DOI 10.1007/s10958-017-3232-8
Journal of Mathematical Sciences, Vol. 221, No. 3, March, 2017

ON INTERSECTION OF PRIMARY SUBGROUPS OF ODD ORDER IN FINITE ALMOST SIMPLE GROUPS

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ABSTRACT. We consider the question of the determination of subgroups A and B such that $A \cap B^g \neq 1$ for any $g \in G$ for a finite almost simple group G and its primary subgroups A and B of odd order. We prove that there exist only four possibilities for the ordered pair (A, B).

Introduction

Let G be a finite group and A and B be its subgroups. By definition, M is the set of subgroups that are minimal by inclusion among all subgroups of type $A \cap B^g$, $g \in G$, and m consists of those elements of the set M whose order is minimal. Set $\mathrm{Min}_G(A, B) = \langle M \rangle$ and $\min_G(A, B) = \langle m \rangle$. Evidently, $\mathrm{Min}_G(A, B) \geq \min_G(A, B)$ and the following three conditions are equivalent:

(1) $A \cap B^g \neq 1$ for any $g \in G$;
(2) $\mathrm{Min}_G(A, B) \neq 1$;
(3) $\min_G(A, B) \neq 1$.

The first author showed that $\mathrm{Min}_G(A, B) \leq F(G)$ for any pair of Abelian subgroups A and B of G [4, Theorem 1], where $F(G)$ is the Fitting subgroup of G (the greatest normal nilpotent subgroup of G). On the other hand, V. I. Zenkov and V. D. Mazurov proved that $\mathrm{Min}_G(A, B) = 1$ for any pair of primary subgroups A and B of a simple non-Abelian group G [3, Theorem 1]. But also in the almost simple group $G \simeq \mathrm{Aut}\big(L_2(7)\big)$ we have that $\mathrm{Min}_G(S, S) = \min_G(S, S) = S$ for a Sylow 2-subgroup S of G. Moreover, it is proved in [5] that for a group G with socle $L_2(q)$, $q > 3$, if subgroups A and B are primary, then the inequality $\mathrm{Min}_G(A, B) \neq 1$ is valid only for $q = 9$ and for the Mersenne prime $q = 2^n - 1$; in these cases subgroups A and B are 2-groups.

In the present paper, we consider the case where A and B are primary subgroups of odd order in a finite almost simple group G. Our main result is the following theorem.

Theorem 1. *Let G be a finite almost simple group and A and B be its primary subgroups of odd order. Then the following are equivalent.*

(1) $\mathrm{Min}_G(A, B) \neq 1$.
(2) *G contains a normal subgroup of index 1 or 2 that is isomorphic to $D_4(3) \leftthreetimes Z_3$, and $(A, B) \in \{S, S_0\}^2$, where S is a Sylow 3-subgroup of G, $S_0 = O_3\big(N_G(P)\big)$ and P is a minimal proper parabolic subgroup of the group $D_4(3)$ corresponding to the central node of the Coxeter graph of type D_4. Moreover, $S_0 = \min_G(S, S)$.*

In Sec. 2, we establish some properties of subgroups of Chevalley groups, which are necessary for the proof of Theorem 1 and, in the opinion of the authors, are also of independent interest.

1. Notation and Preliminary Results

A finite group G is called *almost simple* if $\mathrm{Inn}(K) \leq G \leq \mathrm{Aut}(K)$, where K is a finite simple non-Abelian group and $\mathrm{Inn}(K)$ and $\mathrm{Aut}(K)$ are, respectively, the groups of inner automorphisms and all automorphisms of the group K. In this paper, the following shortcuts and notations are used:

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 19, No. 6, pp. 115–123, 2014.

- $g^h = h^{-1}gh$ for elements g and h of a group G;
- $A^B = \{a^b \mid a \in A,\ b \in B\}$ for subsets A and B of a group G;
- $A \leq G$ denotes that A is a subgroup of a group G;
- $\langle M \rangle$ is the group generated by a set M;
- $N_G(A)$ is the normalizer of a subgroup A in a group G;
- $O_p(G)$ is the maximal normal p-subgroup of a finite group G;
- $A \leftthreetimes B$ is the semidirect product of groups A and B with a normal subgroup A.

Actually, the proof of the implication (1) $\Longrightarrow$ (2) of Theorem 1 is reduced to the analysis of the situation in the group $D_4(3) \leftthreetimes Z_3$ by the usage of the following result of the first author.

Lemma 1 ([5, Theorem B(2a)]). *Let G be a finite almost simple group, p be an odd prime, and S be a Sylow p-subgroup of G. If $S \cap S^g \neq 1$ for any element g of G, then $p = 3$ and G contains a normal subgroup of index 1 or 2 that is isomorphic to $D_4(3) \leftthreetimes Z_3$.*

We will also need the following two technical lemmas, which will be used in the sequel.

Lemma 2 ([5, Lemma 3.1]). *Let G be a finite group and M_1 be a subgroup of G. Let P_1 be a Sylow p-subgroup of M_1 such that $P_1 \cap P_1^k = O_p(M_1)$ for some $k \in M$, and M_2 be a subgroup of G that is conjugate with M_1. Then there exists a Sylow p-subgroup P_2 of M_2 such that $P_1 \cap P_2 \leq O_p(M_1) \cap O_p(M_2)$.*

Lemma 3. *Let A, B, and S be subgroups of a finite group G such that $\mathrm{Min}_G(A, B) \neq 1$, $A \leq S$, and $A \cap B^h = S \cap S^h = T$ for some element $h \in G$ and some cyclic subgroup T of prime order. Then $T^S \leq A$.*

Proof. By the conditions of the lemma, we have that $T^s = S \cap S^{hs} \geq A \cap B^{hs} \neq 1$ for any $s \in S$. Therefore, taking into account that T is a cyclic subgroup of prime order, we deduce the inclusion $T^s \leq A$ for any $s \in S$, i.e., $T^S \leq A$. The lemma is proved. $\square$

2. Some Properties of Intersections of Sylow p-Subgroups of Chevalley Groups over a Finite Field of Characteristic p

Further, Φ is a reduced indecomposable root system, $\Pi = \{r_1, \ldots, r_l\}$ is its set of fundamental roots, Φ^+ is a positive root system with respect to Π, and also $\Phi^- = -\Phi^+$. We always assume that r_1 is a short root and the sum $r_i + r_j$, $i \leq j$, is a root if and only if:

$(i, j) = (l-3, l)$ or $(i, i+1), 1 \leq i \leq l-2$, if $\Phi = E_l$;
$(i, j) = (1, 3)$ or $(i, i+1), 2 \leq i \leq l-1$, if $\Phi = D_l$;
$(i, j) = (i, i+1)$ in all other cases.

We will need the following strengthening of Lemma 3.6.2 from [2, p. 50].

Lemma 4. *Let a fundamental root r_{i_1} be a part with nonzero coefficient of the expression of a root $r \in \Phi^+$ as an integral combination of fundamental roots with nonnegative coefficients. Then r can be expressed as a sum of fundamental roots*

$$r = r_{i_1} + r_{i_2} + \cdots + r_{i_k}$$

in such a way that $r_{i_1} + r_{i_2} + \cdots + r_{i_s}$ is a root for all $s \leq k$.

Proof. Let

$$r = c_1 r_1 + \cdots + c_l r_l$$

be the expression of a root $r \in \Phi^+$ as an integral combination of fundamental roots with nonnegative coefficients. Obviously, we have only the following two cases:

(1) $c_i \leq 1$;
(2) at least one of the numbers c_i is greater than 1.

For any subgraph of the Coxeter graph of type Φ, whose vertices are labeled by fundamental roots $r_{j_1}, \dots, r_{j_m}$, the sum $r_{j_1} + \dots + r_{j_m}$ is a root if and only if this subgraph is connected. Therefore, in the first case the lemma is true. The second case is reduced by induction to the first one by virtue of the following assertion:

$$\text{if } c_i > 1 \text{ for some } i, \text{ then there exists } j \text{ such that } c_j > 1 \text{ and the difference } r - r_j \text{ is a root.} \tag{A}$$

Assertion (A) can be directly verified for every root system.

For the root system of type A_l, only the first case is possible.

For the root system of type B_l in the case (2) we have

$$r = 2r_1 + \dots + 2r_s + r_{s+1} + \dots + r_t,$$

where $1 \leq s < t \leq l$. Here only the one variant $j = s$ is possible, in order that the difference $r - r_j$ is a root.

For the root system of type C_l in the case (2) we have

$$r = r_t + \dots + r_{s-1} + 2r_s + \dots + 2r_{l-1} + r_l,$$

where $1 \leq t \leq s \leq l-1$ (for $s = 1$ we set $r = 2r_1 + \dots + 2r_{l-1} + r_l$). Here also only one variant $j = s$ is possible.

For the root system of type D_l in the case (2) we have

$$r = r_1 + r_2 + 2r_3 + \dots + 2r_s + r_{s+1} + \dots + r_t,$$

where $3 \leq s < t \leq l$. Here also only one variant $j = s$ is possible.

In the Tables V–VIII from [1] for the exceptional types E_l and F_4 there are listed all positive roots that have at least one of the numbers c_j greater than 1. Using these tables, it is not difficult to check the validity of assertion (A) for the types E_l and F_4. Note that here for some roots the parameter j is not uniquely defined.

For the type G_2, correctness of conclusion (A) is easily checked and in this case the parameter j is defined uniquely.

Hence, assertion (A), along with the lemma, is proved. $\square$

Further, $\Phi(q)$ is an adjoint Chevalley group of type Φ of rank l over the finite field $\mathbb{F}_q$ of order $q = p^n$, where p is a prime. The group $\Phi(q)$ is generated by the root subgroups

$$X_r = \langle x_r(t) \mid t \in K \rangle, \quad r \in \Phi,$$

where $x_r(t)$ is the corresponding root element in the group $\Phi(q)$. We will need the following natural subgroups of the group $\Phi(q)$:

- the unipotent subgroups
$$U = \langle X_r \mid r \in \Phi^+ \rangle, \quad V = \langle X_r \mid r \in \Phi^- \rangle,$$
- the monomial subgroup
$$N = \langle n_r(t) \mid r \in \Phi,\ t \in \mathbb{F}_q^* \rangle,$$
- the diagonal subgroup
$$H = \langle h_r(t) \mid r \in \Phi,\ t \in \mathbb{F}_q^* \rangle,$$
- and the Borel subgroup
$$B = UH.$$

Here, $\mathbb{F}_q^*$ is the multiplicative subgroup of the field $\mathbb{F}_q$ and

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t), \quad h_r(t) = n_r(t)n_r(-1).$$

We set also

$$I = \{1, 2, \dots, l\}.$$

Overgroups of the Borel subgroup B and conjugate with them are called *parabolic*. Due to familiar result of J. Tits, parabolic subgroups containing the subgroup B are

$$P_J = \langle B, n_{r_j}(1) \mid j \in J \rangle,$$

where $J \subseteq I$.

Lemma 5. *Fix a monomial element n_0 with the condition $U^{n_0} = V$ and a positive integer $i \in I$. Set $n = n_0 n_{r_i}(1)$. Then $U \cap U^n = X_{r_i}$.*

Proof. The root subgroups X_{r_i} and X_{-r_i} normalize the subgroup

$$V_{r_i} = \langle X_r \mid r \in \Phi^- \setminus \{-r_i\} \rangle$$

and $V = V_{r_i} X_{-r_i}$ [2, Lemma 8.1.1]. Therefore, $U^n = V^{n_{r_i}(1)} = V_{r_i} X_{r_i}$. Clearly, $U \cap V_{r_i} X_{r_i} = X_{r_i}$. The lemma is proved. ☐

For $l = 1$ the root subgroup X_{r_i} coincides with a Sylow p-subgroup of the group $\Phi(q)$ and in this case the element n from Lemma 5 is diagonal.

Lemma 6. *Let $P = P_{I \setminus \{i\}}$ be the parabolic maximal subgroup of the group $\Phi(q)$ of type A_l, D_l, or E_l of rank $l \geq 2$ and the monomial element n as in Lemma 5. Then $U \cap U^n = X_{r_i}$ and $\langle X_{r_i}^U \rangle = O_p(P)$.*

Proof. For Chevalley groups $\Phi(q)$ of any type, the equality

$$O_p(P) = \langle X_r \mid r = c_k r_k + \cdots + c_i r_i + \cdots + c_m r_m,\ 1 \leq k \leq i \leq m \leq l,\ c_j \geq 1 \rangle$$

holds [2, Theorem 8.5.2]. For types A_l, D_l, and E_l, all structure constants of Chevalley's commutator formula are equal to 1. Hence, using Lemma 4, we can obtain the equality $\langle X_{r_i}^U \rangle = O_p(P)$. Really, let $X_r \in O_p(P)$. Then by Lemma 4 as $i = i_1$ for the root r we have the following representation:

$$r = r_{i_1} + r_{i_2} + \cdots + r_{i_k},$$

where the sum $r_{i_1} + r_{i_2} + \cdots + r_{i_s}$ is the root for all $s \leq k$. Therefore, we obtain inclusions

$$\begin{aligned}
&[X_{r_{i_1}}, X_{r_{i_2}}] = X_{r_{i_1}+r_{i_2}} \subset \langle X_{r_i}^U \rangle,\\
&[X_{r_{i_1}+r_{i_2}}, X_{r_{i_3}}] = X_{r_{i_1}+r_{i_2}+r_{i_3}} \subset \langle X_{r_i}^U \rangle,\\
&\quad \cdots\\
&[X_{r_{i_1}+\cdots+r_{i_{k-1}}}, X_{r_k}] = X_{r_{i_1}+\cdots+r_{i_k}} = X_r \subset \langle X_{r_i}^U \rangle.
\end{aligned}$$

Hence, $\langle X_{r_i}^U \rangle = O_p(P)$. Lemma 5 gives the equality $U \cap U^n = X_{r_i}$. The lemma is proved. ☐

Lemma 6 conclusion cannot be adapted in general for each types B_l, C_l, F_4, and G_2. For example, the following result holds.

Lemma 7. *Let $P = P_{I \setminus \{1\}}$ be the parabolic maximal subgroup of the group $\Phi(2)$ of type B_l, $l \geq 2$, over the field of two elements, where r_1 is short root. There is no root subgroup X_r such that $\langle X_r^U \rangle = O_p(P)$.*

Proof. Due to the choice of the parabolic maximal subgroup P, the following equality is valid:

$$O_p(P) = \langle X_s \mid s = c_1 r_1 + \cdots + c_k r_k,\ 1 \leq k \leq l,\ c_j \geq 1 \rangle.$$

It is trivial, that the equality $\langle X_r^U \rangle = O_p(P)$ is admissible only for $r = r_1$. Note that the subgroup $\langle X_{r_1}^U \rangle$ contains the product $x_{r_1+r_2}(1) x_{2r_1+r_2}(1)$, but individually the elements $x_{r_1+r_2}(1)$ and $x_{2r_1+r_2}(1)$ do not belong to $\langle X_{r_1}^U \rangle$. The lemma is proved. ☐

Figure 1 depicts correspondence between nodes of the Coxeter graph and roots from the fundamental root system, which are associated with the Chevalley group of type D_4, is recognized.

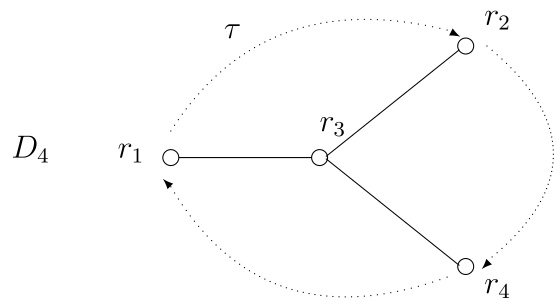

Fig. 1

Lemma 8. *Let $G = D_4(3) \leftthreetimes \langle \tau \rangle$ and $S = U \leftthreetimes \langle \tau \rangle$, where τ is the graph automorphism of order 3 of the group $D_4(3)$ as in Fig. 1, and let the monomial element $n = n_0 n_{r_1}(1) \in D_4(3)$ be as in Lemma 5. Then $S \cap S^n = X_{r_1}$, in particular, $U \cap U^n = X_{r_1}$.*

Proof. In the Weyl group of type D_4, there exists an element w_0 such that $w_0(r) = -r$ for any root r. Moreover, w_0 coincides with the cube c^3 of the Coxeter element

$$c = w_{r_3} w_{r_1} w_{r_2} w_{r_4}.$$

Hence in our case the element n_0 is the preimage of the element w_0 under the homomorphism of the monomial subgroup of the group $D_4(3)$ on the Weyl group of type D_4. Moreover, we can assume that

$$n_0 = \big(n_{r_3}(1) n_{r_1}(1) n_{r_2}(1) n_{r_4}(1)\big)^3.$$

Since the graph automorphism τ centralizes the monomial element

$$n_{r_3}(1) n_{r_1}(1) n_{r_2}(1) n_{r_4}(1)$$

(see Fig. 1 and [2, Proposition 12.2.3]), then

$$\tau^n = \tau^{n_{r_1}(1)} = \tau n_{r_2}(-1) n_{r_1}(1).$$

It is clear that $\tau n_{r_2}(-1) n_{r_1}(1) \notin \langle \tau \rangle$. Therefore, and from the equality $U \cap U^n = X_{r_1}$, which is given by Lemma 5, we obtain the equality $S \cap S^n = X_{r_1}$. The lemma is proved. □

Lemma 9. *Let G, S, and τ be as in Lemma 8. Set $S_0 = U_{r_3} \leftthreetimes \langle \tau \rangle$, where $U_{r_3} = \langle X_r \mid r \in \Phi^+ \setminus \{r_3\} \rangle$. Then $S_0 = \min_G(S, S)$.*

Proof. The subgroup S is a Sylow 3-subgroup of G. Therefore, by Lemma 1 $S \cap S^x \neq 1$ for any $x \in G$, whence $\min_G(A, B) = \langle m \rangle \neq 1$ and by Lemma 5 the set m consists of subgroups of order 3. Further, in the Coxeter graph of type D_4 the roots r_1, r_2, and r_4 coincide with symmetric nodes (see Fig. 1), whence Lemma 8 is valid if the root r_1 is exchanged with the root r_2 or r_4. Therefore, by Lemma 6 $O_3\big(P_{\{i\}}\big) \leq \min_G(S, S)$ for any $i = 1, 2, 4$. Thus, Lemmas 1, 5, 6, and 8 along with the equality

$$\big\langle O_3\big(P_{I\setminus\{1\}}\big), O_3\big(P_{I\setminus\{2\}}\big), O_3\big(P_{I\setminus\{4\}}\big)\big\rangle = O_3\big(P_{\{3\}}\big)$$

give the inclusion $S_0 \leq \min_G(S, S)$.

Suppose that $S_0 < \min_G(S, S)$. Then there exists an element (subgroup) D of the set m such that $D = S \cap S^g \not\leq S_0$ for some $g \in G$. Because of $|S \cap S^g| = 3$ by Lemma 8, we see that $S_0 \cap S_0^g = 1$. The subgroups S and S_0 satisfy the conditions of Lemma 2 as

$$G = D_4(3) \leftthreetimes \langle \tau \rangle, \quad M_1 = N_G\big(P_{\{3\}}\big), \quad P_1 = S, \quad O_3(M_1) = S_0, \quad k = n_{r_3}(1), \quad O_3(M_2) = S_0.$$

Thus, by Lemma 2

$$S \cap S^x \leq S_0 \cap S_0^g = 1$$

for suitable $x \in G$. Hence, $S \cap S^x = 1$. This contradicts Lemma 1. The lemma is proved. □

3. The Proof of Theorem 1

Let a group G satisfy the conditions of Theorem 1. Further, we use the notation of the preceding section for subgroups and elements of the Chevalley group $D_4(3)$.

(1) $\Longrightarrow$ (2). Let $\mathrm{Min}_G(A,B)\neq 1$. Since A and B are primary subgroups, the condition $\mathrm{Min}_G(A,B)\neq 1$ implies that subgroups A and B are p-groups for an odd prime p. Hence, without losing generality, we can assume that the subgroups A and B lie in one fixed Sylow p-subgroup S of the group G. Now again by the condition $\mathrm{Min}_G(A,B)\neq 1$ we get the inequality $S\cap S^g\neq 1$ for any element g of G. Therefore, by Lemma 1 we can assume that $p=3$ and the group G contains the normal subgroup

$$G_0 = D_4(3) \leftthreetimes \langle\tau\rangle$$

of index 1 or 2, where τ is a graph automorphism of order 3 of the group $D_4(3)$. We can assume that

$$S = U \leftthreetimes \langle\tau\rangle.$$

Let

$$n_0 = \big(n_{r_3}(1)n_{r_1}(1)n_{r_2}(1)n_{r_4}(1)\big)^3.$$

Then (see the proof of Lemma 8)

$$S\cap S^{n_0} = \langle\tau\rangle.$$

Since

$$\langle\tau\rangle = S\cap S^{n_0} \geq A\cap B^{n_0}\neq 1,$$

we obtain

$$A\cap B^{n_0} = \langle\tau\rangle.$$

By Lemma 8 there exists a monomial element $n\in D_4(3)$ such that

$$S\cap S^{n} = X_{r_1}.$$

Since

$$X_{r_1} = S\cap S^{n} \geq A\cap B^{n}\neq 1,$$

we have that

$$A\cap B^{n_0} = X_{r_1}.$$

Now by Lemma 3, we have that

$$X_{r_1}^S \leq A.$$

By Lemma 6,

$$\langle X_{r_1}^U\rangle = O_3\big(P_{I\setminus\{1\}}\big).$$

As $\tau\in A$,

$$\big\langle O_3\big(P_{I\setminus\{1\}}\big),\, O_3\big(P_{I\setminus\{1\}}^{\tau}\big),\, O_3\big(P_{I\setminus\{1\}}^{\tau^2}\big)\big\rangle = O_3\big(P_{\{3\}}\big)\leq A.$$

Hence

$$O_3\Big(N_G\big(P_{\{3\}}\big)\Big)\leq A.$$

Suppose that

$$S_0 = O_3\Big(N_G\big(P_{\{3\}}\big)\Big).$$

Note that $|S:S_0|=3$. Therefore, either $A=S_0$ or $A=S$. The condition $A\cap B^g\neq 1$ for any $g\in G$ is equivalent to the condition $B\cap A^{g^{-1}}\neq 1$ for any $g\in G$. Thus,

$$(A,B)\in\{S,S_0\}^2.$$

It remains only to prove the equality $S_0=\mathrm{min}_G(S,S)$. We have two cases only:

(a) $G=G_0$;
(b) $|G:G_0|=2$.

In case (a), the equality $S_0 = \min_G(S, S)$ is valid by Lemma 9.

The case (b) follows from (a) and the invariance of subgroup S with respect to outer (graph) automorphisms of order 2 of the group $D_4(3)$.

(1) $\Longleftarrow$ (2). It is clear that for the pair (S, S) the inequality $S \cap S^g \neq 1$ for any element $g \in G$ follows from the recently found equality $S_0 = \min_G(S, S)$.

It is already known that $S \cap S^g \neq 1$ for any $g \in G$ and $S \cap S^{n_{r_3}(1)} = S_0$. Then by Lemma 2 (see the proof of Lemma 9) we have $S_0 \cap S_0^g \neq 1$ and, moreover, $S_0 \cap S^g \neq 1$ and $S \cap S_0^g \neq 1$ for any $g \in G$. This concludes the proof of our theorem.

The work of the first author is supported by RFBR (project 13-01-00469), by the Program of the Division of Mathematical Sciences of RAS (project 12-T-1-10003), by the Program of the Joint Investigations of RAS with SB RAS (project 12-C-1-1018) and with UB Belarussian National Academy of Sciences (project 12-C-1-1009) and by the Program of the State support of leading universities of Russia (agreement No. 02.A03.210006 from 27.08.2013). The work of the second author is supported by RFBR (project 12-01-00968-a).

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