem (3.24)-(3.25) is considered on long time intervals), then the derivatives of the solution $u(t, x)$ to (3.1)-(3.2) are estimated as

$$(3.26) \quad \left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j} \right| \leq \frac{K}{\varepsilon^{2(i+j)}}, \quad t \in [t_0, T], \quad x \in R^1, \quad 0 < \varepsilon \leq \varepsilon^*.$$

These bounds are natural ones for the problem (3.1)-(3.2) in the singular case.

If one followed the arguments of Lemma 3.1 and Theorem 3.1 in the singular case (i.e., taking the assumptions (i), (3.7), (3.26) instead of (i)-(ii)), an estimate of the form $\frac{K}{C} \frac{h}{\varepsilon^2} (e^{K(T-t_0)/\varepsilon^2} - 1)$ would be obtained for the proposed methods. Due to the big factor $1/\varepsilon^2$ in the exponent, this estimate is meaningless for practical purposes.

Our numerical tests (see Section 7.1) demonstrate that the proposed methods are of essentially better quality than could be predicted by this estimate. Apparently, the methods work fairly well in the singular case because the derivatives are large only in a small domain known as interior layer (see, e.g., [8]). The further theoretical investigation, namely, obtaining a realistic estimate for the errors of the methods proposed, should rest on a stability analysis and on more extensive properties of the solution considered. Recently the similar problem for finite elements methods has been considered in a few papers (see [3] and references therein).

However, in some particular, but important, singular cases of the problem (3.1)-(3.2) we get reasonable estimates (without $1/\varepsilon^2$ in the exponent) for the errors of the proposed methods by the arguments of Lemma 3.1 and Theorem 3.1. As an example, we give the following two theorems here (see their proofs in [20]).

Theorem 3.3. *Assume the coefficients b and σ in (3.1) are independent of u . Let the conditions (i) and (3.7) hold, and let the derivatives $|\partial^{i+j}u/\partial t^i \partial x^j|$, $i = 0, j = 1, 2, 3, 4$; $i = 1, j = 0, 1, 2$; $i = 2, j = 0, 1$; $i = 3, j = 0$, satisfy (3.26). Then for a sufficiently small h/ε^2 , the global error of the explicit layer method (3.21) is estimated as*

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq \frac{K}{C} \frac{h}{\varepsilon^4} (e^{K(T-t_0)} - 1),$$

where the constants C and K do not depend on h, k, x, ε .

In a lot of applications (e.g., in shock waves) the derivatives are significant only in a small interval (interior layer) $(x_*(t), x^*(t))$ of width $|x_*(t) - x^*(t)| \sim \varepsilon^2$:

$$(3.27) \quad \left| \frac{\partial^{i+j}u}{\partial t^i \partial x^j} \right| \leq \frac{K}{\varepsilon^{2(i+j)}}, \quad t \in [t_0, T], \quad x \in (x_*(t), x^*(t)), \quad 0 < \varepsilon \leq \varepsilon^*,$$

and

$$(3.28) \quad \left| \frac{\partial^{i+j}u}{\partial t^i \partial x^j} \right| \leq K, \quad t \in [t_0, T], \quad x \notin (x_*(t), x^*(t)), \quad 0 < \varepsilon \leq \varepsilon^*,$$

$$\int_{x \notin (x_*(t), x^*(t))} \left| \frac{\partial^{i+j}u}{\partial t^i \partial x^j} \right| dx \leq K, \quad i+j \neq 0, \quad t \in [t_0, T], \quad 0 < \varepsilon \leq \varepsilon^*.$$

Theorem 3.4. *Assume the coefficients b and σ in (3.1) are independent of u . Let conditions (i) and (3.7) hold, and let the derivatives $|\partial^{i+j}u/\partial t^i \partial x^j|$, $i = 0, j = 1, 2, 3, 4$; $i = 1, j = 0, 1, 2$; $i = 2, j = 0, 1$; $i = 3, j = 0$, satisfy (3.27) and (3.28). Then for a sufficiently small h/ε^2 , the global error of the explicit layer method (3.21) is estimated in l^1 -norm as*

$$(3.29) \quad \int_{R^1} |u(t_k, x) - \bar{u}(t_k, x)| dx \leq \frac{K}{C} \frac{h}{\varepsilon^2} (e^{K(T-t_0)} - 1),$$

where the constants C and K do not depend on h, k, x, ε .

The analogous theorems for the simpler method (2.8) give the same estimates of its error. However, in our experiments the layer method (3.21) gives better results than (2.8). To show the advantages of the method (3.21) in the singular case theoretically, further investigation is required. Seemingly, a more accurate analysis of the error of the method (3.21) should rest on more extensive properties of the solution $u(t, x)$.

See also Remarks 6.1 and 7.1, where in the singular situation we give reasonable estimates of the errors for some other particular cases of the problem (3.1)-(3.2).

4. NUMERICAL ALGORITHMS BASED ON INTERPOLATION

To calculate $u(t_k, x) \simeq \bar{u}(t_k, x)$ at a certain point x by the explicit layer method (3.21), one can use a recursive procedure. But it is evident that if the number of steps $N = (T - t_0)/h$ is relatively large, the recursive procedure is practically unrealizable due to the huge volume of needed calculations. In [18] another way is proposed, which is based on a discretization in the variable x and on an interpolation of $\bar{u}(t_k, x)$.

Introduce an equidistant space discretization: $\{x_j = x_0 + jh_x, j = 0, \pm 1, \pm 2, \dots\}$, where $x_0 \in R^1$ and h_x is a sufficiently small positive number.

When it does not lead to any misunderstanding, we use the old notation $\bar{u}, \bar{u}^{(1)}$, etc. for new values here.

Theorem 4.1. *Under assumptions (i) and (ii), the numerical algorithm based on the explicit method (3.21) and on the linear interpolation:*

$$\begin{aligned}
 (4.1) \quad & \bar{u}(t_N, x) = \varphi(x), \\
 & \hat{b}_{k,j} = b(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \quad \hat{c}_{k,j} = c(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \\
 & \hat{\sigma}_{k,j} = \sigma(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \\
 & \bar{u}^{(1)}(t_k, x_j) = \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}) + hg(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \\
 & \bar{u}(t_k, x_j) \\
 & = \frac{1}{2}\bar{u}(t_{k+1}, x_j + h[b(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) \\
 & \quad + b(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}))]/2 \\
 & \quad + \varepsilon^2 h\hat{c}_{k,j} + \varepsilon h^{1/2}\hat{\sigma}_{k,j}) \\
 & + \frac{1}{2}\bar{u}(t_{k+1}, x_j + h[b(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) \\
 & \quad + b(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}))]/2 \\
 & \quad + \varepsilon^2 h\hat{c}_{k,j} - \varepsilon h^{1/2}\hat{\sigma}_{k,j}) \\
 & + \frac{1}{2}hg(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + \frac{1}{2}hg(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j})), \\
 & \bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\bar{u}(t_k, x_{j+1}), \quad x_j < x < x_{j+1}, \\
 & \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N - 1, \dots, 0,
 \end{aligned}$$

has global error estimated by $O(h^2 + \varepsilon^2 h)$ if the value of h_x is selected as $h_x = \alpha \min(h^{3/2}, \varepsilon h)$, where α is a positive constant.

Proof. Here we follow the proof of the corresponding theorem in [18].

Introduce the notation

$$\begin{aligned}
 X_{k+1,j}^{(\pm)} & = x_j + \frac{1}{2}hb(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + \frac{1}{2}hb(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j})) \\
 & \quad + \varepsilon^2 h\hat{c}_{k,j} \pm \varepsilon h^{1/2}\hat{\sigma}_{k,j}.
 \end{aligned}$$

Just as in Theorem 3.1 for the implicit method (3.5), it is possible to obtain the expression like (3.20) for the algorithm (4.1) at the nodes x_j :

$$\begin{aligned}
 R(t_k, x_j) & = \frac{1}{2}R(t_{k+1}, X_{k+1,j}^{(+)}) + \frac{1}{2}R(t_{k+1}, X_{k+1,j}^{(-)}) + r(t_k, x_j) + O(h^3 + \varepsilon^2 h^2), \\
 |r(t_k, x_j)| & \leq KhR_{k+1},
 \end{aligned}$$

where

$$R_{k+1} := \max_{-\infty < x < +\infty} |R(t_{k+1}, x)|$$

(unlike (3.20), this formula does not contain $\tilde{r}(t_k, x_j)$, because the algorithm (4.1) is explicit and, for instance, Δb satisfies the inequality $|\Delta b| < K|R(t_{k+1}, x_j)|$).

Hence

$$(4.2) \quad |R(t_k, x_j)| \leq R_{k+1} + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2).$$

We have

$$(4.3) \quad u(t_k, x) = \frac{x_{j+1} - x}{h_x} u(t_k, x_j) + \frac{x - x_j}{h_x} u(t_k, x_{j+1}) + O(h_x^2), \quad x_j < x < x_{j+1},$$

where the interpolation error $O(h_x^2)$ satisfies the inequality $|O(h_x^2)| \leq Ch_x^2$ with C independent of $h, k, h_x, j, x, \varepsilon$.

From the last relation of (4.1) and from (4.3), we get

$$R(t_k, x) = \frac{x_{j+1} - x}{h_x} R(t_k, x_j) + \frac{x - x_j}{h_x} R(t_k, x_{j+1}) + O(h_x^2), \quad x_j < x < x_{j+1},$$

whence, due to (4.2)

$$|R(t_k, x)| \leq R_{k+1} + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2 + h_x^2)$$

with a new constant C .

As $h_x = \alpha \min(h^{3/2}, \varepsilon h)$, we can now obtain the statement of the theorem by the usual arguments. Theorem 4.1 is proved.

Remark 4.1. The way of proving Theorem 4.1 gives us some restriction on the type of interpolation procedure which can be used for constructing the numerical algorithm. The restriction is such that the sum of the absolute values of the coefficients staying at $\bar{u}(t_k, \cdot)$ in the interpolation procedure must be not greater than 1. We can make use of B-splines of the order $O(h_x^2)$, for which this restriction is valid. Cubic interpolation of order $O(h_x^4)$ does not satisfy the restriction. However, our numerical tests give fairly good results in the case of the algorithm based on cubic interpolation. See also Section 7.1, and some details and theoretical explanations in [18].

Remark 4.2. One can use a nonequidistant space discretization. For instance, one can take small h_x in the intervals of x where derivatives of the solution are big, and take relatively large h_x outside these intervals.

5. EXTENSIONS TO THE MULTI-DIMENSIONAL CASE AND TO REACTION-DIFFUSION SYSTEMS WITH SMALL PARAMETER

It is easy to generalize our algorithms to the multi-dimensional case ($d > 1$). For instance, consider the algorithm like (4.1) in the case of $d = 2$. Introduce the equidistant space discretization: $x_j^1 = x_0^1 + jh_{x^1}$, $x_l^2 = x_0^2 + lh_{x^2}$, $j, l = 0, \pm 1, \pm 2, \dots$, $(x_j^1, x_l^2)^\top \in \mathbb{R}^2$, $h_{x^i} = \alpha^i \min(h^{3/2}, \varepsilon h)$, $i = 1, 2$, and the α^i are positive constants.

The algorithm with global error $O(h^2 + \varepsilon^2 h)$ has the form

$$\begin{aligned}
 (5.1) \quad & \bar{u}(t_N, x^1, x^2) = \varphi(x^1, x^2), \\
 & \hat{b}_{k,j,l} = b(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)), \quad \hat{c}_{k,j,l} = c(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)), \\
 & \hat{\sigma}_{k,j,l} = \sigma(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)), \\
 & \bar{u}^{(1)}(t_k, x_j^1, x_l^2) = \bar{u}(t_{k+1}, x_j^1 + h\hat{b}_{k,j,l}^1, x_l^2 + h\hat{b}_{k,j,l}^2) + hg(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)), \\
 & (\tilde{X}_{k+1,j,l}^1, \tilde{X}_{k+1,j,l}^2)^\top = (x_j^1, x_l^2)^\top + \frac{1}{2}hb(t_k, x_j^1, x_l^2, \bar{u}^{(1)}(t_k, x_j^1, x_l^2)) \\
 & \quad + \frac{1}{2}hb(t_{k+1}, x_j^1 + h\hat{b}_{k,j,l}^1, x_l^2 \\
 & \quad \quad + h\hat{b}_{k,j,l}^2, \bar{u}(t_{k+1}, x_j^1 + h\hat{b}_{k,j,l}^1, x_l^2 + h\hat{b}_{k,j,l}^2)) \\
 & \quad + \varepsilon^2 h\hat{c}_{k,j,l}, \\
 & \bar{u}(t_k, x_j^1, x_l^2) \\
 & = \frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}_{k+1,j,l}^1 + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{11} + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{12}, \tilde{X}_{k+1,j,l}^2 \\
 & \quad \quad + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{21} + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{22}) \\
 & \quad + \frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}_{k+1,j,l}^1 - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{11} + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{12}, \tilde{X}_{k+1,j,l}^2 \\
 & \quad \quad - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{21} + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{22}) \\
 & \quad + \frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}_{k+1,j,l}^1 + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{11} - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{12}, \tilde{X}_{k+1,j,l}^2 \\
 & \quad \quad + \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{21} - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{22}) \\
 & \quad + \frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}_{k+1,j,l}^1 - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{11} - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{12}, \tilde{X}_{k+1,j,l}^2 \\
 & \quad \quad - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{21} - \varepsilon h^{1/2}\hat{\sigma}_{k,j,l}^{22}) \\
 & \quad + \frac{1}{2}hg(t_k, x_j^1, x_l^2, \bar{u}^{(1)}(t_k, x_j^1, x_l^2)) \\
 & \quad + \frac{1}{2}hg(t_{k+1}, x_j^1 + h\hat{b}_{k,j,l}^1, x_l^2 + h\hat{b}_{k,j,l}^2, \bar{u}(t_{k+1}, x_j^1 + h\hat{b}_{k,j,l}^1, x_l^2 + h\hat{b}_{k,j,l}^2)), \\
 & \bar{u}(t_k, x^1, x^2) \\
 & = \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_l^2) + \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x_l^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_{l+1}^2) \\
 & \quad + \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_l^2) + \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x_l^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_{l+1}^2), \\
 & \quad x_j^1 \leq x^1 \leq x_{j+1}^1, \quad x_l^2 \leq x^2 \leq x_{l+1}^2, \quad j, l = 0, \pm 1, \pm 2, \dots, \\
 & \quad k = N - 1, \dots, 0.
 \end{aligned}$$

The proposed methods are applicable to the Cauchy problem for systems of reaction-diffusion equations with small parameter as well. For instance, in the case of the system (we take $d = 1$ for simplicity in writing)

$$\begin{aligned}
 (5.2) \quad & \frac{\partial u_q}{\partial t} + \frac{\varepsilon^2}{2}\sigma_q^2(t, x, u) \frac{\partial^2 u_q}{\partial x^2} + (b_q(t, x, u) + \varepsilon^2 c_q(t, x, u)) \frac{\partial u_q}{\partial x} + g_q(t, x, u) = 0, \\
 & t \in [t_0, T], \quad x \in R^1, \quad q = 1, \dots, n, \quad u := (u_1, \dots, u_n), \\
 & u_q(T, x) = \varphi_q(x),
 \end{aligned}$$

the method like (3.21) has the form

$$\begin{aligned}
 (5.3) \quad & \bar{u}_q(t_N, x) = \varphi_q(x), \\
 & \hat{b}_{q,k} = b_q(t_k, x, \bar{u}(t_{k+1}, x)), \hat{c}_{q,k} = c_q(t_k, x, \bar{u}(t_{k+1}, x)), \hat{\sigma}_{q,k} = \sigma_q(t_k, x, \bar{u}(t_{k+1}, x)), \\
 & \bar{u}_i^{(1)}(t_k, x) = \bar{u}_i(t_{k+1}, x + h\hat{b}_{i,k}) + hg_i(t_k, x, \bar{u}(t_{k+1}, x)), \quad i = 1, \dots, n, \\
 & \bar{u}_q(t_k, x) = \frac{1}{2}\bar{u}_q(t_{k+1}, x + h[b_q(t_k, x, \bar{u}^{(1)}(t_k, x)) \\
 & \quad + b_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k}))]/2 + \varepsilon^2 h\hat{c}_{q,k} + \varepsilon h^{1/2}\hat{\sigma}_{q,k}) \\
 & \quad + \frac{1}{2}\bar{u}_q(t_{k+1}, x + h[b_q(t_k, x, \bar{u}^{(1)}(t_k, x)) \\
 & \quad + b_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k}))]/2 + \varepsilon^2 h\hat{c}_{q,k} - \varepsilon h^{1/2}\hat{\sigma}_{q,k}) \\
 & \quad + \frac{1}{2}hg_q(t_k, x, \bar{u}^{(1)}(t_k, x)) + \frac{1}{2}hg_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k})), \\
 & \quad q = 1, \dots, n, \quad k = N - 1, \dots, 0.
 \end{aligned}$$

It is easy to write down the corresponding algorithm like (4.1) on the basis of this method.

See [18] to obtain such methods and algorithms for another type of reaction-diffusion systems.

6. HIGH-ORDER METHODS FOR SEMILINEAR EQUATION WITH SMALL CONSTANT DIFFUSION AND ZERO ADVECTION

Here we restrict ourselves to the case of $d = 1$ for simplicity again. Consider the Cauchy problem

$$(6.1) \quad \frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} + g(t, x, u) = 0, \quad t \in [t_0, T], \quad x \in R^1,$$

$$(6.2) \quad u(T, x) = \varphi(x).$$

We assume that the term $g(t, x, u)$ is a uniformly bounded and sufficiently smooth function and conditions like (ii) from Section 3 are fulfilled for the solution $u(t, x)$ to (6.1)-(6.2). Note that to construct high-order methods we need uniform boundedness of the derivatives of $u(t, x)$ with higher orders than in the assumption (3.8). To realize the methods of this section, we can avoid any interpolation. This is so because when $\sigma \equiv 1, b \equiv 0, c \equiv 0$ we are able to choose a special space discretization. The methods of this section are tested by simulation of the generalized KPP-equation with a small parameter (see Section 7.2).

The probabilistic representation of the solution to (6.1)-(6.2) has the form (see (2.1)-(2.3))

$$(6.3) \quad u(t, x) = E(\varphi(X_{t,x}(T)) + Z_{t,x,0}(T)),$$

where $X_{t,x}(s)$ and $Z_{t,x,z}(s), s \geq t$, satisfy the system

$$\begin{aligned}
 (6.4) \quad dX &= \varepsilon dw(s), \quad X(t) = x, \\
 dZ &= g(s, X, u(s, X))ds, \quad Z(t) = z.
 \end{aligned}$$

Note that (6.4) is a system of differential equations with small *additive* noise.

6.1. Two-layer methods. For completeness of presentation, let us write down the layer methods (2.8) and (3.21) and the second-order layer method from [18] in the case of the problem (6.1)-(6.2).

The explicit layer method (2.8) with error $O(h)$ has the form

$$(6.5) \quad \begin{aligned} \bar{u}(t_N, x_j) &= \varphi(x_j), \\ \bar{u}(t_k, x_j) &= \frac{1}{2}\bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j - \varepsilon h^{1/2}) + hg(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)), \\ x_j &= x_0 + j\varepsilon h^{1/2}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N - 1, \dots, 0. \end{aligned}$$

Note that it coincides with the well-known finite-difference scheme under the special relation of time and space steps ($h_x = \varepsilon h_t^{1/2}$) in the scheme.

In the case of the problem (6.1)-(6.2) the explicit layer method (3.21) with error $O(h^2 + \varepsilon^2 h)$ takes the form

$$(6.6) \quad \begin{aligned} \bar{u}(t_N, x_j) &= \varphi(x_j), \\ \bar{u}^{(1)}(t_k, x_j) &= \bar{u}(t_{k+1}, x_j) + hg(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \\ \bar{u}(t_k, x_j) &= \frac{1}{2}\bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j - \varepsilon h^{1/2}) \\ &\quad + \frac{1}{2}h[g(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j))], \\ x_j &= x_0 + j\varepsilon h^{1/2}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N - 1, \dots, 0. \end{aligned}$$

Using the second-order Runge-Kutta scheme (2.12), the implicit second-order layer method for the semilinear parabolic equation with constant diffusion is constructed in [18]. In the case of the problem (6.1)-(6.2) the implicit layer method with error $O(h^2)$ has the form

$$(6.7) \quad \begin{aligned} \bar{u}(t_N, x_j) &= \varphi(x_j), \\ \bar{u}(t_k, x_j) &= \frac{1}{6}\bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}) + \frac{2}{3}\bar{u}(t_{k+1}, x_j) + \frac{1}{6}\bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}) \\ &\quad + \frac{h}{2}g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{h}{3}g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)) \\ &\quad + \frac{h}{12}g(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2})) \\ &\quad + \frac{h}{12}g(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2})), \\ x_j &= x_0 + j\sqrt{3}\varepsilon h^{1/2}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N - 1, N - 2, \dots, 0. \end{aligned}$$

To solve the algebraic equations obtained at each step of the method (6.7), one can use the Newton method or the method of simple iteration.

Remark 6.1. In the singular case the natural bounds for derivatives of the solution to (6.1)-(6.2) have the form

$$(6.8) \quad \left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j} \right| \leq \frac{K}{\varepsilon^j}, \quad t \in [t_0, T], \quad x \in R^1, \quad 0 < \varepsilon \leq \varepsilon^*.$$

These bounds are obtained using the change of variables $t = t', \quad x = \varepsilon x'$ (cf. Section 3.4).

By the same arguments as in Theorem 3.3 one can prove under (6.8) that the errors of both methods (6.5) and (6.6) are estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq Kh,$$

where the constant K does not depend on x, k, h, ε .

Nevertheless, the method (6.6) gives better results than (6.5) in our experiments. One can explain this by the fact that the constant K of (6.6) is essentially less than the K of (6.5).

Under (6.8) the error of the method (6.7) remains $O(h^2)$.

Note that in Section 7.2 we present results of testing these methods (instead of (6.5) we use a modification of it) on an equation in which g depends on ε and the derivatives of the solution have other bounds than (6.8) (see Remark 7.1 and other details in Section 7.2).

6.2. Three-layer methods. Here we obtain two three-layer methods. Their one-step errors can be estimated by the same arguments as in Lemma 3.1. We do not prove their convergence, which requires stability analysis of multi-layer methods. We test these methods in our experiments, and they give fairly good results.

To calculate $\bar{u}(t_{k+1}, x)$ by a three-layer method, two previous layers are used. So, to start simulations we should know $\bar{u}(t_N, x)$ and $\bar{u}(t_{N-1}, x)$. To simulate $\bar{u}(t_{N-1}, x)$ one can use, e.g., the two-layer method (6.7) with a sufficiently small step. Below we consider this layer to be known, and denote $\psi(x) := \bar{u}(t_{N-1}, x)$.

Apply the special Runge-Kutta scheme (2.13) to approximate (6.4):

$$(6.9) \quad \begin{aligned} X_{t_k, x}(t_{k+1}) &\simeq \bar{X}_{t_k, x}(t_{k+1}) = x + \varepsilon h^{1/2} \xi_k, \\ Z_{t_k, x, z}(t_{k+1}) &\simeq \bar{Z}_{t_k, x, z}(t_{k+1}) \\ &= z + \frac{h}{6} (g(t_k, x, u(t_k, x)) + 2g(t_{k+1/2}, x, u(t_{k+1/2}, x)) \\ &\quad + 2g(t_{k+1/2}, x + \varepsilon h^{1/2} \xi_k, u(t_{k+1/2}, x + \varepsilon h^{1/2} \xi_k)) \\ &\quad + g(t_{k+1}, x + \varepsilon h^{1/2} \xi_k, u(t_{k+1}, x + \varepsilon h^{1/2} \xi_k))), \end{aligned}$$

where ξ_k are i.i.d. variables with the law $P(\xi = 0) = 2/3$, $P(\xi = \pm\sqrt{3}) = 1/6$.

The implicit method with one-step error $O(h^5 + \varepsilon^2 h^3)$ has the form (to get the method we use the scheme (6.9) with the time step $2h$)

$$(6.10) \quad \begin{aligned} \bar{u}(t_N, x_j) &= \varphi(x_j), \quad \bar{u}(t_{N-1}, x_j) = \psi(x_j), \\ \bar{u}(t_k, x_j) &= \frac{1}{6} \bar{u}(t_{k+2}, x_j + \sqrt{6} \varepsilon h^{1/2}) + \frac{2}{3} \bar{u}(t_{k+2}, x_j) + \frac{1}{6} \bar{u}(t_{k+2}, x_j - \sqrt{6} \varepsilon h^{1/2}) \\ &\quad + \frac{h}{3} g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{10h}{9} g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)) \\ &\quad + \frac{h}{9} g(t_{k+1}, x_j + \sqrt{6} \varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j + \sqrt{6} \varepsilon h^{1/2})) \\ &\quad + \frac{h}{9} g(t_{k+1}, x_j - \sqrt{6} \varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j - \sqrt{6} \varepsilon h^{1/2})) \\ &\quad + \frac{h}{18} g(t_{k+2}, x_j + \sqrt{6} \varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j + \sqrt{6} \varepsilon h^{1/2})) + \frac{2h}{9} g(t_{k+2}, x_j, \bar{u}(t_{k+2}, x_j)) \\ &\quad + \frac{h}{18} g(t_{k+2}, x_j - \sqrt{6} \varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j - \sqrt{6} \varepsilon h^{1/2})), \\ x_j &= x_0 + j \sqrt{6} \varepsilon h^{1/2}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N-2, N-3, \dots, 0. \end{aligned}$$

Let us look at the stability properties of this method in the simple case when u and g in (6.1) do not depend on x , i.e., apply the method (6.10) to the ordinary differential equation

$$(6.11) \quad \frac{du}{dt} + g(t, u) = 0, \quad t \leq T, \quad u(T) = \varphi.$$

Recall (see, e.g., [10]) that a linear n -step method for (6.11)

$$\begin{aligned} \alpha_n u_k + \alpha_{n-1} u_{k+1} + \dots + \alpha_0 u_{k+n} &= h \cdot (\beta_n g_k + \dots + \beta_0 g_{k+n}), \\ g_i &= g(t_i, u_i), \quad \alpha_n \neq 0, \quad |\alpha_0| + |\beta_0| > 0, \end{aligned}$$

is zero-stable (D-stable) if the generating polynomial

$$(6.12) \quad \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_0 = 0$$

satisfies the root condition: the roots of (6.12) lie on or within the unit circle, and the roots on the unit circle are simple.

In the case of (6.11) the method (6.10) coincides with the Milne two-step method which is of the order $O(h^4)$ and is zero-stable. Its generating polynomial has two roots: 1 and -1 . As is known [10], the root -1 can be dangerous for some differential equations. The method (6.10) has unstable behavior in our numerical tests on the generalized KPP-equation with a small parameter, (7.10)-(7.12) (Section 7.2). One can see that the method (6.10) does not preserve the property $u \leq 1$ of the problem (7.10)-(7.12) that leads to an unstable behavior of the approximate solutions. We modify the method (6.10) in the experiments: if $\bar{u}(t_k, x_j) > 1$, we put $\bar{u}(t_k, x_j) = 1$. Since locally, in a single step, the resulting difference $0 < \bar{u}(t_k, x_j) - 1$ is not greater than the one-step error of this method, this modification does not change the one-step accuracy order of the method. The modified method turned out to work fairly well if applied to the generalized KPP-equation. However, the modification is based on knowledge of the properties of the solution, and it may be difficult to find such a modification for another problem. Fortunately, we are able to approximate the system (6.4) by another weak scheme and obtain a method for (6.1)-(6.2) with better stability properties in the sense considered above but with the one-step error of lower order (see the method (6.13) below). Let us note that it is possible to reach both the same one-step accuracy $O(h^5 + \varepsilon^2 h^3)$ and the better stability properties by a four-layer method.

Approximate (6.4) by the special scheme with one-step order $O(h^4 + \varepsilon^2 h^3)$:

$$\begin{aligned} X_{t_k, x}(t_{k+1}) &\simeq \bar{X}_{t_k, x}(t_{k+1}) = x + \varepsilon h^{1/2} \xi_k, \\ Z_{t_k, x, z}(t_{k+1}) &\simeq \bar{Z}_{t_k, x, z}(t_{k+1}) = z + \frac{h}{12} (5g(t_k, x, u(t_k, x)) + g(t_{k+1}, x, u(t_{k+1}, x)) \\ &\quad + 7g(t_{k+1}, x + \varepsilon h^{1/2} \xi_k, u(t_{k+1}, x + \varepsilon h^{1/2} \xi_k)) \\ &\quad - g(t_{k+2}, x + \varepsilon h^{1/2} \xi_k, u(t_{k+2}, x + \varepsilon h^{1/2} \xi_k))), \end{aligned}$$

where the ξ_k are i.i.d. variables with the law $P(\xi = 0) = 2/3, P(\xi = \pm\sqrt{3}) = 1/6$.

The three-layer implicit method with one-step error $O(h^4 + \varepsilon^2 h^3)$ has the form

$$\begin{aligned}
 (6.13) \quad & \bar{u}(t_N, x_j) = \varphi(x_j), \quad \bar{u}(t_{N-1}, x_j) = \psi(x_j), \\
 & \bar{u}(t_k, x_j) = \frac{1}{6}\bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}) + \frac{2}{3}\bar{u}(t_{k+1}, x_j) + \frac{1}{6}\bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}) \\
 & \quad + \frac{5h}{12}g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{17h}{36}g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)) \\
 & \quad + \frac{7h}{72}g(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2})) \\
 & \quad + \frac{7h}{72}g(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2})) \\
 & \quad - \frac{h}{72}g(t_{k+2}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j + \sqrt{3}\varepsilon h^{1/2})) \\
 & \quad - \frac{h}{18}g(t_{k+2}, x_j, \bar{u}(t_{k+2}, x_j)) - \frac{h}{72}g(t_{k+2}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j - \sqrt{3}\varepsilon h^{1/2})), \\
 & \quad x_j = x_0 + j\sqrt{3}\varepsilon h^{1/2}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = N-2, N-3, \dots, 0.
 \end{aligned}$$

For (6.11) this method coincides with one of the implicit two-step Adams methods of order $O(h^3)$, and the roots of its generating polynomial are 1 and 0. One can expect that in the case of the problem (6.1)-(6.2) the method (6.13) also possesses better stability properties than (6.10). In our numerical tests on the generalized KPP-equation with a small parameter (Section 7.2) the method (6.13) has stable behavior.

To solve the algebraic equations obtained at each step of the methods (6.10) and (6.13), one can use the Newton method or the method of simple iteration.

Remark 6.2. The methods of this section can be extended for problems of a higher dimension or for systems of reaction-diffusion equations. Additionally using some other weak approximations to SDE with small additive noise, new layer methods can be constructed. For instance, three- and four-layer methods with the one-step error $O(h^5 + \varepsilon^2 h^2)$ can be obtained. It is also not difficult to get an implicit four-layer method with the one-step error $O(h^5 + \varepsilon^2 h^3)$ for (6.1)-(6.2) possessing good stability properties in the above sense, or an explicit four-layer method with the one-step error $O(h^4 + \varepsilon^2 h^3)$, and so on.

Remark 6.3. In the preprint [20] another special layer method for some nonlinear problems is constructed. To approximate SDE, it attracts the *exact simulation* of the Brownian motion instead of the weak schemes used in Sections 2-6. In [17] a few methods with exact simulation of some components of SDE are proposed for linear problems. It was shown that these methods are preferable to weak schemes in some situations. In the nonlinear case layer methods, using the exact simulation of the Brownian motion, possess some preferable properties as well.

7. NUMERICAL TESTS

We start with two short digressions of a general nature.

In the previous sections we dealt with semilinear parabolic equations with *negative* direction of time t : the equations are considered for $t < T$ and the “initial” conditions are given at $t = T$. This form of equation is suitable for the probabilistic approach which we use to construct numerical methods. Of course, our methods

are adaptable to semilinear parabolic equations with *positive* direction of time, and this adaptation is particularly easy in the autonomous case.

Consider the Cauchy problem for autonomous semilinear parabolic equations with positive direction of time

$$(7.1) \quad \frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2} \sigma^2(x, u) \frac{\partial^2 u}{\partial x^2} + (b(x, u) + \varepsilon^2 c(x, u)) \frac{\partial u}{\partial x} + g(x, u), \quad t > 0, x \in R^1,$$

$$(7.2) \quad u(0, x) = \varphi(x).$$

Note that if we substitute the solution $u(t, x)$ of this problem in the coefficients σ , b , c , and g , the equation (7.1) becomes nonautonomous. Nevertheless, it is not difficult to obtain numerical procedures with positive direction of time which correspond to the algorithms given in the previous sections. In our numerical tests we use algorithms with positive direction of time (see, e.g., (7.8), (7.9) below).

We noted in Remark 4.1 that the algorithms based on *cubic interpolation* give quite good results. We use the advantage of the cubic interpolation in our numerical tests on the Burgers equation. Let us recall that a sufficiently smooth function $f(x)$, $x \in R^1$, can be interpolated by cubic interpolation as

$$(7.3) \quad f(x) \simeq \bar{f}(x) = \sum_{i=0}^3 \Phi_{j,i}(x) f(x_{j+i}), \quad x_{j+1} < x < x_{j+2},$$

$$\Phi_{j,i}(x) = \prod_{m=0, m \neq i}^3 \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},$$

where $x_j = x_0 + j \cdot h_x$, $x_0 \in R^1$, $j = 0, \pm 1, \pm 2, \dots$, and h_x is a positive number.

The error of the cubic interpolation (7.3) is estimated by

$$|\bar{f}(x) - f(x)| \leq \frac{3}{128} \max_{x_j < x < x_{j+3}} \left| \frac{\partial^4 u}{\partial x^4} \right| \cdot h_x^4, \quad x_{j+1} < x < x_{j+2}.$$

Recall (see Theorem 4.1) that the algorithm (4.1), based on the layer method (3.21) and on linear interpolation, has error estimated by $O(h^2 + \varepsilon^2 h)$ provided $h_x = \min(h^{3/2}, \varepsilon h)$. One can expect that under assumptions (i) and (ii) from Section 3 the algorithm based on the layer method (3.21) and on the cubic interpolation (7.3) can achieve the same accuracy $O(h^2 + \varepsilon^2 h)$ with h_x taken equal to $\min(h^{3/4}, \sqrt{\varepsilon h})$ only. Our numerical tests on the Burgers equation support this supposition. See the theoretical explanations in [18] as well.

As mentioned in the Introduction, all the methods have been tested through computer experiments. Some of them are presented below.

7.1. The Burgers equation with small viscosity. The one-dimensional Burgers equation with small viscosity has the form

$$(7.4) \quad \frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad t > 0, x \in R^1,$$

$$(7.5) \quad u(0, x) = \varphi(x).$$

By means of the Cole-Hopf transformation, one can find the explicit solution of the problem (7.4)-(7.5):

$$u(t, x) = \frac{\int_{-\infty}^{\infty} K(t, x, y) \varphi(y) \exp\left(-\frac{1}{\varepsilon^2} \int_0^y \varphi(\xi) d\xi\right) dy}{\int_{-\infty}^{\infty} K(t, x, y) \exp\left(-\frac{1}{\varepsilon^2} \int_0^y \varphi(\xi) d\xi\right) dy},$$

$$K(t, x, y) = \frac{1}{\sqrt{2\pi\varepsilon^2 t}} \exp\left(-\frac{(x-y)^2}{2\varepsilon^2 t}\right).$$

Let us take the initial condition $\varphi(x)$ of the form

$$(7.6) \quad \varphi(x) = \begin{cases} c & x < l_0, \\ \lambda(x), & l_0 \leq x \leq l_0 + l, \\ d, & x > l_0 + l, \end{cases}$$

where c, d, l_0, l are numbers, $c > d, l \geq 0$, $\lambda(x)$ is a bounded measurable function, and $d \leq \lambda(x) \leq c$.

Recall some theoretical facts concerning the problem (7.4)-(7.5), (7.6) (the details are given, e.g., in [9, 28]).

The solution $u(t, x)$ to (7.4)-(7.5), (7.6) is uniformly bounded:

$$d \leq u(t, x) \leq c, \quad x \in R^1, 0 \leq t, 0 \leq \varepsilon \leq \varepsilon^*.$$

Let the initial condition $\varphi(x)$ be a sufficiently smooth function. Introduce the time moment T such that the solution of the hyperbolic problem obtained from (7.4)-(7.5), (7.6) for $\varepsilon = 0$ is smooth at $t < T$ and discontinuous at $t \geq T$. The solution $u(t, x)$ to (7.4)-(7.5), (7.6) is regular for $t \leq t_* < T$:

$$\left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j}(t, x) \right| \leq K, \quad x \in R^1, 0 \leq t \leq t_*, 0 \leq \varepsilon \leq \varepsilon^*.$$

If $t \geq T$, then the solution is singular in an interval $(x_*(t), x^*(t))$ with width $|x^*(t) - x_*(t)| \sim \varepsilon^2$:

$$\left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j}(t, x) \right| \leq \frac{K}{\varepsilon^{2(i+j)}}, \quad x \in (x_*(t), x^*(t)), t \geq T, 0 < \varepsilon \leq \varepsilon^*,$$

$$\left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j}(t, x) \right| \leq K, \quad x \notin (x_*(t), x^*(t)), t \geq T, 0 < \varepsilon \leq \varepsilon^*.$$

In our experiments we take $\lambda(x)$ equal to

$$(7.7) \quad \lambda(x) = a - b \sin \frac{\pi x}{\mu},$$

$$\mu > 0, b > 0, \text{ and } c = a + b, d = a - b, l = \mu, l_0 = -\frac{\mu}{2}.$$

For this $\lambda(x)$ the moment T can easily be found: $T = \mu/\pi b$.

We compare the behavior of two algorithms. The first one is based on the layer method (3.21) with the cubic interpolation (7.3). In the case of the problem (7.4)-(7.5) it has the following form:

$$\begin{aligned}
 (7.8) \quad & \bar{u}(0, x) = \varphi(x), \\
 & \bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j))) + \varepsilon h^{1/2} \\
 & \quad + \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j))) - \varepsilon h^{1/2}, \\
 & \bar{u}(t_k, x) = \sum_{i=0}^3 \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_{j+1} < x < x_{j+2}, \\
 & \Phi_{j,i}(x) = \prod_{m=0, m \neq i}^3 \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}}, \\
 & j = 0, \pm 1, \pm 2, \dots, \quad k = 0, \dots, N - 1,
 \end{aligned}$$

where $x_j = x_0 + j \cdot h_x$.

The second algorithm is based on the layer method (2.8) proposed in [18] and on the cubic interpolation (7.3):

$$\begin{aligned}
 (7.9) \quad & \bar{u}(0, x) = \varphi(x), \\
 & \bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j)) + \varepsilon h^{1/2} + \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j)) - \varepsilon h^{1/2}, \\
 & \bar{u}(t_k, x) = \sum_{i=0}^3 \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_{j+1} < x < x_{j+2}, \\
 & \Phi_{j,i}(x) = \prod_{m=0, m \neq i}^3 \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}}, \\
 & j = 0, \pm 1, \pm 2, \dots, \quad k = 0, \dots, N - 1.
 \end{aligned}$$

Table 1 gives the results of simulation of the problem (7.4)-(7.5) with $\varphi(x)$ from (7.6), (7.7) in the case of the regular solution. In this case assumptions (i) and (ii) from Section 3 are fulfilled, and the algorithm (7.8) has error $O(h^2 + \varepsilon^2 h)$ while (7.9) has error $O(h)$. The value of h_x is taken equal to $h^{3/4}$. We present the errors of the approximate solutions \bar{u} in the discrete Chebyshev norm and in the l^1 -norm:

$$\begin{aligned}
 err^c(t) &= \max_{x_i} |\bar{u}(t, x_i) - u(t, x_i)|, \\
 err^l(t) &= \sum_i |\bar{u}(t, x_i) - u(t, x_i)| \cdot h_x.
 \end{aligned}$$

One can infer from Table 1 that the proposed special algorithm (7.8) with error $O(h^2 + \varepsilon^2 h)$ requires less computational effort than the algorithm (7.9) with error $O(h)$, and that the experimental data conform to the orders of accuracy of the algorithms given by the theoretical results.

To find the solution $u(t, x)$ to the problem (7.4)-(7.5), (7.6) for $t > T$, when the solution is singular, we realize the following numerical procedure: we simulate the problem by the algorithms (7.8) and (7.9) with a sufficiently big time step h and with $h_x = h^{3/4}$ up to the time moment $t_* < T$; then we change the time step h to a smaller one h_* , take $h_x = h_*$, and continue the simulations.

TABLE 1. The Burgers equation (regular solution). Dependence of the errors $err^c(t_*)$ and $err^l(t_*)$ on h and ε for the algorithms (7.8) and (7.9) when $a = b = 0.5$, $\mu = 8$, and $t_* = 4$ ($T \approx 5.09$).

ε	h	algorithm (7.8)		algorithm (7.9)	
		$err^c(t_*)$	$err^l(t_*)$	$err^c(t_*)$	$err^l(t_*)$
0.3	0.3	$0.1351 \cdot 10^{-1}$	$0.1531 \cdot 10^{-1}$	0.1130	0.1397
	0.1	$0.2146 \cdot 10^{-2}$	$0.3347 \cdot 10^{-2}$	$0.3978 \cdot 10^{-1}$	$0.4628 \cdot 10^{-1}$
	0.01	$0.2295 \cdot 10^{-3}$	$0.3874 \cdot 10^{-3}$	$0.4221 \cdot 10^{-2}$	$0.4799 \cdot 10^{-2}$
	0.001	$0.2265 \cdot 10^{-4}$	$0.3947 \cdot 10^{-4}$	$0.4244 \cdot 10^{-3}$	$0.4814 \cdot 10^{-3}$
0.1	0.3	$0.2325 \cdot 10^{-1}$	$0.2051 \cdot 10^{-1}$	0.1539	0.1519
	0.1	$0.4255 \cdot 10^{-2}$	$0.2287 \cdot 10^{-2}$	$0.6084 \cdot 10^{-1}$	$0.5007 \cdot 10^{-1}$
	0.03	$0.3489 \cdot 10^{-3}$	$0.2396 \cdot 10^{-3}$	$0.2029 \cdot 10^{-1}$	$0.1553 \cdot 10^{-1}$
	0.01	$0.4444 \cdot 10^{-4}$	$0.5442 \cdot 10^{-4}$	$0.6751 \cdot 10^{-2}$	$0.5169 \cdot 10^{-2}$
	0.001	$0.5529 \cdot 10^{-5}$	$0.6374 \cdot 10^{-5}$	$0.6806 \cdot 10^{-3}$	$0.5189 \cdot 10^{-3}$

TABLE 2. The Burgers equation (singular solution). The errors $err^c(t)$ and $err^l(t)$ for $t = 8$ ($T \approx 5.09$). Other parameter values are the same as in Table 1. The time steps h and h_* are used when $t \leq t_*$ and $t > t_*$ correspondingly.

ε	h	h_*	algorithm (7.8)		algorithm (7.9)	
			$err^c(t)$	$err^l(t)$	$err^c(t)$	$err^l(t)$
0.3	0.1	0.01	$0.6322 \cdot 10^{-2}$	$0.2713 \cdot 10^{-2}$	0.1693	$0.6555 \cdot 10^{-1}$
	0.01	0.001	$0.4036 \cdot 10^{-3}$	$0.2482 \cdot 10^{-3}$	$0.1771 \cdot 10^{-1}$	$0.6782 \cdot 10^{-2}$
	0.001	0.0001	$0.5977 \cdot 10^{-4}$	$0.3760 \cdot 10^{-4}$	$0.1776 \cdot 10^{-2}$	$0.6931 \cdot 10^{-3}$
0.1	0.1	0.001	$0.5553 \cdot 10^{-1}$	$0.2351 \cdot 10^{-2}$	> 0.5	$0.6594 \cdot 10^{-1}$
	0.03	0.001	$0.1219 \cdot 10^{-1}$	$0.4699 \cdot 10^{-3}$	> 0.5	$0.3189 \cdot 10^{-1}$
		0.0001	$0.3955 \cdot 10^{-2}$	$0.1718 \cdot 10^{-3}$	0.4029	$0.1716 \cdot 10^{-1}$
	0.01	0.0001	$0.7047 \cdot 10^{-3}$	$0.3007 \cdot 10^{-4}$	0.1687	$0.6828 \cdot 10^{-2}$
	0.001	0.0001	$0.4139 \cdot 10^{-3}$	$0.2312 \cdot 10^{-4}$	$0.5461 \cdot 10^{-1}$	$0.2185 \cdot 10^{-2}$

Table 2 gives the results of simulation of the problem (7.4)-(7.5) with $\varphi(x)$ from (7.6), (7.7) when $t > T$. One can see that in the singular case the behavior of the algorithm (7.8) is also better than the behavior of (7.9).

In connection with this example, see the numerical experiments in [1] and [18] as well.

7.2. The generalized KPP-equation with a small parameter. Consider the problem

$$(7.10) \quad \frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} + g(x, u; \varepsilon), \quad t > 0, x \in R^1,$$

$$(7.11) \quad u(0, x) = \chi_-(x) = \begin{cases} 1, & x < 0, \\ 1/2, & x = 0, \\ 0, & x > 0, \end{cases}$$

and take

$$(7.12) \quad \begin{aligned} g(x, u; \varepsilon) &= \frac{1}{\varepsilon^2} c(x) u(1-u), \\ c(x) &= c + \frac{a}{\pi} \operatorname{arctg} \alpha(x-b). \end{aligned}$$

Here $\varepsilon > 0$ is a small parameter, $\alpha > 0$ is a big number, c , a , and b are positive constants, and $a/2 < c < 3a/2$.

The problem (7.10)-(7.12) is a generalization of the KPP-equation. The theoretical results for this problem obtained in [5] give the following. For $t < T_0 \approx \frac{b\sqrt{2a}}{c+0.5a}$, the wave propagates to the right of the domain $G_0 = \{x < 0\}$ with the velocity $\sqrt{2c-a}$, “taking no notice” of the fact that after $x = b$ the coefficient $c(x)$ takes a larger value $c + \frac{a}{2}$. But at the time T_0 , a new “source” arises at the point $x = b$, away from which the front starts propagating in both directions: to the left with a velocity close to $\sqrt{2c-a}$ and to the right with a velocity close to $\sqrt{2c+a}$.

Figure 1, obtained in our numerical experiments, demonstrates this phenomenon. Under the parameters used here (see the figure caption), $T_0 \approx 5.65$. As soon as the time t is close to T_0 , the velocity of the new front to the right is greater than $\sqrt{2c+a} \approx 2.06$ (see Figures 1c and 1d); and with an increase of time the velocity tends to $\sqrt{2c+a}$ (see Figures 1e and 1f). One can explain this by the fact that when the new “source” arises the value of the solution u before the front is greater than the corresponding value of u when the shape of the wave takes its limit form.

Let us note that for our values of the parameters ($\varepsilon = 0.1$; see the caption to Figure 1)

$$\min_{-\infty < x < b} \bar{u}(5.75, x) \approx 10^{-71}$$

while there is already the new front at $x = b$ (see Figure 1c). So, the “channel” through which the new “source” is initialized is very narrow. This fact has to be taken into account for realizing numerical procedures on a computer. For instance, when $\varepsilon = 0.04$

$$\min_{-\infty < x < b} \bar{u}(5.75, x) \approx 10^{-439},$$

which is less than the smallest positive number ($\sim 10^{-308}$) realizable by many compilers. To observe the phenomenon in this case, one has to compose a special numerical procedure or use a special compiler.

An additional confirmation of the high sensitivity of our model is given, for instance, by the following experiment. If we put $u(0, x) = \mu$ for $x > 0$ with a small positive μ , e.g. 10^{-15} , in the initial condition (7.11) and take the other parameters as in Figure 1, the new “source” arises at a moment $t \ll 1$.

Here we compare five numerical methods: the methods (6.6), (6.7), (6.10), (6.13) given in Section 6 (of course, we take their versions adapted to problems with positive direction of time) and the first-order method written below. The results of the numerical tests are shown in Figures 2 and 3.

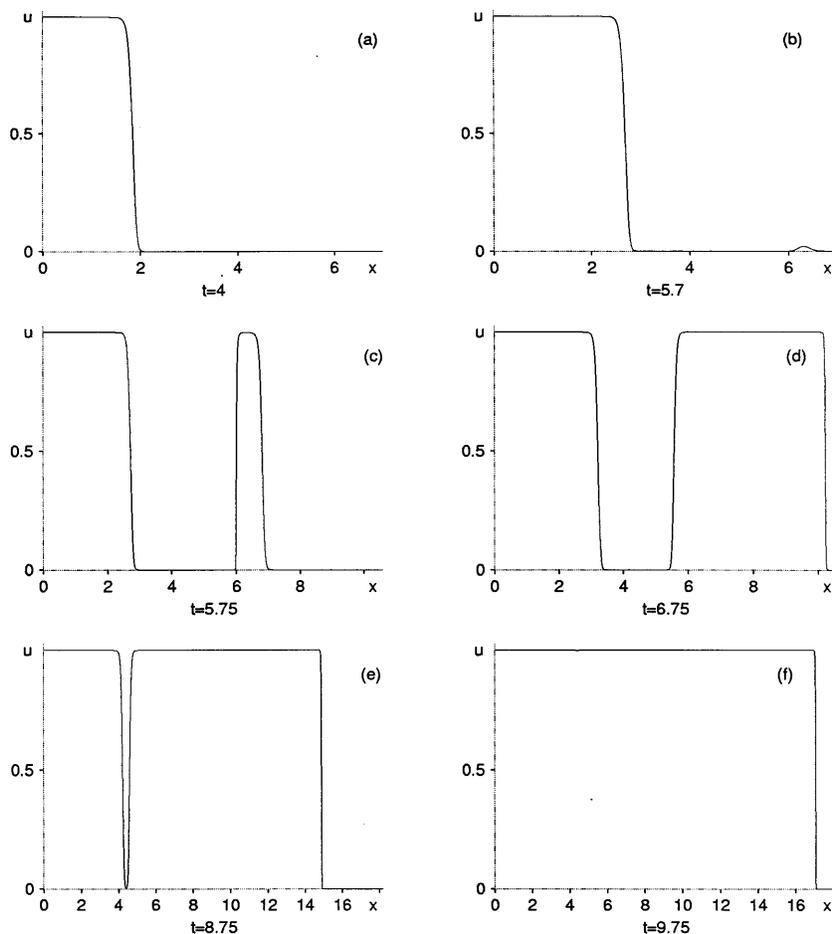


FIGURE 1. The KPP-equation. Evolution of the solution $u(t, x)$ to problem (7.10)-(7.12) for $\varepsilon = 0.1$, $c = 1.125$, $a = 2$, $b = 6$, $\alpha = 150$ simulated by the method (6.13) with $h = 0.0001$.

The first-order method is

$$\begin{aligned}
 (7.13) \quad & \bar{u}(0, x_j) = \chi_-(x_j), \\
 & \bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_k, x_j - \varepsilon h^{1/2}) \\
 & \quad + \frac{h}{2}(g(x_{j-1}, \bar{u}(t_k, x_{j-1})) + g(x_{j+1}, \bar{u}(t_k, x_{j+1}))), \\
 & x_j = x_0 + j\varepsilon h^{1/2}, \quad j = 0, \pm 1, \dots, \quad k = 0, \dots, N-1.
 \end{aligned}$$

It can be checked that for a sufficiently small h this method preserves the monotonicity property of the solution. The first-order method (6.5), which has $hg(x_j, \bar{u}(t_k, x_j))$ instead of $h(g(x_{j-1}, \bar{u}(t_k, x_{j-1})) + g(x_{j+1}, \bar{u}(t_k, x_{j+1}))) / 2$, does not preserve the monotonicity property and has unstable behavior for this problem.

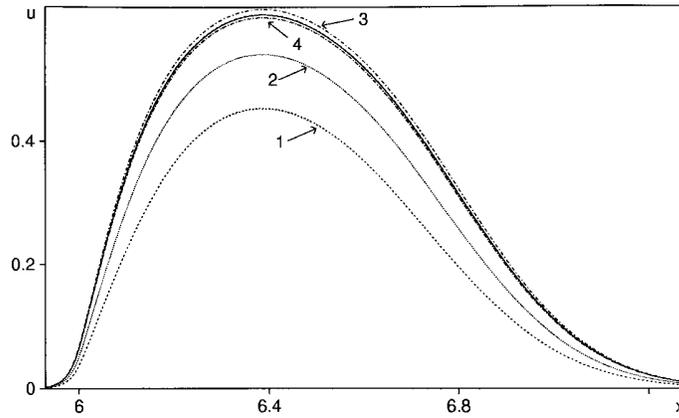


FIGURE 2. The KPP-equation (new source appearance). Comparison of the methods. The solid curve is simulated by (6.7) and (6.13) with $h = 0.0001$, and it visually coincides with the exact solution. The curves 1, 2 are simulated by (7.13) and (6.6) with $h = 0.0001$; the curves 3, 4 by (6.7) and (6.13) with $h = 0.001$. Here $\varepsilon = 0.2$, $t = 5.8$, and the other parameter values are the same as in Figure 1.

The algebraic equations arising in the implementation of the methods (6.7), (6.10), and (6.13) at each step are quadratic ones and are solved exactly. The results of testing the three-layer method (6.10) are discussed in Section 6.2.

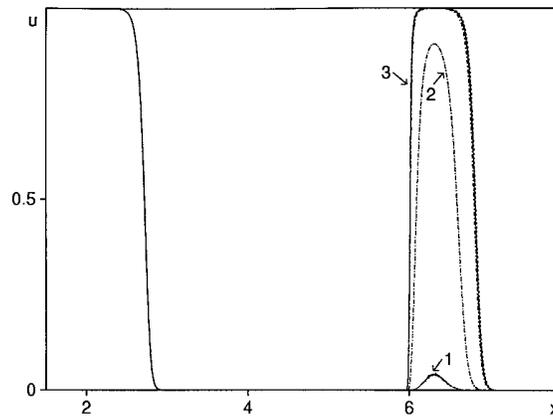


FIGURE 3. The KPP-equation. Comparison of the methods. The curve 3 is simulated by (6.7) and (6.13) with $h = 0.0005$ and $h = 0.0001$, and it visually coincides with the exact solution. The curves 1, 2 are simulated by (7.13) and (6.6) with $h = 0.0001$. Here $t = 5.75$, and the other parameter values are the same as in Figure 1.

Remark 7.1. Derivatives of the solution to (7.10)-(7.11) can be estimated as

$$(7.14) \quad \left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j} \right| \leq \frac{K}{\varepsilon^{2(i+j)}}, \quad t \in [t_0, T], \quad x \in R^1, \quad 0 < \varepsilon \leq \varepsilon^*.$$

By the same arguments as in Theorem 3.3 one can prove that, under (6.8), for a sufficiently small h/ε^2 the errors of both methods (6.5) and (6.6) are estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K \frac{h}{\varepsilon^4},$$

and the error of the method (6.7) is estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K \frac{h^2}{\varepsilon^6},$$

where the constant K does not depend on x, k, h, ε .

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