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Approximation of a Multivalued Solution of the Hamilton–Jacobi Equation

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Abstract—The paper deals with the construction of a multivalued solution of the Cauchy problem for the Hamilton–Jacobi equation with discontinuous Hamiltonian with respect to the phase variable. The constructed multivalued solution is approximated by a sequence of continuous solutions of auxiliary Cauchy problems of the Hamilton–Jacobi equation with Hamiltonian which is Lipschitz with respect to the phase variable. The results of the study are illustrated by an example.

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1. INTRODUCTION

The paper deals with the study of generalized solutions of the Hamilton–Jacobi equation in the case of a discontinuous Hamiltonian. Such settings arise in the study of hierarchical differential games [1] and in optimal control problems.

The theory of solutions of the Hamilton–Jacobi equation was first developed for the case of a continuous Hamiltonian. Here we can note two approaches to the definition of a solution: minimax and viscosity solutions [2], [3]. The notion of minimax solution is based on the weak invariance property of the graph of the solution with respect to a differential inclusion. The viscosity approach is that the Hamilton–Jacobi equation is replaced by a pair of inequalities for test functions. Under these approaches, existence and uniqueness theorems for a solution in the class of continuous functions were proved. The equivalence of the viscosity and minimax approaches was proved by Subbotin [3]. In the case of a sufficiently smooth Hamiltonian, numerical methods for constructing these solutions were developed in [3]–[6].

The theory of generalized solutions for discontinuous Hamilton–Jacobi equations was developed following the theory of minimax and viscosity solutions and, to some extent, generalizes this approach. Here we can also mention several approaches to the definition of a generalized solution. Ishii proposed the viscosity approach [7] and found a continuous solution of the discontinuous Hamilton–Jacobi equation. Subbotin and Lakhtin proposed the notion of an M-solution, which is based on the weak invariance property of the graph of the solution with respect to a differential inclusion [8]; an M-solution belongs to the class of multivalued mappings. In that paper, the conditions on the Hamiltonian are weaker compared to those in the problem considered in [7]. Another approach uses the relationship between the Hamilton–Jacobi equation and the corresponding quasilinear equation of first order [9]. The authors of [9] constructed a continuous viscosity solution on the basis of the entropy solution of the corresponding quasilinear equation. They assumed that the Hamiltonian is convex with respect to the conjugate variable and that the set of points of discontinuity of the Hamiltonian consist of a finite number of points. These conditions ensure the existence and uniqueness of a continuous solution of the Hamilton–Jacobi equation.

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In [10], it was shown that a viscosity solution and an M-solution are equivalent in the following sense: the upper (lower) envelope of an M-solution coincides with an upper (lower) viscosity solution of the Hamilton–Jacobi equation.

However, the existence theorems proved in [10] do not yield approaches to the construction of the solution. In connection with this, the construction of numerical methods and approximations of multivalued solutions of discontinuous Hamilton–Jacobi equations are of current interest.

In this paper, we propose an approximation of a multivalued M-solution of a discontinuous (with respect to the phase variable) Hamilton–Jacobi equation by using the limit of a sequence of continuous solutions of Hamilton–Jacobi equations that are continuous in all variables.

The paper is organized as follows. In Sec. 2, we describe the setting of the main and auxiliary Cauchy problems for the Hamilton–Jacobi equation. In Sec. 3, we present the definition and the properties of solutions of the auxiliary Cauchy problem. We construct a multivalued solution of the original Cauchy problem as the convex hull of upper limits in the sense of Kuratowski of a sequence of solutions of the auxiliary Cauchy problem. In Sec. 4, we study an example in which a multivalued solution and its approximation by continuous solutions of auxiliary Cauchy problems is constructed.

2. STATEMENT OF THE PROBLEM

Consider the Cauchy problem for the Hamilton–Jacobi equation of first order

$$\frac{\partial w(t,x)}{\partial t} + H(t,x,D_x w(t,x)) = 0, \qquad w(T,x) = \sigma(x). \tag{2.1}$$

Here $t \in [0,T]$ and $x \in \mathbb{R}^n$.

The problem is solved under the following assumptions.

Assumption 1. The function H is Lipschitz in the variables t and s:

$$|H(t,x,s_1) - H(t,x,s_2)| \le \lambda(1+|x|)|s_1 - s_2|, \quad \lambda > 0, \qquad \forall\,(t,x) \in [0,T]\times\mathbb{R}^n.$$

The function H is discontinuous in x on a set of points of measure zero.

Assumption 2. The following sublinear growth condition holds:

$$|H(t,x,0)| \le \nu(1+|x|), \qquad \nu > 0.$$

Assumption 3. Any function $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable.

It follows from Assumptions 1 and 2 that

$$|H(t,x,s)| \le \lambda(1+|x|)|s| + \nu(1+|x|) \le \vartheta(1+\|x\|)(1+\|s\|), \qquad \vartheta > 0.$$

To formulate the definition of a generalized solution of problem (2.1), we consider the set

$$E(t,x,s) = \big\{(f,g) : |f| \le \lambda(1+|x|), \langle f,s\rangle - g \in [H_*(t,x,s), H^*(t,x,s)] \ \forall\, s \in \mathbb{R}^n\big\}. \tag{2.2}$$

Here

$$H_*(t,x,s) = \liminf_{\xi\to x} H(t,\xi,s), \qquad H^*(t,x,s) = \limsup_{\xi\to x} H(t,\xi,s),$$

and the symbol $\langle\,\cdot\,,\cdot\,\rangle$ denotes the inner product of vectors in $\mathbb{R}^n$.

The notion of a multivalued generalized solution of problem (2.1) was proposed by Subbotin in [8].

Definition 1. Let $W \subset [0,T]\times\mathbb{R}^n\times\mathbb{R}$ be a closed set. We say that W is *weakly invariant* with respect to the characteristic differential inclusion $(\dot{x},\dot{z}) \in E(t,x,s)$, where E is defined by (2.2), if, for all $(t_0,x_0,z_0) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}$ and $s \in \mathbb{R}^n$, there exists a $\tau > 0$ and a trajectory of the differential inclusion $(x(\,\cdot\,), z(\,\cdot\,))$ such that

$$x(t_0) = x_0, \quad z(t_0) = z_0, \qquad (t, x(t), z(t)) \in W, \quad t \in [t_0, \tau].$$

Definition 2. We say that $w\colon [0,T]\times\mathbb{R}^n \rightrightarrows \mathbb{R}$ is an *M-solution* of problem (2.1) if

(1) $\operatorname{gr} w$ is weakly invariant with respect to the characteristic differential inclusion $(\dot{x}, \dot{z}) \in E(t, x, s)$, where E is defined by (2.2);

(2) w satisfies the boundary condition $w(T, x) = \sigma(x)$ for all $x \in \mathbb{R}^n$;

(3) $\operatorname{gr} w$ is maximal by inclusion among all multivalued mappings satisfying (1) and (2). In other words, for an arbitrary multivalued mapping $Y\colon [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ weakly invariant with respect to the differential inclusion (2.2), the inclusion $Y(T, \cdot\,) \subset \operatorname{gr} w(T, \cdot\,)$ holds.

Remark 1. We note that the solutions $(x(\,\cdot\,), z(\,\cdot\,))$ of the differential inclusion (2.2) are extended to time $t = T$ by virtue of Assumption 2.

It was shown in [8] that, under Assumptions 1–3, there exists an M-solution of problem (2.1), and it belongs to the class of multivalued mappings. The goal of the present paper is to construct and approximate an M-solution of problem (2.1) by a sequence of continuous solutions of auxiliary Cauchy problems for a Lipschitz (with respect to the phase variable) Hamilton–Jacobi equation.

2.1. The Auxiliary Cauchy Problem for the Hamilton–Jacobi Equation

Let us define the Hamiltonian H_k as follows:

$$H_k(t, x, s) = \int_{\mathbb{R}^n} H(t, y, s)\eta_k(x - y)\, dy = H * \eta_k. \tag{2.3}$$

Here η_k is a mollifier. The family of mollifiers η_k possesses the following properties [11]:

(1) $\eta_k(x) \ge 0$ for all $x \in \mathbb{R}^n$;

(2) $\int_{\mathbb{R}^n} \eta_k(x)\, dx = 1$;

3) for all $\delta > 0$, $\int_{|x|>\delta} \eta_k(x)\, dx \to 0$ as $k \to \infty$.

Further, we shall consider mollifiers from the class $\eta_k \in C_0$, the space of Lipschitz compactly supported functions. We assume that the support $\operatorname{supp} \eta_k$ is a subset of $B(x; \varepsilon_k)$, where $B(x; \varepsilon_k)$ is the ball of radius ε_k centered at the point $x \in \operatorname{supp} \eta_k$, where $\lim_{k\to\infty} \varepsilon_k = 0$.

Consider the Cauchy problem for the Hamilton–Jacobi equation

$$\frac{\partial w_k(t, x)}{\partial t} + H_k(t, x, D_x w_k(t, x)) = 0, \qquad w_k(T, x) = \sigma(x). \tag{2.4}$$

2.2. The Properties of the Hamiltonian H_k

Proposition 1. *A function H_k of the form* (2.3) *possesses the following properties*:

(1) *the function H_k is Lipschitz in s with Lipschitz constant $\lambda(1 + \|x\| + \varepsilon_k)$ and satisfies the sublinear growth condition in s*;

(2) *the function H_k is Lipschitz in x with Lipschitz constant $\kappa(t, s) > 0$.*

Proof. (1) Let us find the Lipschitz constant of the function H_k with respect to the variable s. Consider

$$\begin{aligned} |H_k(t, x, s_1) - H_k(t, x, s_2)| &\le \int_{\mathbb{R}^n} |H(t, y, s_1) - H(t, y, s_2)|\eta_k(x - y)\, dy \\ &\le \lambda|s_1 - s_2| \int_{\mathbb{R}^n} (1 + |y|)\eta_k(x - y)\, dy \\ &\le \lambda|s_1 - s_2| \int_{\mathbb{R}^n} (1 + |x| + |z|)\eta_k(z)\, dz. \end{aligned}$$

We note that $|z| \le \varepsilon_k$, because $\operatorname{supp}\eta_k \in B(z;\varepsilon_k)$. Then

$$\lambda|s_1 - s_2| \int_{\mathbb{R}^n} (1+|x|+|z|)\eta_k(z)\,dz \le \lambda(1+|x|+\varepsilon_k)|s_1-s_2| \int_{\mathbb{R}^n} \eta_k(z)\,dz \le \lambda(1+|x|+\varepsilon_k)|s_1-s_2|.$$

The function H_k possesses the sublinear growth property with respect to x. Indeed,

$$|H_k(t,x,0)| = \left|\int_{\mathbb{R}^n} H(t,y,0)\eta_k(x-y)\,dy\right| \le \left|\int_{\mathbb{R}^n} \nu(1+|x|+|z|)\eta_k(z)\,dz\right| \le \nu(1+|x|+\varepsilon_k)\left|\int_{\mathbb{R}^n} \eta_k(z)\,dz\right| = \nu(1+|x|+\varepsilon_k).$$

We note that the constants λ and ν are independent of the choice of k.

(2) Let us show that H_k is Lipschitz in x for fixed t and s. Consider a compact domain $D \subset \mathbb{R}^n$. Let $x_1, x_2 \in D$. Consider

$$|H_k(t,x_1,s) - H_k(t,x_2,s)| = \left|\int_{\mathbb{R}^n} H(t,y,s)\eta_k(x_1-y) - H(t,y,s)\eta_k(x_2-y)\,dy\right|.$$

We note that $\operatorname{supp}\eta_k(x_1-y) \subset D + B_{\varepsilon_k}$ and $\operatorname{supp}\eta_k(x_2-y) \subset D + B_{\varepsilon_k}$. We denote the Lipschitz constant of the function η_k by the symbol L_k. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} H(t,y,s)(\eta_k(x_1-y) - \eta_k(x_2-y))\,dy\right| \\
&\qquad = \left|\int_{D+B_{\varepsilon_k}} H(t,y,s)(\eta_k(x_1-y) - \eta_k(x_2-y))\,dy\right| \\
&\qquad \le \left|\int_{D+B_{\varepsilon_k}} H(t,y,s)L_k|x_1-x_2|\,dy\right| \\
&\qquad \le \left|\int_{D+B_{\varepsilon_k}} \theta(1+|y|)(1+|s|)L_k|x_1-x_2|\,dy\right| \le \kappa(t,s)|x_1-x_2|,
\end{aligned}$$

where

$$\kappa(t,s) = \left|\int_{D+B_{\varepsilon_k}} \theta(1+|y|)(1+|s|)L_k\,dy\right|.$$

It follows that H_k is Lipschitz in x. Here the Lipschitz constant κ depends on k. □

2.3. The Solution of the Auxiliary Cauchy Problem for the Hamilton–Jacobi Equation

To formulate a generalized solution of problem (2.4), we consider the following sets:

$$E_k(t,x,s) = \{(f,g) : |f| \le \lambda(1+|x|+\varepsilon_k),\ g = \langle f,s\rangle - H_k(t,x,s)\} \qquad \forall\, s \in \mathbb{R}^n. \tag{2.5}$$

Let us recall the definition of a solution of problem (2.4); see [3].

Definition 3. By a *minimax/viscosity solution* of problem (2.4) we mean a continuous function $w_k\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ whose graph is weakly invariant with respect to the characteristic differential inclusion $(\dot{x},\dot{z}) \in E_k(t,x,s)$, where E_k is defined by (2.5) and the boundary condition $w_k(T,x) = \sigma(x)$ holds for all $x \in \mathbb{R}^n$.

In [3], it was shown that, under Assumptions 1–3, there exists a unique generalized solution of problem (2.4) for the class of continuous functions.

3. CONSTRUCTION OF AN M-SOLUTION OF THE CAUCHY PROBLEM

For subsequent constructions, we recall the following definition [12].

Definition 4. By the *upper limit* of a sequence of sets $\{w_k\}_{k=1}^{\infty}$, where w_k is a minimax solution of problem (2.4), we mean the set

$$\operatorname{Ls}(w_k) = \Big\{(t,x,z) : \exists\,\{(t_i,x_i,z_i)\}_{k=1}^{\infty}\ \exists\,\{k_i\}_{i=1}^{\infty} \text{ with properties } \lim_{i\to\infty}(t_i,x_i,z_i) = (t,x,z),\ (t_i,x_i,z_i) \in \operatorname{gr} w_{k_i}\Big\}.$$

Lemma 1. *Let* $E\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n\times\mathbb{R}$ *be the multivalued mapping defined by* (2.2)*, and let* $E_k\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n\times\mathbb{R}$ *be the mapping defined by* (2.5)*. Then*

$$\lim_{k\to\infty} E_k(t,x,s) \subset E(t,x,s).$$

Proof. Consider the mapping E_k of the form (2.5). Note that the set of all f_k satisfying the inequality $\|f_k\| \le \lambda(1+\|x\|+\varepsilon_k)$ lies in the set $\|f\| \le \lambda(1+\|x\|)+\varepsilon$. Since $\varepsilon_k \to 0$ as $k\to\infty$, it follows that there exists a number K such that, for all $k > K$, $\varepsilon_k < \varepsilon$. This yields $\|f_k\| \le \lambda(1+\|x\|)+\varepsilon$.

By virtue of the construction of H_k, we have

$$H_*(t,x,s) \le H_k(t,x,s) \le H^*(t,x,s).$$

Then

$$g_k = \langle f_k, s\rangle - H_k(t,x,s) \in [\langle f,s\rangle - H^*(t,x,s), \langle f,s\rangle - H_*(t,x,s)] + B_\varepsilon.$$

We showed that, for an arbitrary element $(f_k,g_k) \in E_k(t,x,s)$, we have $(f_k,g_k) \in E(t,x,s) + B_\varepsilon$ beginning with some number K. Then $E_k(t,x,s) \subset E(t,x,s) + B_\varepsilon$. □

Subbotin and Lakhtin proved the following theorem [4].

Theorem 1. *Let W be a closed set in* $[0,T]\times\mathbb{R}^n\times\mathbb{R}$*. Let* $W(t,x) = \{z\in\mathbb{R} : (t,x,z)\in W\} \ne \varnothing$*, and let*

$$W_*(t,x) = \inf_{z\in W(t,x)} z, \qquad W^*(t,x) = \sup_{z\in W(t,x)} z.$$

The mapping W is an M-solution of Eq. (2.1) *if and only if* $\operatorname{epi} W_*$ *and* $\operatorname{hypo} W^*$ *are M-solutions of Eq.* (2.1).

Let us recall that $\operatorname{co} A$ denotes the convex hull of the set A.

Theorem 2. *An M-solution w of problem* (2.1) *can be represented as*

$$\operatorname{co}\operatorname{Ls} w_k = \operatorname{gr} w, \qquad k\to\infty,$$

where the w_k are minimax/viscosity solutions of problem (2.4).

Proof. We denote $w^* = \operatorname{Ls} w_k$. Our goal is to prove the equality $\operatorname{co} w^* = w$, where w is an M-solution of problem (2.1). First, let us show that the set w^* is not empty. Let $B \subset [0,T]\times\mathbb{R}^n$ be a compact set, and let $(\tau,\xi)\in B$. Consider $\zeta_k = w_k(\tau,\xi)$. By the definition of a minimax solution, $\operatorname{gr} w_k$ is weakly invariant with respect to the differential inclusion (2.5) for all $s\in\mathbb{R}^n$. Let $s = 0$. It follows from $(\tau,\xi,\zeta_k)\in\operatorname{gr} w_k$ that there exists a solution $(x_k(\cdot), z_k(\cdot))$ of the differential inclusion (2.5) with initial conditions $x_k(\tau)=\xi$, $z_k(\tau)=\zeta_k$, and $(T, x_k(T), z_k(T)) \in \operatorname{gr} w_k$. Thus, $z_k(T) = \sigma(x_k(T))$. Let us recall that

$$\dot{x}_k \le \lambda(1+|x_k|+\varepsilon_k), \qquad \dot{z}_k = H_k(t,x_k(t),0) \le \nu(1+|x_k(t)|+\varepsilon_k).$$

By Gronwall's lemma, we have

$$|x_k(T)| \le (1+\varepsilon_k)(e^{\lambda(T-\tau)}-1) + |\xi|e^{\lambda(T-\tau)}.$$

We assume that $\varepsilon_k < 1$ for any k. Then

$$|x_k(T)| \le 2e^{\lambda(T-\tau)} + |\xi| e^{\lambda(T-\tau)}.$$

Therefore,

$$\begin{aligned} |w_k(\tau,\xi)| = |z_k(\tau)| &= \left|\sigma(x_k(T)) + \int_T^\tau H_k(t, x_k(t), 0)\,dt\right| \\ &\le \left|\sigma(x_k(T)) + \int_T^\tau \nu(1 + |x_k(t)| + \varepsilon_k)\,dt\right| \\ &\le |\sigma(x_k(T))| + \nu(2 + |x_k(T)|)(T-\tau) \le r. \end{aligned}$$

The constant r is constructed from the constants τ, ξ, λ, ν, and T and is independent of k. It was shown that $|w_k(\tau,\xi)| \le r$ for arbitrary k and $(\tau,\xi) \in B$. It follows from the uniform boundedness of w_k that we can choose any convergent subsequence $\{w_{k_i}\}_{i=1}^{\infty}$. Therefore, the set $w^* = \operatorname{Ls} w_k$ is not empty.

Let us prove that w^* is weakly invariant with respect to the differential inclusion (2.2). We choose points $(t_0, x_0, z_0) \in w^*$ and $s \in \mathbb{R}^n$. By the definition of w^*, there exists a subsequence

$$\{(\tau_{k_j}, \xi_{k_j}, \zeta_{k_j})\} \in \operatorname{gr} w_{k_j}, \qquad \lim_{j\to\infty} (\tau_{k_j}, \xi_{k_j}, \zeta_{k_j}) = (t_0, x_0, z_0).$$

The sets $\operatorname{gr} w_{k_j}$ are weakly invariant with respect to the differential inclusion (2.5); therefore, the solution $(x_{k_j}(\,\cdot\,), z_{k_j}(\,\cdot\,))$ of (2.5) with initial condition $x_{k_j}(\tau_{k_j}) = \xi_{k_j}$, $z_{k_j}(\tau_{k_j}) = \zeta_{k_j}$ exists and

$$(t, x_{k_j}(t), z_{k_j}(t)) \in \operatorname{gr} w_{k_j}, \qquad t \in [\tau_{k_j}, T].$$

The sequence $(x_{k_j}(\,\cdot\,), z_{k_j}(\,\cdot\,))$ is uniformly bounded and equicontinuous. Its equicontinuity follows from the fact that $\dot{x}_{k_j}$ and $\dot{z_{k_j}}$ are bounded independently of k:

$$\begin{aligned} &\forall\, \varepsilon > 0 \quad \exists\, \delta > 0: \\ &\quad |t_1 - t_2| < \delta \quad \Longrightarrow \quad \|x_{k_j}(t_1) - x_{k_j}(t_2)\| \le \|\dot{x}_{k_j}(\theta)(t_1 - t_2)\| \\ &\qquad\qquad\qquad\qquad \le \lambda(1 + \|x_{k_j}(\theta)\| + \varepsilon_{k_j})|t_1 - t_2| < \varepsilon. \end{aligned}$$

Similarly, for z_{k_j}, we have

$$\begin{aligned} &\forall\, \varepsilon > 0 \quad \exists\, \delta > 0: \\ &\quad |t_1 - t_2| < \delta \quad \Longrightarrow \quad \|z_{k_j}(t_1) - z_{k_j}(t_2)\| \le \|\dot{z}_{k_j}(\theta)(t_1 - t_2)\| \\ &\qquad\qquad\qquad\qquad \le |\langle \dot{x}_{k_j}(\theta), s\rangle - H_{k_j}(\theta, x_{k_j}(\theta), s)| \cdot |t_1 - t_2| < \varepsilon. \end{aligned}$$

Without loss of generality, we may assume that the sequence $(x_{k_j}(\,\cdot\,), z_{k_j}(\,\cdot\,))$ itself converges to some motion $(x^*(\,\cdot\,), z^*(\,\cdot\,))$ by the Arzelà–Ascoli theorem.

The limit function $(x^*(\,\cdot\,), z^*(\,\cdot\,))$ is Lipschitz by the Arzelà–Ascoli theorem. Therefore, for almost all $\tau \in [t_0, T]$, $(\dot{x}^*(\tau), \dot{z}^*(\tau))$ exists. Since the sets E_k and E are upper semicontinuous mappings and Lemma 1 holds, it follows that, for a sufficiently large K and a small $\delta > 0$, the following inclusions hold:

$$\begin{aligned} E_k(t, x_k(t), s) \subseteq E(\tau, x_k(t), s) + B_\alpha &\subseteq E(\tau, x^*(\tau), s) + B_{2\alpha} \\ \forall\, k \ge K \quad \forall\, t \in [\tau - \delta, \tau + \delta], &\quad \alpha > 0. \end{aligned} \tag{3.1}$$

Let $\tau' \in [\tau - \delta, \tau + \delta]$, $\tau' \ne \tau$. From inclusion (3.1) and the convexity of $E(\tau, x^*(\tau), s) + B_{2\alpha}$ we obtain

$$\left(\frac{x_k(\tau') - x_k(\tau)}{\tau' - \tau}, \frac{z_k(\tau') - z_k(\tau)}{\tau' - \tau}\right) \in E(\tau, x^*(\tau), s) + B_{2\alpha}.$$

In the limit as $k \to \infty$,

$$\left(\frac{x^*(\tau') - x^*(\tau)}{\tau' - \tau}, \frac{z^*(\tau') - z^*(\tau)}{\tau' - \tau}\right) \in E(\tau, x^*(\tau), s) + B_{2\alpha}.$$

Since $E(\tau, x^*(\tau), s) + B_{2\alpha}$ is a closed set, we have

$$(\dot{x}^*(\tau), \dot{z}^*(\tau)) \in E(\tau, x^*(\tau), s) + B_{2\alpha}.$$

This inclusion holds for all α; therefore,

$$(\dot{x}^*(\tau), \dot{z}^*(\tau)) \in E(\tau, x^*(\tau), s).$$

This means that $\operatorname{gr} w^*$ is weakly invariant with respect to the differential inclusion (2.2).

We choose a point $(T, x, z) \in w^*$. Consider the subsequence

$$\{(T, x_{k_j}, z_{k_j})\}_{j=1}^{\infty} \in \operatorname{gr} w_{k_j}, \qquad \lim_{j\to\infty}(T, x_{k_j}, z_{k_j}) = (T, x, z).$$

We note that $z_{k_j} = \sigma(x_{k_j})$. Then

$$\lim_{j\to\infty}(T, x_{k_j}, \sigma(x_{k_j})) = (T, x, \sigma(x)) \in w^* \qquad \forall\, x \in \mathbb{R}^n.$$

Obviously, $w^* \subset \operatorname{gr} w$, because $\operatorname{gr} w$ is a maximal (by inclusion) weakly invariant set.

Let us show that $\operatorname{co} w^*$ is weakly invariant with respect to the differential inclusion (2.2). We construct the mappings

$$W_*(t, x) = \min_{y \in \operatorname{co} w^*(t,x)} y, \qquad W^*(t, x) = \max_{y \in \operatorname{co} w^*(t,x)} y.$$

Let us show that $\operatorname{epi} W_*$ is weakly invariant with respect to (2.2). Let us fix a point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Consider the points

$$z_0^1 = W_*(t_0, x_0), \qquad z_0^2 = W^*(t_0, x_0).$$

Since $\operatorname{gr} w^*$ is weakly invariant with respect to the differential inclusion (2.2), it follows that there exists a solution $(x(\,\cdot\,), z(\,\cdot\,))$ of (2.2) with initial conditions $x(t_0) = x_0$ and $z(t_0) = z_0^1$ that survives in the graph of w^*. Let us construct the solution $(x(\,\cdot\,), z(\,\cdot\,))$ of the differential inclusion (2.2) with initial conditions $x(t_0) = x_0$ and $z(t_0) = \overline{z} \in [z_0^1, z_0^2]$. For all $\overline{z} \in [z_0^1; z_0^2]$, the following inequality holds:

$$z(t, \overline{z}) = \int_{t_0}^{t} \langle f, s\rangle - h(\tau, x(\tau), s)\, d\tau + \overline{z} \geq \int_{t_0}^{t} \langle f, s\rangle - h(\tau, x(\tau), s)\, d\tau + z_0^1 = z(t, z_0^1).$$

Here $h(\tau, x(\tau), s) \in [H_*(\tau, x(\tau), s), H^*(\tau, x(\tau), s)]$. For $t = T$, we have

$$z(T, \overline{z}) \geq \sigma(x(T)).$$

It follows that the solution $(x(\,\cdot\,), z(\,\cdot\,))$ of the differential inclusion (2.2) survives in $\operatorname{epi} W_*$ for any initial point $(t_0, x_0, \overline{z}) \in \operatorname{epi} W_*$. Similarly, we prove that $\operatorname{hypo} W^*$ is weakly invariant with respect to the differential inclusion (2.2) for all $(t_0, x_0, \overline{z}) \in \operatorname{hypo} W^*$.

By Theorem 1, $\operatorname{co} w^* = \operatorname{hypo} W^* \cap \operatorname{epi} W_*$ is an M-solution of problem (2.1), and so we obtain $\operatorname{co} w^*(T, x) = \{\sigma(x)\}$ for all $x \in \mathbb{R}^n$. By construction, the M-solution $\operatorname{co} w^*$ is maximal by inclusion. □

4. EXAMPLE

Consider the following Cauchy problem with discontinuous Hamiltonian:

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\operatorname{sgn} x + \frac{1}{4} = 0, \qquad w(T, x) = x.$$

Here $x \in \mathbb{R}$ and $t \in [t_0, T]$. It is easy to verify that the problem has an M-solution of the form

$$w(t, x) = \begin{cases} -\dfrac{5}{4}(t - T) + x, & x \geq 0, \\[2mm] \left[\dfrac{3}{4}(t - T), -\dfrac{5}{4}(t - T)\right], & x = 0, \\[2mm] \dfrac{3}{4}(t - T) + x, & x \leq 0. \end{cases} \tag{4.1}$$

We verify that the proposed multivalued mapping satisfies the definition of an M-solution. The right-hand side of the differential inclusion is of the form

$$E(t,x,s)=\left\{(f,g),|f|\le 1, fs-g\in\left[-s+\frac{1}{4}, s+\frac{1}{4}\right]\right\}.$$

Let us fix a point (t_0,x_0,z_0), $x_0>0$. Then

$$\dot{x}=1,\qquad \dot{z}=-\frac{1}{4}.$$

Integrating this system of ordinary differential equations, we obtain

$$x(t)=x_0+t-t_0,\qquad z(t)=z_0-\frac{1}{4}(t-t_0).$$

If $z_0=-(5/4)(t_0-T)+x_0$, then

$$z(t)=-t_0+\frac{5}{4}T-\frac{1}{4}t=-\frac{5}{4}(t-T)+x(t)-x_0.$$

Similarly, we consider another case. If $x_0<0$, then the solutions of the system

$$\dot{x}=-1,\qquad \dot{z}=-\frac{1}{4}$$

survive in the graph of w. Let

$$x_0=0,\qquad z_0=\beta\frac{3}{4}(t_0-T)-(1-\beta)\frac{5}{4}(t_0-T),\quad \beta\in[0,1].$$

We choose

$$\dot{x}=0,\qquad \dot{z}=-\frac{1}{4}.$$

Then there exists an instant of time $t_1\in[t_0,T]$ for which the solutions

$$x(t)\equiv 0,\qquad z(t)=-\frac{1}{4}(t-t_0)+z_0$$

satisfy the relations

$$z(t_1)=-\frac{5}{4}(t_1-T)\qquad\text{or}\qquad z(t_1)=\frac{3}{4}(t_1-T).$$

Further, for $t\in[t_1,T]$, we pass to the previous case. Thus, the constructed solutions survive in the graph of the M-solution.

Let us construct the auxiliary Hamiltonian H_k:

$$H_k(t,x,s)=\frac{1}{d_k}\int_{B_{r_k}}H(t,x+y,s)\,dy=s\,\frac{|x+r_k|-|x-r_k|}{2r_k}+\frac{1}{4}.$$

Here d_k denotes the diameter of the ball of radius r_k centered at zero. In more detail, the Hamiltonian H_k is of the form

$$H_k(t,x,s)=\begin{cases} s+\dfrac{1}{4}, & x\ge r_k,\\ \dfrac{xs}{r_k}+\dfrac{1}{4}, & |x|\le r_k,\\ -s+\dfrac{1}{4}, & x\le -r_k.\end{cases}$$

The mollifier η_k is of the form

$$\eta_k(x)=\begin{cases}\dfrac{1}{d_k}, & x\in B_{r_k},\\ 0, & x\notin B_{r_k}.\end{cases}$$

The right-hand side of the differential inclusion in the definition of a minimax solution is of the form

$$E_k(t,x,s) = \{(f,g) : |f| \le 1,\ g = fs - H_k(t,x,s)\}.$$

Consider the Cauchy problem with Lipschitz Hamiltonian

$$\frac{\partial w_k}{\partial t} + H_k(t,x,D_x w_k(t,x)) = 0, \qquad w_k(T,x) = x. \tag{4.2}$$

We denote $a = e^{(t-T)/r_k}$. The auxiliary problem has a minimax solution of the form

$$w_k(t,x) = \begin{cases} -\dfrac{5}{4}(t-T) + x, & x \ge r_k, \\ r_k \ln\dfrac{x}{r_k} - \dfrac{5}{4}(t-T) + r_k, & r_k a < x < r_k, \\ x e^{-(t-T)/r_k} - \dfrac{1}{4}(t-T), & |x| \le r_k a, \\ -r_k \ln\dfrac{x}{-r_k} + \dfrac{3}{4}(t-T) - r_k, & -r_k < x < -r_k a, \\ \dfrac{3}{4}(t-T) + x, & x \le -r_k. \end{cases}$$

Let us verify that the function w_k is a minimax solution of problem (4.2). Let us fix a point (t_0, x_0, z_0), $x_0 > r_k$; then

$$\dot{x} = 1, \qquad \dot{z} = -\frac{1}{4}.$$

Integrating this system of ordinary differential equations, we obtain

$$x(t) = x_0 + t - t_0, \qquad z(t) = z_0 - \frac{1}{4}(t - t_0).$$

If $z_0 = -(5/4)(t_0 - T) + x_0$, then

$$z(t) = -t_0 + \frac{5}{4}T - \frac{1}{4}t = -\frac{5}{4}(t-T) + x(t).$$

Let $t_0 > 0$, let $r_k a < x_0 < r_k$, and let $z_0 = r_k \ln(x_0/r_k) - 5/4(t_0 - T) + r_k$. The differential inclusion is of the form

$$\dot{x} = \frac{x}{r_k}, \qquad \dot{z} = -\frac{1}{4}.$$

Integrating this system of differential equations, we obtain

$$x(t) = x_0 e^{(t-t_0)/r_k}, \qquad z(t) = -\frac{1}{4}(t-t_0) + z_0.$$

These characteristics survive in the graph of w_k, because

$$z(t) = -\frac{1}{4}(t-t_0) + r_k \ln\frac{x_0}{r_k} - \frac{5}{4}(t_0 - T) + r_k = r_k \ln\frac{x(t)}{r_k} - \frac{5}{4}(t-T) + r_k.$$

Let $t_0 > 0$, let $|x_0| \le r_k a$, and let $z_0 = x_0 e^{-(t_0-T)/r_k} - 1/4(t_0 - T)$. The differential inclusion is of the form

$$\dot{x} = \frac{x}{r_k}, \qquad \dot{z} = -\frac{1}{4}.$$

Integrating this system of differential equations, we obtain

$$x(t) = x_0 e^{(t-t_0)/r_k}, \qquad z(t) = -\frac{1}{4}(t-t_0) + z_0.$$

These characteristics survive in the graph of w_k, because

$$z(t) = \frac{1}{4}(t - t_0) + x_0 e^{-(t_0 - T)/r_k} - \frac{1}{4}(t_0 - T) = x(t) e^{-(t-T)/r_k} - \frac{1}{4}(t - T).$$

The other cases are verified in a similar way. The boundary condition holds.

It is easy to verify that

$$w^*(t, +0) = \lim_{k\to\infty,\, x_k \to +0} w_k(t, x_k) = -\frac{5}{4}(t - T),$$

$$w^*(t, -0) = \lim_{k\to\infty,\, x_k \to -0} w_k(t, x_k) = \frac{3}{4}(t - T).$$

Then the set $\operatorname{co} w^*$ coincides with a solution of the form (4.1).

Let $\widetilde{w}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a selector of the multivalued mapping w. Let us estimate the quantity $\int_{\mathbb{R}} |w_k(t,x) - \widetilde{w}(t,x)|\, dx$. We note that, in the domain

$$\{(t,x) : [t \in [0,T] \times \{x \geq r_k\}] \cup [t \in [0,T] \times \{x \leq -r_k\}]\},$$

the solutions w and w_k coincide. Then

$$\int_{[0,T]\times[-r_k, r_k]} |w_k(t,x) - w(t,x)|\, dx\, dt \approx C r_k,$$

where $C > 0$.

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