Discrete Mathematics 268 (2003) 59–80
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Counting Mal'tsev clones on small sets

Andrei A. Bulatov[a], Paweł M. Idziak[b,c,*,1]

[a] *Department of Algebra and Discrete Mathematics, Ural State University, Lenin av. 51, 620083 Ekaterinburg, Russia*
[b] *Wyzsza-Szkola Biznesu, National Louis University, Nowy Sacz, Poland*
[c] *Computer Science Department, Jagiellonian University, Nawojki 11, PL-30-072 Kraków, Poland*

Received 25 August 2000; received in revised form 6 June 2002; accepted 17 June 2002

Abstract

Idziak (Int. J. Algebra Comput. 9 (1999) 213) has shown that there are only 14 Mal'tsev polynomial clones on a 3 element set and constructed infinitely many such clones on a 4 element set. Here we improve this result by showing that there are, in fact, only countably many Mal'tsev polynomial clones on a 4 element set. A description of all such clones is included.

MSC: 08A40; 03B50

1. Introduction

By a *clone* on a set A we mean a set of operations on A which is closed under superposition and contains all projection operations. The well-known result of Post [11] gives a description of all clones on a two-element set. If A has three or more elements then there are continuum many clones on A (see [12] or [5]). This fact leaves no hope for a good classification of clones in this case. The gap between the number of clones on two- and three-element sets is much greater if we consider clones containing all constant operations, i.e., so called *polynomial clones*. From the result of Post it follows that there are only 7 polynomial clones on a two-element set. On the other hand, Ágoston et al. [1] constructed uncountably many polynomial clones on a three-element set.

* Corresponding author. Computer Science Department, Jagiellonian University, ul. Nawojki 11, Krakow 30-072, Poland. Fax: +48-12-634-1865.
E-mail addresses: andrei.bulatov@usu.ru (A.A. Bulatov), idziak@ii.uj.edu.pl (P.M. Idziak).
1 Supported by a research grant from Polish KBN.

0012-365X/03/$ - see front matter
PII: S0012-365X(02)00681-7

The clone of term [polynomial] operations of an algebra **A** is denoted by $\mathsf{Clo}(\mathbf{A})$ [$\mathsf{Pol}(\mathbf{A})$]. Algebras **A** and **B** with the same universe are *term* [*polynomially*] *equivalent* if $\mathsf{Clo}(\mathbf{A})=\mathsf{Clo}(\mathbf{B})$ [$\mathsf{Pol}(\mathbf{A})=\mathsf{Pol}(\mathbf{B})$]. Thus, the number of clones (polynomial clones) of a certain kind on a set A is equal to the number of term [polynomially] inequivalent algebras of that kind with the universe A. In this paper, we restrict ourselves to such clones that the corresponding algebras generate congruence permutable varieties. They need to contain a ternary operation $\mathbf{d}(x,y,z)$ satisfying $\mathbf{d}(x,x,y)=y=\mathbf{d}(y,x,x)$. Such an operation **d** is called a Mal'tsev operation. A clone containing at least one Mal'tsev operation is called a *Mal'tsev clone*. By $\mathcal{O}_A$ we denote the clone of all operations on a set A. For $D\subseteq\mathcal{O}_A$ the clone generated by operations from D and constant operations is denoted by $\langle D\rangle$.

In the following table we summarize our present knowledge on the number of clones satisfying the most popular congruence conditions. For a fixed number of elements in A the top elements in the row shows the number of all clones on A with a given Mal'tsev condition, while the bottom elements in the row shows the number of polynomial clones of this type. The symbols CD, CP and C3P denote congruence distributivity, congruence permutability and congruence 3-permutability. The entries for $|A|=2$ follow from Post's classification. The remaining entries can be filled up by results of [1,6,9,10,12] and of the present paper. Note that since each cloned that is both congruence distributive and congruence permutable contains a majority operation (Pixley term), by Baker and Pixley [2], we know that such a clone is determined by a set of binary relations on A. Thus, there are only finitely many such clones.

$\|A\|$	all clones	CD	CD and C3P	CP	CD and CP
2	ω	ω	ω	11	6
	7	2	1	2	1
3	2^ω	2^ω	2^ω	?	fin
	2^ω	2^ω	2^ω	14	4
4	2^ω	2^ω	2^ω	$\geqslant\omega$	fin
	2^ω	2^ω	2^ω	ω	fin
5	2^ω	2^ω	2^ω	$\geqslant\omega$	fin
	2^ω	2^ω	2^ω	$\geqslant\omega$	fin
⋮	⋮	⋮	⋮	⋮	⋮
	⋮	⋮	⋮	⋮	⋮

2. Result and tools

The main result of this paper is the following

Theorem 1. *There are only countably many Mal'tsev polynomial clones on a 4-element set.*

In the proof of Theorem 1 we will see that the most complicated situations are connected with clones that give rise to algebras with nilpotent but non-Abelian congruences. In particular, we will show that there are exactly countably many such clones.

Our proofs make heavy use of the main developments in universal algebra: modular commutator theory (see [3]) and the tame congruence theory of Hobby and McKenzie [4]. A basic knowledge of these two theories, or at least an access to the above-mentioned books, is assumed.

Moreover, we will use the notions and techniques developed by Idziak and Słomczyńska [7]. Following that paper an algebra **A** is called *polynomially rich* if every operation preserving the congruences of **A** and the labels of their prime quotients (in the sense of the tame congruence theory) is a polynomial of **A**.

To state the definition of polynomial richness we start with recalling that Tame Congruence Theory is a tool to study the local structure of finite algebras. Instead of considering the whole algebra and all its operations at once, the theory allows us to localize to small subsets on which the structure is much simpler to understand and handle. There are only five possible ways a finite algebra can behave locally with respect to this theory. It can be either one of the following:

1. a finite set with a group action on it,
2. a finite vector space over a finite field,
3. a two element Boolean algebra,
4. a two element lattice,
5. a two element semilattice.

Moreover, the theory allows to label all prime quotients in the congruence lattice of a finite algebra by one of the types 1–5. The type of the prime quotient $\alpha \prec \beta$ in the congruence lattice $\mathsf{Con}(\mathbf{A})$ is denoted by $\mathsf{typ}_{\mathbf{A}}(\alpha,\beta)$, or simply $\mathsf{typ}(\alpha,\beta)$ if **A** is clear from the context.

Now, if **A** is a finite algebra then by an **A**-admissible mapping we mean a function of the form $f: A^s \to A$ and such that f preserves congruences of **A** and their labels,

or more formally

if $\mathbf{A} + f$ denotes the algebra **A** expanded by a new s-ary operation f then $\mathsf{Con}(\mathbf{A}+f) = \mathsf{Con}(\mathbf{A})$ and for $\alpha,\beta \in \mathsf{Con}(\mathbf{A})$ with $\alpha \prec \beta$ we have $\mathsf{typ}_{\mathbf{A}+f}(\alpha,\beta) = \mathsf{typ}_{\mathbf{A}}(\alpha,\beta)$.

A finite algebra **A** is said to be *polynomially rich* if every **A**-admissible mapping is a polynomial of **A**. It is *hereditarily polynomially rich* if all its homomorphic images are also.

We will often use the characterization of Mal'tsev polynomially rich algebras described in [7].

Theorem 2 (Idziak and Słomczyńska [7]). *Let* **A** *be a finite algebra that has a Mal'tsev polynomial. Then* **A** *is hereditarily polynomially rich iff for every* *subdirectly irreducible homomorphic image* **D** *of* **A** *the following conditions hold*:

(SC1) *the centralizer of the monolith of* **D** *is not bigger than the monolith itself*,
(GFp) *if the monolith* μ *of* **D** *is Abelian*, *then the* $(\mathbf{0},\mu)$*-minimal sets are polynomially equivalent to a one-dimensional vector space over a field* $\mathbf{GF}(p)$ *for some prime number* p.

Note that in general, if the monolith of a subdirectly irreducible algebra **A** is Abelian then $(\mathbf{0},\mu)$-minimal sets may consists of several traces each of which is polynomially equivalent to a (one-dimensional) vector space over the field $\mathbf{GF}(p^k)$. Thus, condition (GFp) says that $(\mathbf{0},\mu)$-minimal sets have only one trace and that the corresponding field is of prime order.

The possible label of the prime quotient $\alpha \prec \beta$ of congruences of a finite algebra with a Mal'tsev polynomial is either **2** or **3**. The type $\mathsf{typ}(\alpha,\beta)$ of this prime quotient is obviously determined by the commutator square $[\beta,\beta]$:

$$\mathsf{typ}(\alpha,\beta)=\begin{cases}\mathbf{2} & \text{if } [\beta,\beta]\leqslant\alpha,\\ \mathbf{3} & \text{otherwise.}\end{cases}$$

If, in addition, the algebra is hereditarily polynomially rich, or more generally if all its subdirectly irreducible quotients satisfy condition (SC1), then the following strong converse can be found in [7].

Lemma 3. *Suppose that a finite algebra* **A** *belongs to a congruence modular variety and that all its subdirectly irreducible homomorphic images satisfy* (SC1). *Then the commutator of congruences of* **A** *is uniquely determined by the labeled congruence lattice* $\mathsf{Con}(\mathbf{A})$ *using the following formula*:

$$[\alpha,\beta]=([\alpha,\alpha]\wedge\beta)\vee(\alpha\wedge[\beta,\beta]),$$

where the commutator square $[\alpha,\alpha]$ *is either the intersection of all subcovers* δ *of* α *such that* $[\alpha,\alpha]\leqslant\delta$, *or* α, *if there are no such subcovers.*

We will usually say that a clone $\mathscr{C}$ on A enjoys a particular property if the algebra $(A;\mathscr{C})$ enjoys this property. For example, we will call clones [hereditarily] polynomially rich, nilpotent, and so on. Moreover, by the congruence lattice of a clone $\mathscr{C}$ we mean the congruence lattice of the algebra $(A;\mathscr{C})$.

We conclude this section by showing that on a finite set there are only finitely many hereditarily polynomially rich Mal'tsev clones. Actually, we state this result in a more general setting.

Following Kiss [8] by a 4-difference term in a congruence modular variety $\mathscr{V}$ we mean a term $\mathbf{q}(x,y,z,w)$ such that

- $\mathbf{q}(x,y,x,y)=x$ and $\mathbf{q}(x,x,w,w)=w$ are identities of $\mathscr{V}$,

- if $\mathbf{A} \in \mathscr{V}$, $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$ and $a, b, c, c', d \in A$ are such that

$$a \overset{\alpha}{\equiv} b \overset{\beta}{\equiv} d \overset{\alpha}{\equiv} c \overset{\beta}{\equiv} a \quad \text{and} \quad d \overset{\alpha}{\equiv} c' \overset{\beta}{\equiv} a$$

then $\mathbf{q}(a, b, c, d) \overset{[\alpha,\beta]}{\equiv} \mathbf{q}(a, b, c', d)$.

In Theorem 3.8 of [8], Kiss has shown that every congruence modular variety $\mathscr{V}$ has a 4-difference term $\mathbf{q}$. Moreover, from Theorem 3.8(iii) of [8] it easily follows that for $\mathbf{A} \in \mathscr{V}$ and $\alpha, \beta, \gamma \in \mathsf{Con}(\mathbf{A})$ with $\gamma \leqslant \alpha, \beta$ the condition $[\alpha, \beta] \leqslant \gamma$ can be equivalently expressed by saying that for every basic operation $\mathbf{f}$ of $\mathbf{A}$ and all quadruples $(a_i, b_i, c_i, d_i) \in A^4$ satisfying

$$a_i \overset{\alpha}{\equiv} b_i \overset{\beta}{\equiv} d_i \overset{\alpha}{\equiv} c_i \overset{\beta}{\equiv} a_i,$$

the following holds:

$$\mathbf{f}(\mathbf{q}(a_1, b_1, c_1, d_1), \ldots, \mathbf{q}(a_n, b_n, c_n, d_n)) \overset{\gamma}{\equiv} \mathbf{q}(\mathbf{f}(\bar{a}), \mathbf{f}(\bar{b}), \mathbf{f}(\bar{c}), \mathbf{f}(\bar{d})).$$

Now let $\mathbf{A}$ be a finite algebra from a congruence modular variety and let $\mathbf{q}$ be its 4-difference polynomial. For $\alpha, \beta, \gamma \in \mathsf{Con}(\mathbf{A})$ satisfying $[\alpha, \beta] \leqslant \gamma \leqslant \alpha, \beta$ we define the 5-ary relation $R_{\alpha,\beta,\gamma} \subseteq A^5$ by

$$R_{\alpha,\beta,\gamma} = \{(a, b, c, d, u) \in A^5 : a \overset{\alpha}{\equiv} b \overset{\beta}{\equiv} d \overset{\alpha}{\equiv} c \overset{\beta}{\equiv} a \text{ and } u \overset{\gamma}{\equiv} \mathbf{q}(a, b, c, d)\}.$$

Now let $\mathscr{C}$ be the clone of all operations on the set A that preserve the congruences of $\mathbf{A}$ and the relations of the form $R_{\alpha,\beta,\gamma}$. It should be clear that the clone $\mathscr{C}$ is the largest polynomial clone on $\mathbf{A}$ with the same congruences as $\mathbf{A}$ and the same commutator operation for congruences. Thus we have the following.

Lemma 4. *Let* $\mathbf{A}$ *be a finite algebra from a congruence modular variety and let* $\mathsf{Concom}(\mathbf{A}) = (\mathsf{Con}(\mathbf{A}); \wedge, \vee, [\cdot,\cdot])$ *denote its congruence lattice endowed with the commutator operation. Then there is a largest clone* $\mathscr{C}$ *on* A *containing* $\mathsf{Pol}(\mathbf{A})$ *and such that* $\mathsf{Concom}(A; \mathscr{C}) = \mathsf{Concom}(\mathbf{A})$.

The largest clone described in Lemma 4 depends on the 4-difference term $\mathbf{q}$. For example if $+$ denotes addition modulo 4 in the set $\{0, 1, 2, 3\}$ then the clones $\langle x + y \rangle$ and $\langle x + y + 2xy, 2x \rangle$ are both Abelian and have the same 3-element chain as the congruence lattices. However the join of these clones is no longer Abelian.

The usefulness of the concept of polynomial richness in counting Mal'tsev polynomial clones follows from the next lemma.

Lemma 5. *On a finite set there are only finitely many hereditarily polynomially rich Mal'tsev clones. More precisely, if* $\mathbf{A}$ *is a finite algebra with a Mal'tsev polynomial, then there is at most one hereditarily polynomially rich clone* $\mathscr{R}$ *containing* $\mathsf{Pol}(\mathbf{A})$ *such that the algebra* $(A; \mathscr{R})$ *has the same congruences and the same labels of their prime quotients as* $\mathbf{A}$.

Proof. Note that every Mal'tsev polynomial $\mathbf{d}(x, y, z)$ of $\mathbf{A}$ gives rise to a 4-difference polynomial $\mathbf{q}(x, y, z, w) = \mathbf{d}(x, y, w)$. Now let $\mathscr{C}$ be the largest clone on A containing $\mathsf{Pol}(\mathbf{A})$ and such that $\mathsf{Concom}(A; \mathscr{C}) = \mathsf{Concom}(\mathbf{A})$. This clone is supplied by Lemma 4.

Using the labels of the prime quotients of congruences of $\mathbf{A}$ define a new binary operation $[\cdot,\cdot]'$ on $\mathsf{Con}(\mathbf{A})$ in a way described in Lemma 3. Let $\mathscr{C}'$ be the clone of all operations preserving congruences of $\mathbf{A}$ and the relations $R_{\alpha,\beta,\gamma}$ with $[\alpha,\beta]' \leqslant \gamma \leqslant \alpha, \beta$. Then the clone $\mathscr{C}'$ satisfies (SC1), or more precisely every subdirectly irreducible homomorphic image of the algebra $(A; \mathscr{C}')$ satisfies (SC1). Since $[\alpha,\beta] \subseteq [\alpha,\beta]'$ we get that $\mathscr{C} \subseteq \mathscr{C}'$.

Now if $\mathscr{R}$ is a hereditarily polynomially rich clone with the same congruences and labels as in $\mathbf{A}$ then all operations in $\mathscr{R}$ preserve congruences and the 5-ary relations $R_{\alpha,\beta,\gamma}$ that determine $\mathscr{C}'$. Consequently $\mathscr{R} \subseteq \mathscr{C}'$. From the maximality of $\mathscr{R}$ we get $\mathscr{R} \subseteq \mathscr{C}'$. Consequently $\mathscr{C}'$ is the only possible hereditarily polynomially rich extension of $\mathsf{Pol}(\mathbf{A})$ with the same congruences and the labels of their prime quotients as in $\mathbf{A}$. $\square$

In [6] it was shown that every Mal'tsev polynomial clone on a 3 element set is hereditarily polynomially rich. The same is trivially true for a 2 element set.

3. Mal'tsev clones on a 4-element set

Fix a four element set $A = \{0, 1, 2, 3\}$. We will prove Theorem 1 by showing that for every lattice $\mathbf{L}$ of permuting equivalence relations on A the number of Mal'tsev polynomial clones with the congruence lattice $\mathbf{L}$ is either finite or countable. If $\mathbf{L}$ consists of permuting equivalences then it is modular. By inspecting sublattices of the lattice of equivalence relations on A one checks that Fig. 1 gives a complete (up to the permutations of A) list of such lattices. (Hyphens separate cosets of an equivalence relation. By $\mathbf{0}$ and $\mathbf{1}$ we denote the identity relation and the total relation on A.)

We split the nine situations presented in Fig. 1 into five cases.

Case 3.1. *There are only three*, *up to polynomial equivalence*, *simple Mal'tsev algebras on the four element set*.

Proof. Suppose that $\mathbf{A}$ is a simple algebra. If it is non-Abelian then it is polynomially rich and therefore it is primal. If $\mathbf{A}$ is Abelian then the argument splits into two cases depending on whether $\mathbf{A}$ is minimal or not. Since $\mathbf{A}$ is simple then $(\mathbf{0}, \mathbf{1})$-minimal sets consist of a single trace. Thus if $\mathbf{A}$ is minimal it has to be polynomially equivalent to a one-dimensional vector space over $\mathbf{GF}(4)$. On the other hand, if $\mathbf{A}$ is not minimal then the $(\mathbf{0}, \mathbf{1})$-minimal sets have either 2 or 3 elements, so that both conditions (SC1) and (GFp) hold. Therefore $\mathbf{A}$ is polynomially rich. Moreover $\mathbf{A}$ is Abelian and therefore it is polynomially equivalent to a module over a finite ring. In particular it has a binary polynomial $x \oplus y$ such that $(A; \oplus, 0)$ is one of the Abelian groups $\mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_4$.

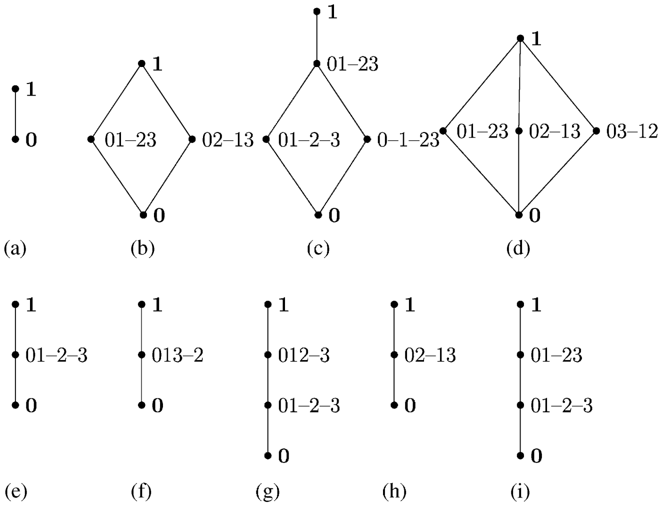

Fig. 1. Lattices of permuting equivalence relations on A.

We will show that the later case is in fact impossible. Indeed, suppose otherwise and let 2 be the unique non-zero element of A with $2 \oplus 2 = 0$. Since the pair $(0, 1)$ gets collapsed by the congruence generated by $(0, 2)$ there is a unary polynomial $\mathbf{p}$ of $\mathbf{A}$ that takes 0 to 0 and 2 to 1. Since $\mathbf{A}$ is polynomially equivalent to a module, every unary polynomial $\mathbf{f}$ of $\mathbf{A}$ with $\mathbf{f}(0) = 0$ is linear i.e., $\mathbf{f}(x \oplus y) = \mathbf{f}(x) \oplus \mathbf{f}(y)$. In particular $1 \oplus 1 = \mathbf{p}(2) \oplus \mathbf{p}(2) = \mathbf{p}(2 \oplus 2) = \mathbf{p}(0) = 0$. This means that $(A; \oplus, 0)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. Consequently $\mathbf{A}$ is the unique polynomially rich simple Abelian algebra on A. Note that then $\mathsf{Pol}(\mathbf{A})$ cannot contain addition $+$ modulo 4, as otherwise the polynomial $x \oplus y \oplus (x + y)$ would witness that $\mathbf{A}$ is not Abelian. □

Case 3.2. *Every Mal'tsev algebra with the congruence lattice of type* (b), (c) *or* (d) *is hereditarily polynomially rich.*

Proof. If $\mathbf{A}$ has a congruence lattice of type (b), (c) or (d) then any of its subdirectly irreducible homomorphic images has either 2 or 3 elements. Thus, by Idziak [6] the algebra $\mathbf{A}$ is hereditarily polynomially rich. □

Case 3.3. *Every Mal'tsev algebra with the congruence lattice of type* (e), (f) *or* (g) *is hereditarily polynomially rich.*

Proof. Suppose that $\mathbf{A}$ is subdirectly irreducible and has the congruence lattice of type (e), (f) or (g). Let μ denote the monolith of $\mathbf{A}$ while α is the unique cover of μ in $\mathsf{Con}(\mathbf{A})$.

As previously observed each proper homomorphic image of $\mathbf{A}$ is small enough to be hereditarily polynomially rich. Moreover, since μ has exactly one non-trivial coset and $\mathbf{A}$ is not simple we know that $(\mathbf{0}, \mu)$-minimal sets consist of a single trace and can have at most three elements. Consequently (GFp) must hold. Thus to conclude that **A** itself is polynomially rich we have to check that α does not centralize μ. From $(0,1)\in\mu$ and the fact that the μ-class containing 2 is a singleton we easily infer that

$$\mathbf{d}(0,0,2)=2=\mathbf{d}(1,0,2).$$

Now since $(0,2)\in\alpha$ the assumption that $[\mu,\alpha]=\mathbf{0}$ would give

$$0=\mathbf{d}(0,0,0)=\mathbf{d}(1,0,0)=1,$$

a contradiction. □

Case 3.4. *Every Mal'tsev algebra with the congruence lattice of type* (h) *is either hereditarily polynomially rich or has a binary group polynomial.*

Proof. Suppose that the congruence lattice of an algebra **A** has type (h) and let μ denote the monolith of **A**. If $[\mu,\mathbf{1}]\neq\mathbf{0}$ then **A** satisfies (SC1). Moreover **A** is not nilpotent and therefore not E-minimal, so that each $(\mathbf{0},\mu)$-minimal set consists of one two element trace. Since the only non-trivial proper quotient of **A** has two elements, Theorem 2 applies and we get that **A** is hereditarily polynomially rich.

Now suppose that $[\mu,\mathbf{1}]=\mathbf{0}$. This obviously gives $\mathsf{typ}(\mathbf{0},\mu)=\mathbf{2}$. The binary polynomial $\mathbf{s}(x,y)=\mathbf{d}(x,0,y)$ preserves μ and respects $[\mu,\mu]=\mathbf{0}$. This determines all but four values of this polynomial and we have

$\mathbf{s}(x,y)$	0	1	2	3
0	0	1	2	3
1	1		3	
2	2	3	0	1
3	3		1	

Moreover, all four missing entries are uniquely determined by $\mathbf{s}(1,1)$. Obviously, $\mathbf{s}(1,3)\overset{\mu}{\equiv}\mathbf{s}(1,1)$ and if $\mathbf{s}(1,3)=\mathbf{s}(1,1)$ then $[\mu,\mathbf{1}]=\mathbf{0}$ would give $3=\mathbf{s}(0,3)=\mathbf{s}(0,1)=1$. An argument of this kind gives

$$\mathbf{s}(1,1)=\mathbf{s}(3,3)\overset{\mu-\mathbf{0}}{\equiv}\mathbf{s}(1,3)=\mathbf{s}(3,1).$$

In particular if $\mathbf{s}(1,1)=0$ then $(A,\mathbf{s})$ is the group $\mathbf{Z}_2\times\mathbf{Z}_2$, while $\mathbf{s}(1,1)=2$ gives that $(A,\mathbf{s})$ is the group $\mathbf{Z}_4$.

In the remaining cases, i.e., for $\mathbf{s}(1,1)\in\{1,3\}$ we immediately get $[\mathbf{1},\mathbf{1}]=\mathbf{1}$ so that $\mathsf{typ}(\mu,\mathbf{1})=\mathbf{3}$. In particular, we know that for $a=\mathbf{s}(1,1)$ the set $U=\{0,a\}$ is $(\mu,\mathbf{1})$-minimal and $\mathbf{e}(x)=\mathbf{s}(x,x)$ is a unary idempotent polynomial of **A** with range U. We will consider the case $a=1$ as $a=3$ can be treated by switching 1 and 3. Put $\mathbf{e}'(x)=\mathbf{d}(x,\mathbf{e}(x),0)$ and observe that $\mathbf{e}'(0)=\mathbf{e}'(1)=0$ and $\mathbf{e}'(2)=2$. To compute $\mathbf{e}'(3)$ first note that

$$\mathbf{d}(1,0,0)=1\neq 3=\mathbf{d}(3,0,0)$$

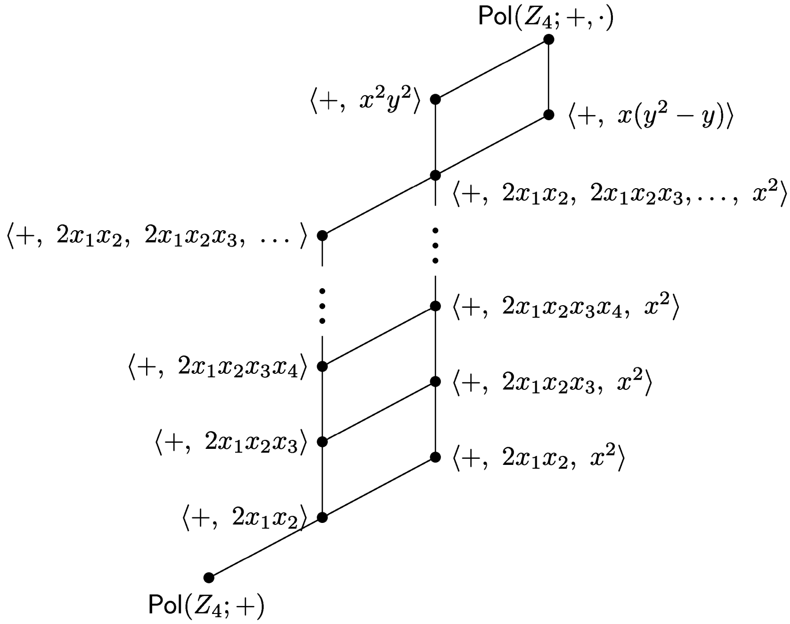

Fig. 2. Clones between the group and the ring of integers modulo 4.

so that $[\mu, \mathbf{1}] = \mathbf{0}$ gives

$$0 = \mathbf{d}(1,1,0) \neq \mathbf{d}(3,1,0) = \mathbf{e}'(3).$$

As $\mathbf{e}'(3) \in \mathbf{e}'(1)/\mu = \{0,2\}$ we get $\mathbf{e}'(3) = 2$.

On the other hand, we know that there is a binary polynomial $\mathbf{f}(x,y)$ of $\mathbf{A}$ such that $(U, \mathbf{f}|_U)$ is a Boolean group. Now a simple calculation shows that the polynomial $\mathbf{d}(\mathbf{f}(\mathbf{e}(x), \mathbf{e}(y)), 0, \mathbf{e}'(\mathbf{s}(x,y)))$ determines the addition of a Boolean group on the set A. □

To conclude that there are only countably many Mal'tsev polynomial clones in Case 3.4 we need the following two lemmas. The first one is due to Krokhin et al. [9].

Lemma 6 (Krokhin et al. [9]). *There are countably many polynomial clones between the clone* $\mathsf{Pol}(Z_4;+)$ *of polynomials of the group of integers modulo* 4 *and the clone* $\mathsf{Pol}(Z_4;+,\cdot)$ *of polynomials of the ring of integers modulo* 4. *Fig. 2 presents the lattice of such intermediate clones.*

It is easy to check that the clone $\langle +, 2x_1x_2, 2x_1x_2x_3, \ldots, x^2\rangle$ is the largest nilpotent clone containing $\mathsf{Pol}(Z_4;+)$ and contained in $\mathsf{Pol}(Z_4;+,\cdot)$, while $\langle +, 2x_1x_2, 2x_1x_2x_3, \ldots\rangle$ is the largest E-minimal such clone. Both of the clones $\langle +, x(y^2-y)\rangle$ and $\mathsf{Pol}(Z_4;+,\cdot)$ are polynomially rich with an Abelian monolith μ—in the former one we have

$\mathsf{typ}(\mu, \mathbf{1}) = \mathbf{2}$, i.e., the resulting algebra is solvable but non-nilpotent, while in the latter $\mathsf{typ}(\mu, \mathbf{1}) = \mathbf{3}$.

Now, suppose that the clone $\mathscr{C} \supseteq \mathsf{Pol}(Z_4; +)$ has a congruence lattice of type (h) and that $\mathsf{typ}(\mathbf{0}, \mu) = \mathbf{2}$. The second part of Lemma 5 tells us that all clones containing the Mal'tsev operation $x - y + z$ of the group $\mathbf{Z}_4$ have a unique common polynomially rich extension. We already know, that the clones $\langle +, x(y^2 - y)\rangle$ and $\mathsf{Pol}(Z_4; +, \cdot)$ are polynomially rich. Therefore $\mathscr{C}$ is contained in one of them and thus in $\mathsf{Pol}(Z_4; +, \cdot)$. Consequently, if the clone $\mathscr{C}$ satisfies $\mathsf{Pol}(Z_4; +) \subseteq \mathscr{C} \nsubseteq \mathsf{Pol}(Z_4; +, \cdot)$ then $\mathsf{typ}(\mathbf{0}, \mu) = \mathbf{3}$ and therefore $\mathscr{C}$ is polynomially rich. Thus there are at most two clones of type (h) that contain the addition modulo 4 and are not listed in Fig. 2. Let $*$ denote the binary operation on the set $\{0, 1, 2, 3\}$ defined by $2 * 2 = 2$ and $x * y = 0$ otherwise. Then one can easily check that the clones

$$\mathscr{H}_{\mathbf{2}}^{\mathbf{2}} = \langle +, x(y^2 - y)\rangle,$$
$$\mathscr{H}_{\mathbf{2}}^{\mathbf{3}} = \langle +, xy)\rangle = \mathsf{Pol}(Z_4; +, \cdot),$$
$$\mathscr{H}_{\mathbf{3}}^{\mathbf{2}} = \langle +, x * y\rangle,$$
$$\mathscr{H}_{\mathbf{3}}^{\mathbf{3}} = \langle +, x * y, xy\rangle,$$

are polynomially rich and for $\mathscr{H}_{\mathbf{i}}^{\mathbf{j}}$ we have $\mathsf{typ}(\mathbf{0}, \mu) = \mathbf{i}$ and $\mathsf{typ}(\mu, \mathbf{1}) = \mathbf{j}$. Moreover $\mathscr{H}_{\mathbf{i}}^{\mathbf{j}} \subseteq \mathscr{H}_{\mathbf{i}'}^{\mathbf{j}'}$ iff $\mathbf{i} \leqslant \mathbf{i}'$ and $\mathbf{j} \leqslant \mathbf{j}'$.

Our second Lemma deals with clones of type (h) that contain the addition $\oplus$ of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ but do not extend $\mathsf{Pol}(Z_4; +)$.

Lemma 7. *There are exactly two clones with the congruence lattice of type* (h) *containing* $\oplus$ *but not the addition* $+$ *modulo* 4. *Both of them are Abelian.*

Proof. Let $\mathbf{A}$ be an algebra whose congruence lattice is of type (h) and have a binary polynomial $\oplus$ of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ given by the following table:

$x \oplus y$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

We have already noticed that all four polynomially rich clones with the congruence lattice of type (h) extending $\mathsf{Pol}(Z_4; +)$ contain the polynomial $2xy$ and therefore the polynomial $x \oplus y = x + y + 2xy$. Consequently, Lemma 5 gives that if the algebra $\mathbf{A}$ with the polynomial $\oplus$ is (hereditarily) polynomially rich then it also has addition $+$ modulo 4 as a polynomial. Therefore we may assume that $\mathbf{A}$ is not polynomially rich. In particular this gives that

Claim 1. $[\mathbf{1}, \mu] = \mathbf{0}$.

Now we will show that

Claim 2. *There is no unary polynomial* $\mathbf{f} \in \mathsf{Pol}(\mathbf{A})$ *with the range* $\{0,2\}$ *and* $\mathbf{f}(a)=0$ *for an odd number of elements* $a \in A$.

Proof. Suppose otherwise. Evaluating the polynomial $\mathbf{f}(x \oplus y)$ in the following way

$$\begin{array}{ll} \mathbf{f}(1)=\mathbf{f}(0\oplus 1) & \mathbf{f}(0\oplus 2)=\mathbf{f}(2) \\ \mathbf{f}(3)=\mathbf{f}(2\oplus 1) & \mathbf{f}(2\oplus 2)=\mathbf{f}(0) \end{array}$$

and using Claim 1 we get that if the two entries in one row are equal then also the entries in the other row coincide. This gives $\mathbf{f}(1)=\mathbf{f}(2)$ iff $\mathbf{f}(0)=\mathbf{f}(3)$. Analogously considering columns instead of rows we get that $\mathbf{f}(1)=\mathbf{f}(3)$ iff $\mathbf{f}(0)=\mathbf{f}(2)$. This suffices to establish Claim 2. □

Since $(0,2)$ gets collapsed whenever $(0,1)$ does, there is a unary polynomial $\mathbf{p} \in \mathsf{Pol}(\mathbf{A})$ with $(0,2)=(\mathbf{p}(0),\mathbf{p}(1))$. Since $\mathbf{p}(2) \stackrel{\mu}{\equiv} \mathbf{p}(0)$ and $\mathbf{p}(3) \stackrel{\mu}{\equiv} \mathbf{p}(1)$, the range of $\mathbf{p}$ is contained in $\{0,2\}$.

Using Claim 2 we know that $\mathbf{p}$ is either $\mathbf{p}_1$ or $\mathbf{p}_2$ from the following table:

x	$\mathbf{p}_1(x)$	$\mathbf{p}_2(x)$	$\mathbf{p}_3(x)$	$\mathbf{q}(x)$
0	0	0	0	0
1	2	2	0	1
2	0	2	2	0
3	2	0	2	1

Analogously, there is a unary polynomial $\mathbf{f} \in \mathsf{Pol}(\mathbf{A})$ that sends 0 to 0 and 3 to 2. Now since $\mathbf{p}_1(x)=\mathbf{f}(x \oplus \mathbf{p}_2(x))$ we get

Claim 3. $\mathbf{p}_1$ *is a polynomial of* $\mathbf{A}$.

From

$$\begin{aligned} \mathbf{p}_3(x) &= \mathbf{p}_1(x) \oplus \mathbf{p}_2(x), \\ \mathbf{q}(x) &= x \oplus \mathbf{p}_3(x), \\ \mathbf{p}_2(x) &= x \oplus \mathbf{p}_1(x) \oplus \mathbf{q}(x), \end{aligned}$$

we immediately get

Claim 4. *If one of* $\mathbf{p}_2, \mathbf{p}_3, \mathbf{q}$ *is a polynomial of* $\mathbf{A}$ *then all are.*

From Claim 1 we know that the monolith μ of $\mathbf{A}$ is Abelian. However the algebra $\mathbf{A}$ may not be Abelian. First we consider the case when $\mathbf{A}$ is Abelian. Then we will show that if $\mathbf{A}$ fails to be Abelian then it has the addition modulo 4 as a polynomial.

If $\mathbf{A}$ is Abelian then it is polynomially equivalent to a module over a finite ring. Consequently $\mathsf{Pol}(\mathbf{A})$ is generated by $\oplus$, the constants and the unary polynomials $\mathbf{f}$ of $\mathbf{A}$ with $\mathbf{f}(0)=0$.

The proof will split into 2 cases depending on whether $\mathbf{A}$ is $(\mathbf{0},\mu)$-minimal. First suppose that $\mathbf{A}$ is $(\mathbf{0},\mu)$-minimal. Then, by Lemma 4.30 of [4] $\mathbf{A}$ is also $(\mu,\mathbf{1})$-minimal. Consequently, every unary polynomial of $\mathbf{A}$ is either a permutation or collapses μ to $\mathbf{0}$ and has the range contained in a single μ-class. It is easy to see that the only such non-constant and non-permutational unary mapping $f:A\to A$ with $f(0)=0$ is the polynomial $\mathbf{p}_1$. On the other hand, if f is a permutation of A that preserves μ and such that $f(0)=0$ then f is either the identity function or $f(x)=x\oplus\mathbf{p}_1(x)$. Consequently $\mathsf{Pol}(\mathbf{A})=\langle\oplus,\mathbf{p}_1\rangle$, which means that there is only one Abelian clone on A which is $(\mathbf{0},\mu)$-minimal.

Now suppose that $\mathbf{A}$ is not $(\mathbf{0},\mu)$-minimal, i.e., that $U=\{0,2\}$ is a $(\mathbf{0},\mu)$-minimal set. Let $\mathbf{e}$ be a unary idempotent polynomial of $\mathbf{A}$ with range U. In view of Claim 2 it is either $\mathbf{p}_2$ or $\mathbf{p}_3$. Consequently $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{q}$ are polynomials of $\mathbf{A}$. We will show that $\mathsf{Pol}(\mathbf{A})=\langle\oplus,\mathbf{q}\rangle$. To see this it suffices to show that every map $f:A\to A$ preserving μ and such that $f(0)=0$ is in $\langle\oplus,\mathbf{q}\rangle$. Since $x=\mathbf{p}_3(x)\oplus\mathbf{q}(x)$ it suffices to show the last sentence under the assumption that the range of f is contained in either U or $\{0,1\}$. In view of Claim 2 there are exactly 3 nonconstant mappings $f:A\to\{0,2\}$ with $f(0)=0$, namely $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$, and all of them are in $\langle\oplus,\mathbf{q}\rangle$. On the other hand if $f:A\to\{0,1\}$ is a nonconstant mapping preserving μ and such that $f(0)=0$ then $f(2)=0$ and $f(1)=f(3)=1$, i.e., $f=\mathbf{q}$. This shows that $\mathsf{Pol}(\mathbf{A})=\langle\oplus,\mathbf{q}\rangle$, or in other words that there is exactly one Abelian clone on A which is not $(\mathbf{0},\mu)$-minimal.

We will show that if $\mathbf{A}$ is not Abelian then the binary mapping $2xy$ is a polynomial of $\mathbf{A}$ and therefore $\mathsf{Pol}(\mathbf{A})$ contains the addition modulo 4 as it can be defined by $x+y=x\oplus y\oplus 2xy$.

Since $\mathbf{A}$ fails to be Abelian there is a binary polynomial $\mathbf{s}(x,y)$ and $a,b,c,d\in A$ with $\mathbf{s}(a,c)=\mathbf{s}(a,d)$ and $\mathbf{s}(b,c)\neq\mathbf{s}(b,d)$. Obviously, $\mathbf{A}$ has unary polynomials $\mathbf{f}_1,\mathbf{f}_2$ that map the pair $(0,1)$ to (a,b) and (c,d), respectively. Replacing $\mathbf{s}(x,y)$ by $\mathbf{s}(\mathbf{f}_1(x),\mathbf{f}_2(y))$ we get that

$$\mathbf{s}(0,0)=\mathbf{s}(0,1),$$
$$\mathbf{s}(1,0)\neq\mathbf{s}(1,1).$$

Replacing $\mathbf{s}(x,y)$ by $\mathbf{s}(x,y)\oplus\mathbf{s}(x,0)$ we get that $0=\mathbf{s}(0,0)=\mathbf{s}(0,1)=\mathbf{s}(1,0)\neq\mathbf{s}(1,1)$. If, for this new $\mathbf{s}$ we have $\mathbf{s}(1,1)\in\{1,3\}$ then we once more modify $\mathbf{s}$ by replacing it with $\mathbf{p}_1\mathbf{s}(x,y)$. Then we have

$$0=\mathbf{s}(0,0)=\mathbf{s}(0,1),$$
$$0=\mathbf{s}(1,0)\neq\mathbf{s}(1,1)=2.$$

Since $\mathbf{s}(x,y)$ is μ related to $\mathbf{s}(a,b)$ for some $a,b\in\{0,1\}$ we know that the range of $\mathbf{s}$ is $\{0,2\}$. Now we look at the polynomials $\mathbf{s}(x,1)$ and $\mathbf{s}(1,x)$.

If one of them sends 2 to 2 then, by Claim 2, it must be $\mathbf{p}_2$, so that Claim 4 gives that $\mathbf{q}$ is a polynomial of $\mathbf{A}$. But then the polynomial $\mathbf{s}(\mathbf{q}(x),\mathbf{q}(y))$ is nothing else but $2xy$.

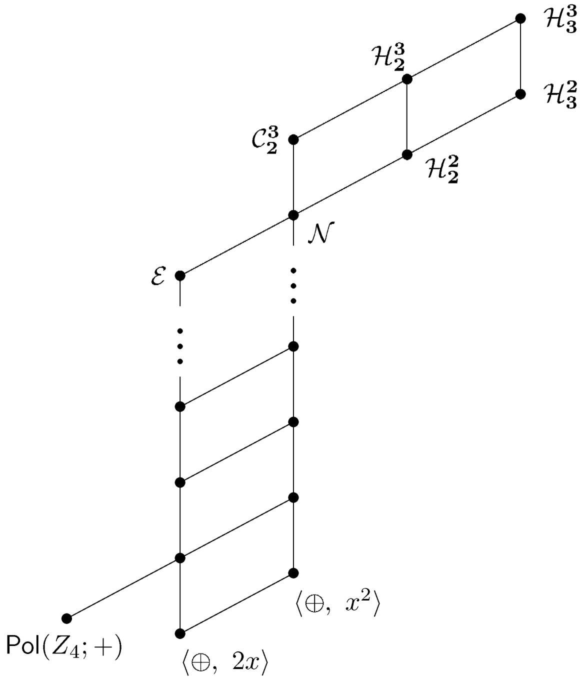

Fig. 3. Clones with the congruence lattice of type (h).

Now suppose that $\mathbf{s}(2,1)=0=\mathbf{s}(1,2)$ so that $\mathbf{s}(x,1)=\mathbf{p}_1(x)=\mathbf{s}(1,x)$. In particular $\mathbf{s}(3,1)=2=\mathbf{s}(1,3)$. Therefore Claim 1 gives $\mathbf{s}(2,0)=0$ and $\mathbf{s}(3,0)=0$. Analogously $\mathbf{s}(0,2)=0=\mathbf{s}(0,3)$.

Since μ is Abelian and $\mathbf{s}(0,0)=\mathbf{s}(0,2)$ then $0=\mathbf{s}(2,0)=\mathbf{s}(2,2)$. Analogously we get $\mathbf{s}(3,3)=2$. The remaining two entries in the table of $\mathbf{s}$ can be easily filled by 0 using Claim 1. Consequently, we see that $2xy=\mathbf{s}(x,y)$ is a polynomial of $\mathbf{A}$, as required. □

Summing up we know that there are exactly four polynomial clones of type (h) that are not described in Lemma 6, i.e., in Fig. 1. These are the Abelian $(\mathbf{0},\mu)$-minimal clone $\langle\oplus,2x\rangle$, the Abelian not $(\mathbf{0},\mu)$-minimal clone $\langle\oplus,x^2\rangle$, and two polynomially rich clones $\mathscr{H}_{\mathbf{3}}^{\mathbf{2}}$ and $\mathscr{H}_{\mathbf{3}}^{\mathbf{3}}$, where $\mathbf{i}$ in $\mathscr{H}_{\mathbf{i}}^{\mathbf{j}}$ shows the type $\mathsf{typ}(\mathbf{0},\mu)$ and $\mathbf{j}=\mathsf{typ}(\mu,\mathbf{1})$.

We have already noticed that the clones $\mathscr{H}_{\mathbf{2}}^{\mathbf{2}}=\langle +,x(y^2-y)\rangle$ as well as $\mathscr{H}_{\mathbf{2}}^{\mathbf{3}}=\mathsf{Pol}(Z_4;+,\cdot)$ are polynomially rich; the largest nilpotent clone is $\mathscr{N}=\langle +,2x_1x_2, 2x_1x_2x_3,\ldots,x^2\rangle$ while the largest E-minimal clone is $\mathscr{E}=\langle +,2x_1x_2,2x_1x_2x_3,\ldots\rangle$. Moreover $\mathscr{C}_{\mathbf{2}}^{\mathbf{3}}=\langle +,x^2y^2\rangle$ is the unique non-solvable clone with an Abelian monolith satisfying $[\mathbf{1},\mu]=\mathbf{0}$.

Fig. 3 shows the complete picture of the clones of type (h).

Case 3.5. *There are countably many Mal'tsev polynomial clones on A whose congruence lattice is of type* (i).

Proof. Let $\mathbf{A}$ be an algebra whose congruence lattice of type (i). Denote by μ the monolith of $\mathbf{A}$ and by α its unique cover.

It can be easily seen that if v denotes the monolith of a non-trivial quotient $\mathbf{D}$ of $\mathbf{A}$ then $(\mathbf{0}, v)$-minimal sets of $\mathbf{D}$ have exactly 2 elements, so that condition (GFp) holds trivially for $\mathbf{A}$. Consequently, hereditarily polynomially richness of $\mathbf{A}$ reduces to the commutator condition (SC1). On the other hand, there are exactly, 3 prime quotients in $\mathsf{Con}(\mathbf{A})$ and $\mathsf{typ}\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}\}$. Therefore, in this case, for every fixed Mal'tsev polynomial $\mathbf{d}$ there are at most 8 hereditarily polynomially rich clones containing $\mathbf{d}$.

In the rest of the proof we assume that $\mathbf{A}$ is not hereditarily polynomially rich. Since $\mathbf{A}/\mu$ is a 3 element algebra with a Mal'tsev polynomial it must be hereditarily polynomially rich. In particular $[\mathbf{1}, \alpha] = \alpha$. Moreover $\mathbf{d}(0,0,0) \neq \mathbf{d}(1,0,0)$, while $\mathbf{d}(1,0,2) \stackrel{\mu}{\equiv} \mathbf{d}(0,0,2) = 2$, i.e., $\mathbf{d}(0,0,2) = \mathbf{d}(1,0,2)$ which gives that $[\mathbf{1}, \mu] = \mu$. Since $\mathbf{A}/\mu$ is hereditarily polynomially rich and $\mathbf{A}$ is not then $[\alpha, \mu] = \mathbf{0}$. Consequently α is the centralizer of μ.

Since $U = \{0, 1\}$ is the only non-trivial class of μ it is the $(\mathbf{0}, \mu)$-minimal set (and in fact a trace). Thus $\mathbf{A}|_U$ is (polynomially equivalent to) the group $\mathbf{Z}_2 = (U; +)$. Moreover there is a unary idempotent polynomial $\mathbf{e}_0(x)$ of $\mathbf{A}$ with range U. One easily checks that the polynomials

$$\mathbf{e}'(x) = \mathbf{d}(x, \mathbf{e}_0(x), 0),$$
$$\mathbf{e}(x) = \mathbf{d}(x, \mathbf{e}'(x), 0),$$
$$\mathbf{e}''(x) = \mathbf{d}(0, x, \mathbf{d}(x, \mathbf{e}_0(x), 1))$$

take the following values:

x	$\mathbf{e}(x)$	$\mathbf{e}'(x)$	$\mathbf{e}''(x)$
0	0	0	1
1	1	0	0
2	0	2	0
3	0	3	0

In particular we have $x = \mathbf{d}(\mathbf{e}(x), 0, \mathbf{e}'(x))$. Consequently for any operation $f : A^n \to A$ and $\bar{x} \in A^n$, we have $f(\bar{x}) = \mathbf{d}(\mathbf{e}(f(\bar{x})), 0, \mathbf{e}'(f(\bar{x})))$. Since the range of $\mathbf{e}$ is U and the range of $\mathbf{e}'$ is $\{0, 2, 3\}$, the following claim shows that the understanding of the clone $\mathsf{Pol}(\mathbf{A})$ can be reduced to the understanding of its operations that whose range is contained in U:

Claim 1. *Every* $\mathbf{A}$*-admissible mapping* $f : A^n \to \{0, 2, 3\}$ *is a polynomial of* $\mathbf{A}$.

Proof. For an operation $f : A^n \to A$ preserving congruences of $\mathbf{A}$ and the labels $\mathsf{typ}(\mu, \alpha)$ and $\mathsf{typ}(\alpha, \mathbf{1})$ the mapping

$$f/\mu : (A/\mu)^n \ni (x_1/\mu, \ldots, x_n/\mu) \mapsto f(x_1, \ldots, x_n)/\mu \in A/\mu$$

is well defined and $(\mathbf{A}/\mu)$-admissible. Since $\mathbf{A}/\mu$ is polynomially rich, $f/\mu \in \mathsf{Pol}(\mathbf{A}/\mu)$.

This means that there is a polynomial $\mathbf{q}$ of $\mathbf{A}$ with $f(x_1,\ldots,x_n) \overset{\mu}{\equiv} \mathbf{q}(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in A$. Consequently,

$$f(x_1,\ldots,x_n) = \mathbf{e}' f(x_1,\ldots,x_n) \overset{\mu}{\equiv} \mathbf{e}'\mathbf{q}(x_1,\ldots,x_n)$$

which together with the fact that $\mu|_{\{0,2,3]\}} = \mathbf{0}$ gives that $f(x_1,\ldots,x_n) = \mathbf{e}'\mathbf{q}(x_1,\ldots,x_n)$, i.e., f is a polynomial of $\mathbf{A}$. □

From Claim 1 it follows that if $\mathbf{B}$ is the unique polynomially rich extension of $\mathbf{A}$ then both $\mathbf{A}$ and $\mathbf{B}$ have the same polynomials with ranges contained in $\{0,2,3\}$. In particular, there are only finitely many possible sets of mappings that can serve as the set of all polynomials of $\mathbf{A}$ with the range contained in $\{0,2,3\}$.

Claim 2. *The mappings* $\mathbf{w},\mathbf{p} : A \to U$ *given by the table*

x	$\mathbf{w}(x)$	$\mathbf{p}(x)$
0	0	0
1	0	1
2	0	0
3	1	1

are polynomials of $\mathbf{A}$.

Proof. Since $(0,1)$ lies in the congruence α generated by $(2,3)$ and $\mathbf{A}$ is Mal'tsev there is a unary polynomial $\mathbf{w}'$ of $\mathbf{A}$ that takes 2 to 0 and 3 to 1. By replacing $\mathbf{w}'$ by $\mathbf{e}\mathbf{w}'$ we may assume that the range of $\mathbf{w}'$ is U. Now one easily checks that the polynomials

$$\mathbf{w}(x) = \begin{cases} \mathbf{w}'(x), & \text{if } \mathbf{w}'(0)=0 \text{ and } \mathbf{w}'(1)=0, \\ \mathbf{w}'(x)+\mathbf{e}''(x), & \text{if } \mathbf{w}'(0)=1 \text{ and } \mathbf{w}'(1)=0, \\ \mathbf{w}'(x)+\mathbf{e}(x), & \text{if } \mathbf{w}'(0)=0 \text{ and } \mathbf{w}'(1)=1, \\ \mathbf{w}'(x)+\mathbf{e}''(x)+\mathbf{e}(x), & \text{if } \mathbf{w}'(0)=1 \text{ and } \mathbf{w}'(1)=1, \end{cases}$$

and $\mathbf{p}(x) = \mathbf{w}(x) + \mathbf{e}(x)$ witness our claim. □

Now for every positive integer n define $w_n : A^n \to U$ by

$$w_n(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } x_1 = \cdots = x_n = 3, \\ 0 & \text{otherwise,} \end{cases}$$

or in other words

$$w_n(x_1,\ldots,x_n) = \begin{cases} \mathbf{p}(x_1)\cdot\ldots\cdot\mathbf{p}(x_n) & \text{if } x_1,\ldots,x_n \in \{2,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

where the multiplication in the last display is taken in the field $\mathbf{GF}(2) = (U;+,\cdot)$. Note that $w_1 = \mathbf{w} \in \mathsf{Pol}(\mathbf{A})$.

A triple $(d;\mathbf{i},\mathbf{j})$, where d is a Mal'tsev operation on A, $\mathbf{i},\mathbf{j}\in\{\mathbf{2},\mathbf{3}\}$, is *admissible* if there is an algebra $\mathbf{B}[d;\mathbf{i},\mathbf{j}]$ with the same universe and congruences as $\mathbf{A}$ and with $\mathsf{typ}(\mu,\alpha)=\mathbf{i}$ and $\mathsf{typ}(\alpha,\mathbf{1})=\mathbf{j}$.

In particular, we know that if $\mathbf{d}$ is a Mal'tsev polynomial of $\mathbf{A}$ then every triple of the form $(\mathbf{d};\mathsf{typ}_{\mathbf{A}}(\mu,\alpha),\mathsf{typ}_{\mathbf{A}}(\alpha,\mathbf{1}))$ is admissible.

For every admissible triple $(d;\mathbf{i},\mathbf{j})$ we pick and fix an algebra $\mathbf{B}[d;\mathbf{i},\mathbf{j}]$ that witnesses this fact. Since $\mathbf{B}[d;\mathbf{i},\mathbf{j}]/\mu$ is a 3-element algebra then from [6] it follows that it is polynomially rich. This gives that the polynomials of $\mathbf{B}[d;\mathbf{i},\mathbf{j}]$ are uniquely determined modulo μ, i.e., if $\mathbf{B}'[d;\mathbf{i},\mathbf{j}]$ is another algebra witnessing admissibility of $(d;\mathbf{i},\mathbf{j})$ then for any $\mathbf{f}\in\mathsf{Pol}(\mathbf{B}[d;\mathbf{i},\mathbf{j}])$ there is $\mathbf{g}\in\mathsf{Pol}(\mathbf{B})'[d;\mathbf{i},\mathbf{j}]$ such that $\mathbf{f}(\bar{x})\overset{\mu}{\equiv}\mathbf{g}(\bar{x})$ for all tuples $\bar{x}$. In particular, the algebras $\mathbf{B}[d;\mathbf{i},\mathbf{j}]$ and $\mathbf{B}'[d;\mathbf{i},\mathbf{j}]$ have the same polynomials with ranges contained in $\{0,2,3\}$. Let $\mathscr{B}[d;\mathbf{i},\mathbf{j}]=\{f\in\mathsf{Pol}(\mathbf{B})[d;\mathbf{i},\mathbf{j}]:\mathsf{rg}(f)\subseteq\{0,2,3\}\}$. Now for any $n=1,2,3,\ldots,\omega$ we define a clone $\mathscr{W}_n[d;\mathbf{i},\mathbf{j}]$ by putting

$$\mathscr{W}_n[d;\mathbf{i},\mathbf{j}]=\langle\{w_n,\mathbf{d},\mathbf{e},\mathbf{p}\}\cup\mathscr{B}[d;\mathbf{i},\mathbf{j}]\rangle,$$

$$\mathscr{W}_\omega[d;\mathbf{i},\mathbf{j}]=\bigcup_{n\geqslant 1}\mathscr{W}_n[d;\mathbf{i},\mathbf{j}].$$

Since $w_n(x_1,\ldots,x_n)=w_{n+1}(x_1,\ldots,x_n,x_n)$ then

$$\mathscr{W}_1[d;\mathbf{i},\mathbf{j}]\subseteq\mathscr{W}_2[d;\mathbf{i},\mathbf{j}]\subseteq\cdots\subseteq\mathscr{W}_\omega[d;\mathbf{i},\mathbf{j}].$$

Moreover, if $\mathbf{d}$ is a Mal'tsev polynomial of $\mathbf{A}$ then

$$\mathscr{W}_1[\mathbf{d};\mathsf{typ}_{\mathbf{A}}(\mu,\alpha),\mathsf{typ}_{\mathbf{A}}(\alpha,\mathbf{1})]\subseteq\mathsf{Pol}(\mathbf{A}).$$

We will show that in fact $\mathsf{Pol}(\mathbf{A})=\mathscr{W}_n[\mathbf{d};\mathsf{typ}_{\mathbf{A}}(\mu,\alpha),\mathsf{typ}_{\mathbf{A}}(\alpha,\mathbf{1})]$ for some $n=1,2,\ldots,\omega$.

Claim 3. *For every $\in A$ there is a binary polynomial $\mathbf{s}_d(x,y)$ of $\mathbf{A}$ with the range contained in U and such that*

- $\mathbf{s}_d(x,0)=0$ *for all* $x\in A$,
- $\mathbf{s}_d(x,u)=u$ *for all* $x\in d/\alpha$ *and* $u\in U$,
- $\mathbf{s}_d(x,u)=0$ *for all* $x\notin d/\alpha$ *and* $u\in U$.

Proof. From $[\mathbf{1},\mu]\neq\mathbf{0}$ we get that $[\mathrm{Cg}^{\mathbf{A}}(a,d),\mathrm{Cg}^{\mathbf{A}}(0,1)]\neq\mathbf{0}$ whenever $a\in A-d/\alpha$. Thus there is a binary polynomial $\mathbf{t}_{ad}(x,y)$ of $\mathbf{A}$ such that

$$\mathbf{t}_{ad}(a,0)=\mathbf{t}_{ad}(a,1),$$

$$\mathbf{t}_{ad}(d,0)\neq\mathbf{t}_{ad}(d,1).$$

Since $(\mathbf{t}_{ad}(d,0),\mathbf{t}_{ad}(d,1))\in\mu-\mathbf{0}$ then we can additionally assume that the range of $\mathbf{t}_{ad}$ is contained in U, or replace $\mathbf{t}_{ad}(x,y)$ by $\mathbf{e}(\mathbf{t}_{ad}(x,y))$.

Moreover, replacing $\mathbf{t}_{ad}(x,y)$ by $\mathbf{t}_{ad}(x,y)-\mathbf{t}_{ad}(x,0)$ we may additionally arrange that $\mathbf{t}_{ad}(x,0)=0$ for all $x\in A$.

For every $x\in A$ the mapping

$$U\ni u\mapsto\mathbf{t}_{ad}(x,u)\in U$$

is either constant or a permutation of U. Thus, iterating in the second variable, we may assume that every such mapping is either constant or the identity on U.

Up to now we constructed a family $\{\mathbf{t}_{ad} : a \notin d/\alpha\}$ of binary polynomials of $\mathbf{A}$ such that

$$\mathbf{t}_{ad}(a,u) = 0 \quad \text{for all } u \in U,$$
$$\mathbf{t}_{ad}(d,u) = u \quad \text{for all } u \in U,$$
$$\mathbf{t}_{ad}(x,0) = 0 \quad \text{for all } x \in A.$$

Using our assumptions that $[\alpha,\mu] = \mathbf{0}$ and $U = 0/\mu$, the first two of the above properties of the $\mathbf{t}_{ad}$'s can be strengthened to

$$\mathbf{t}_{ad}(x,u) = 0 \quad \text{for all } x \in a/\alpha \text{ and } u \in U,$$
$$\mathbf{t}_{ad}(x,u) = u \quad \text{for all } x \in d/\alpha \text{ and } u \in U.$$

Therefore if $a \in A - d/\alpha$ then the polynomial

$$\mathbf{s}_d(x,y) = \mathbf{t}_{ad}(x,y)$$

witnesses Claim 3. □

Claim 4. *For any sequence $B_1,\ldots,B_n \in A/\alpha$ there is an $(n+1)$-ary polynomial $\mathbf{s}_{B_1,\ldots,B_n}(x_1,\ldots,x_n,y)$ of $\mathbf{A}$ such that for any $u \in U$ we have*

$$\mathbf{s}_{B_1,\ldots,B_n}(x_1,\ldots,x_n,u) = \begin{cases} u & \text{if } (x_1,\ldots,x_n) \in B_1 \times \cdots \times B_n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Pick $(b_1,\ldots,b_n) \in B_1 \times \cdots \times B_n$ and put $\mathbf{s}_{B_1,\ldots,B_n}(x_1,\ldots,x_n,y) = \mathbf{s}_{b_1}(x_1,\mathbf{s}_{b_2}(x_2,\ldots,\mathbf{s}_{b_n}(x_n,y)\ldots))$. □

By a block we mean a direct product of α-cosets. Now for any n-ary polynomial f of $\mathbf{A}$ with range contained in U, and any block $B = B_1 \times \cdots \times B_n$ observe that the mapping $f_B(\bar{x}) = \mathbf{s}_{B_1,\ldots,B_n}(\bar{x}, f(\bar{x}))$ is a polynomial of $\mathbf{A}$. Since

$$f(\bar{x}) = \sum_{B \text{ is a block}} f_B(\bar{x}),$$

then $\mathsf{Pol}(\mathbf{A})$ is in fact determined by its polynomials with the one-block property, i.e., those polynomials $f : A^n \to U$ for which $f^{-1}(1)$ is contained in a single block.

Note that if $f^{-1}(1)$ is contained in the block B then for any $\bar{a} \in U^n$ there is a unique $\bar{b} \in B$ with $\mathbf{p}(b_i) = a_i$ for all $i = 1,\ldots,n$. Consequently, we may define $f' : U^n \to U$ by putting $f'(\bar{a}) = f(\bar{b})$, where $\bar{b}$ is chosen as above. Obviously f' is a polynomial of the field $\mathbf{GF}(2)$, i.e., it can be represented as

$$f'(\bar{x}) = \sum_{\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in \{0,1\}^n} f_\varepsilon \cdot x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$$

for an appropriate choice of the f_ε's in $\{0,1\}$. (Here $x^0 = 1$ and $x^1 = x$.)

Then obviously

$$f(\bar{x}) = \sum_{\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n} f_\varepsilon \cdot \mathbf{p}(x_1)^{\varepsilon_1} \cdot \ldots \cdot \mathbf{p}(x_n)^{\varepsilon_n}.$$

Each of the summands of the form $f_\varepsilon \cdot \mathbf{p}(x_1)^{\varepsilon_1} \cdot \ldots \cdot \mathbf{p}(x_n)^{\varepsilon_n}$ in the above display with $f_\varepsilon \neq 0$ is called a monomial of f. Obviously, f lies in the clone determined by all of its monomials. But we also have the following converse.

Claim 5. *Let $f : A^n \to U$ be a polynomial of* $\mathbf{A}$ *with one-block property. Then all of the monomials of f are in the clone determined by f.*

Proof. To prove this claim we induct on the rank of f, i.e., the cardinality of the set $\{\varepsilon : f_\varepsilon \neq 0\}$. Pick an ε with $f_\varepsilon \neq 0$ and such that $\varepsilon^{-1}(1) = \{i : \varepsilon_i = 1\}$ has minimal cardinality. Let $B = B_1 \times \cdots \times B_n$ be the block containing $f^{-1}(1)$. For each $i = 1,\ldots,n$ let c_i denote the unique element in the set $B_i \cap \{0,2\}$. Define $\bar{f_\varepsilon}(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$, where

$$y_i = \begin{cases} x_i & \text{if } \varepsilon_i = 1, \\ c_i & \text{if } \varepsilon_i = 0. \end{cases}$$

Obviously $\bar{f_\varepsilon}$ depends only on those variables x_i for which $\varepsilon_i = 1$. Moreover by our minimality condition on $|\varepsilon^{-1}(1)|$ we get that in fact $\bar{f_\varepsilon}(x_1,\ldots,x_n) = f_\varepsilon \cdot \mathbf{p}(x_1)^{\varepsilon_1} \cdot \ldots \cdot \mathbf{p}(x_n)^{\varepsilon_n}$. Consequently, this 'minimal' monomial of f as well as $f - \bar{f_\varepsilon}$ are in the clone determined by f. However $f - \bar{f_\varepsilon}$ has the one-block property and has a smaller rank than f. Thus, by the induction hypothesis, all of the monomials of f are in the clone determined by f. □

Claim 6. *Let for the polynomial $f : A^n \to U$ of* $\mathbf{A}$ *the set $f^{-1}(1)$ be contained in the block $B = B_1 \times \cdots \times B_n$. Then for every monomial $f_\varepsilon \cdot \mathbf{p}(x_1)^{\varepsilon_1} \cdot \ldots \cdot \mathbf{p}(x_n)^{\varepsilon_n}$ of f, the set $\varepsilon^{-1}(1)$ either has at most one element or is contained in $\{i : B_i = \{2,3\}\}$.*

Proof. Suppose otherwise, i.e., that there are $k \neq l$ with $\varepsilon_k = 1 = \varepsilon_l$ and $B_k = U$. Consider a binary polynomial $g(x_k, x_l)$ obtained from the monomial $f_\varepsilon \cdot \mathbf{p}(x_1)^{\varepsilon_1} \cdot \ldots \cdot \mathbf{p}(x_n)^{\varepsilon_n}$ by fixing the variables x_i (for $i \neq k, l$) by either the constant 1 or 3 depending on whether $B_i = U$ or not. Now if $B_l = \{c,d\}$ then $g(0,c) = 0 = g(0,d)$ while $g(1,c)\mathbf{p}(c) \neq \mathbf{p}(d) = g(1,d)$. This violates the condition $[\mu,\mu] = \mathbf{0}$ if $B_l = U$ or the condition $[\mu,\alpha] = \mathbf{0}$ if $B_l = \{2,3\}$. □

From the above claim we know that all of the non-constant monomials of the polynomial $f : A^n \to U$ with the one block property are either of the form $w_k(x_{i_1},\ldots,x_{i_k})$ or $\mathbf{s}_B(x_i, \mathbf{p}(x_i))$ and therefore are in the clone $\mathscr{W}_m[\mathbf{d}; \mathsf{typ}(\mu,\alpha), \mathsf{typ}(\alpha,\mathbf{1})]$, where $m = \deg(f)$ is the degree of the polynomial f, i.e., the maximal number of the form $|\varepsilon^{-1}(1)|$ with $f_\varepsilon = 1$. Consequently we have the following:

Claim 7. *Suppose that $m \in \{1,2,\ldots,\omega\}$ is the least upper bound for the degrees $\deg(f_B)$ with f ranging over all polynomials of* $\mathbf{A}$ *with the range contained in U and*

B ranging over all possible blocks (of the arity of f). Then $\mathsf{Pol}(\mathbf{A}) = \mathscr{W}_m[\mathbf{d}; \mathsf{typ}(\mu, \alpha), \mathsf{typ}(\alpha, \mathbf{1})]$.

Proof. From what has been already said we get that the clone $\mathsf{Pol}(\mathbf{A})$ is contained in the other one. To see the converse recall that

$$\mathscr{W}_1[\mathbf{d}; \mathsf{typ}(\mu, \alpha), \mathsf{typ}(\alpha, \mathbf{1})] \subseteq \mathsf{Pol}(\mathbf{A}).$$

Thus we may suppose that $k = \mathsf{deg}(f_B) \geqslant 2$ for some polynomial f of $\mathbf{A}$ with the range contained in U and some block B. Picking a monomial of f with the degree k and applying Claim 6 we see that this monomial is w_k. Now, by Claim 5, $w_k \in \mathsf{Pol}(\mathbf{A})$ and we are done. □

This finishes the proof of Case (i). □

To complete our description of clones we will show that

Claim 8. $\mathscr{W}_1[d; \mathbf{2}, \mathbf{j}] \subseteq \mathscr{W}_2[d; \mathbf{2}, \mathbf{j}] \subset \mathscr{W}_3[d; \mathbf{2}, \mathbf{j}] \subset \cdots \subset \mathscr{W}_\omega[d; \mathbf{2}, \mathbf{j}]$.

Claim 9. $\mathscr{W}_1[d; \mathbf{3}, \mathbf{j}] = \mathscr{W}_2[d; \mathbf{3}, \mathbf{j}] = \mathscr{W}_3[d; \mathbf{3}, \mathbf{j}] = \cdots = \mathscr{W}_\omega[d; \mathbf{3}, \mathbf{j}]$.

Claim 10. $\mathscr{W}_\omega[d; \mathbf{2}, \mathbf{j}] \subset \mathscr{W}_\omega[d; \mathbf{3}, \mathbf{j}]$.

Proof. Let $\mathbf{A}$ be an algebra with $\mathsf{Pol}(\mathbf{A}) = \mathscr{W}_1[d; \mathbf{i}, \mathbf{j}]$ and $[\alpha, \mu] = \mathbf{0}$. In particular, we know that $\mathsf{typ}(\mathbf{0}, \mu) = \mathbf{2}$ and that $\mathbf{e}, \mathbf{w}, \mathbf{p}$ are polynomials of $\mathbf{A}$. Therefore $U = \{0, 1\} = \mathbf{e}(A)$ is a $(\mathbf{0}, \mu)$-minimal set. Moreover, since μ is Abelian, the induced algebra $\mathbf{A}|_U$ is polynomially equivalent to a vector space $(U; +, 0)$ over the two element field.

Since the pair $(2, 0)$ gets collapsed whenever $(0, 3)$ does, there is a unary polynomial $\mathbf{q}$ of $\mathbf{A}$ that maps $(0, 3)$ onto $(2, 0)$. Now the polynomial $\mathbf{r}(x) = d(\mathbf{q}(x), 0, \mathbf{e}'(x))$ is idempotent and sends everything, but 3, to 2. In particular, $V = \{2, 3\} = \mathbf{r}(A)$ is the unique (μ, α)-minimal set.

To see Claim 8 we assume that $n \geqslant 2$ and $\mathbf{i} = \mathbf{2}$. This gives that the induced algebra $\mathbf{A}|_V$ is polynomially equivalent to a vector space $\mathbf{V} = (V; \oplus, 3)$ over the two element field. We may assume that 3 is the zero element of this vector space, i.e., $x \oplus x = 3$.

To show that w_{n+1} is not in the clone $\mathscr{W}_n[d; \mathbf{2}, \mathbf{j}]$ we will construct a subuniverse D_n of $\mathbf{A}^{V^{n+1}}$ that is closed under the operations that were used to generate the clone $\mathscr{W}_n[d; \mathbf{2}, \mathbf{j}]$ and such that D_n is not closed under w_{n+1}. The set D_n will contain all four constant functions $\hat{0}, \hat{1}, \hat{2}, \hat{3} : V^{n+1} \to A$.

Let H_n be the set of all linear mappings from $\mathbf{V}^{n+1}$ to $\mathbf{V}$. Put $H_n' = H_n \cup \{\xi \oplus \hat{2} : \xi \in H_n\}$. In particular H_n' is closed under $\oplus$. Let

$$D_n = \left\{ \xi \in U^{V^{n+1}} : \sum_{v \in V^{n+1}} \xi(v) = 0 \right\} \cup H_n' \subseteq A^{V^{n+1}}.$$

The presence of the constant functions $\hat{0}, \hat{1}, \hat{2}, \hat{3}$ in D_n implies that all polynomials of $\mathbf{A}$ can be extended to polynomials of the subalgebra $\mathbf{E}_n$ of $\mathbf{A}^{V^{n+1}}$ generated by D_n. We are going to show that D_n is closed under the operations of $\mathbf{A}$, or in other words that $E_n = D_n$.

All of the generators of $\mathbf{E}_n$ are α-constant so that all of the elements in $\mathbf{E}_n$ are so. Therefore $E_n \subseteq U^{V^{V^{n+1}}} \cup V^{V^{n+1}}$. Moreover Lemma 6.14 of [4] guarantees that

$$E_n \cap V^{V^{n+1}} = H_n' \subseteq D_n.$$

It remains to show that if $\mathbf{f}$ is a basic operation of $\mathbf{A}$ and $\zeta = \mathbf{f}(\zeta_1,\ldots,\zeta_k) \in U^{V^{n+1}}$ for some $\zeta_1,\ldots,\zeta_k \in D_n$ then $\sum_{v\in V^{n+1}} \zeta(v) = 0$. First assume that $\mathbf{f} \in \mathscr{B}[d,\mathbf{2},\mathbf{j}]$. Then $\zeta \in U^{V^{n+1}}$ gives $\zeta = \hat{0}$ and we are done. Now if $\mathbf{f} = \mathbf{e}$ then the condition $\mathbf{e}(\zeta)^{-1}(1) = \zeta^{-1}(1)$ easily does the job. For $\mathbf{f} = \mathbf{w}$ we have $\mathbf{w}(\zeta)^{-1}(1) = \zeta^{-1}(3)$, and the latter set is either empty (if $\zeta \in U^{V^{n+1}} \cup \{\hat{2}\}$), or is V^{n+1} (if $\zeta = \hat{3}$), or $\zeta(v) = 3$ for exactly 2^n vectors $v \in V^{n+1}$. Before proceeding with $\mathbf{f}$ being the Mal'tsev operation d we will show that D_n is closed under w_n (and therefore under all of the w_k's with $k \leqslant n$). Let $\zeta_1,\ldots,\zeta_n \in D_n$. Remind that $\mathbf{r}$ is the idempotent unary polynomial of $\mathbf{A}$ that sends everything but 3 to 2. Since $w_n(x_1,\ldots,x_n) = w_n(\mathbf{r}(x_1),\ldots,\mathbf{r}(x_n))$ we may assume that $\zeta_1,\ldots,\zeta_n \in H_n'$. Obviously

$$(w_n(\zeta_1,\ldots,\zeta_n))^{-1}(1) = \zeta_1^{-1}(3) \cap \cdots \cap \zeta_n^{-1}(3).$$

Moreover $\zeta_i^{-1}(3)$ is a coset of a linear subspace of V^{n+1} of codimension at most 1. Consequently the intersection in the last display is a coset of a linear subspace of V^{n+1} of dimension $k \geqslant 1$. Therefore $w_n(\zeta_1,\ldots,\zeta_n)$ takes the value 1 exactly 2^k times and therefore it belongs to D_n, as required.

Finally, let $\mathbf{f}$ be the Mal'tsev operation d. Thus $\zeta = d(\zeta_1,\zeta_2,\zeta_3)$ for $\zeta_1,\zeta_2,\zeta_3 \in D_n$. Now if $B = B_1 \times B_2 \times B_3$ is a block containing $(\zeta_1,\zeta_2,\zeta_3)$ then $\zeta = \mathbf{s}_{B_1,B_2,B_3}(\zeta_1,\zeta_2,\zeta_3,\mathbf{e}d(\zeta_1,\zeta_2,\zeta_3))$, where $\mathbf{s}_{B_1,B_2,B_3}$ is a polynomial of $\mathbf{A}$ supplied by Claim 4. The polynomial $d_B(x,y,z) = \mathbf{s}_{B_1,B_2,B_3}(x,y,z,\mathbf{e}d(x,y,z))$ has the one-block property, actually $(\mathbf{e}d)^{-1}(1) \subseteq B$. Since $d(V,V,V) \subseteq V$ we get that the degree of d_B is at most 2. Thus we can repeat the argument for w_n to show that ζ is in H_n'.

Up to now we have shown that D_n is a sub-universe of $\mathbf{A}^{V^{n+1}}$ that is closed under w_n. To see that D_n is not closed under w_{n+1} pick a basis $\{v_1,\ldots,v_{n+1}\}$ of V^{n+1} and define $\eta_i \in H_n$ by putting

$$\eta_i(v_j) = \begin{cases} 2 & \text{if } j = i, \\ 3 & \text{otherwise.} \end{cases}$$

One easily checks that $(w_{n+1}(\eta_1,\ldots,\eta_{n+1}))^{-1}(1)$ is the trivial subspace of V^{n+1} so that the value 1 is taken exactly once. Thus $w_{n+1}(\eta_1,\ldots,\eta_{n+1})$ does not belong to D_n.

This proves Claim 8. From the careful analysis of the above proof we know that $\mathscr{W}_1[d;\mathbf{2},\mathbf{j}] \neq \mathscr{W}_2[d;\mathbf{2},\mathbf{j}]$ if and only if the degree of the polynomials of the form d_B is at most 1. But this is equivalent to saying that the clone $\mathscr{W}_1[d;\mathbf{2},\mathbf{j}]$ has the Abelian congruence α.

Now we prove Claim 9. We know that there is an idempotent polynomial $\mathbf{r}$ that sends everything but 3 to 2. The range $V = \{2,3\}$ of this polynomial is the unique (μ,α)-minimal set. Since $\mathsf{typ}(\mu,\alpha) = \mathbf{3}$ we get that the induced polynomial structure on $\mathbf{A}|_V$ is Boolean. In particular there is a binary polynomial $\mathbf{m} \in \mathscr{W}_1[d;\mathbf{3},\mathbf{j}]$ with $\mathbf{m}(2,2) = \mathbf{m}(2,3) = \mathbf{m}(3,2) = 2$ and $\mathbf{m}(3,3) = 3$. Moreover

$$w_n(x_1,\ldots x_n) = \mathbf{w}(\mathbf{m})(\ldots(\mathbf{m}(\mathbf{m}(\mathbf{r}(x_1),\mathbf{r}(x_2)),\mathbf{r}(x_3))\ldots\mathbf{r}(x_n))),$$

so that $w_n \in \mathscr{W}_1[d;\mathbf{3},\mathbf{j}]$, and we are done with Claim 9.

Claim 10 should be clear, as adding a binary operation $\mathbf{m}$ on A to the clone $\mathscr{W}_\omega[d;\mathbf{2},\mathbf{j}]$ such that $\mathbf{m}(3,3) = 3$ and $\mathbf{m}(x,y) = 2$ for other entries, we change the type of the prime quotient $\mu \prec \alpha$ from $\mathbf{2}$ to $\mathbf{3}$ and, since $\mathbf{m}$ is constant modulo α, we keep the type $\mathsf{typ}(\alpha,\mathbf{1})$ unchanged. Moreover one easily shows that in the clone $\langle \mathscr{W}_\omega[d;\mathbf{2},\mathbf{j}] \cup \{\mathbf{m}\}\rangle$ we have $[\alpha,\mu] = \mathbf{0}$, e.g., by checking that $\mathbf{m}$ commutes with the Mal'tsev operation d on appropriate entries, as required by Proposition 5.7 in [3], i.e., that

$$d(\mathbf{m}(x_1,x_2),\mathbf{m}(y_1,y_2),\mathbf{m}(z_1,z_2)) = \mathbf{m}(d(x_1,y_1,z_1),d(x_2,y_2,z_2))$$

holds whenever $x_i \overset{\mu}{\equiv} y_i \overset{\alpha}{\equiv} z_i$. (This is a special case of Kiss' condition cited just before Lemma 4.) □

Fig. 4 presents polynomial clones containing a given Mal'tsev operation d and having the congruence lattice of type (i). The polynomially rich clone with the

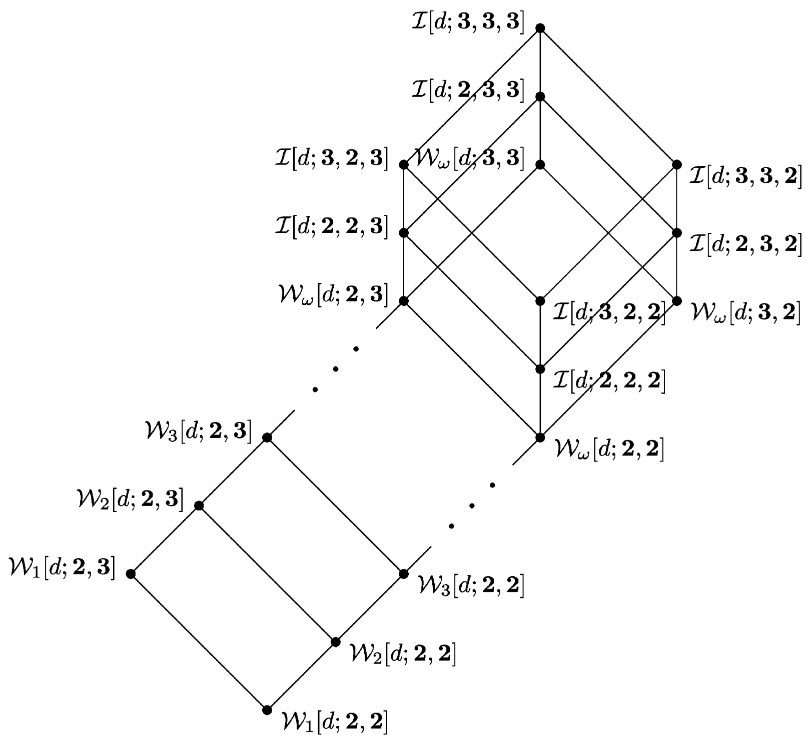

Fig. 4. Clones with the congruence lattice of type (i).

Mal'tsev operation d and $\mathsf{typ}(\mathbf{0},\mu)=\mathbf{k}$, $\mathsf{typ}(\mu,\alpha)=\mathbf{i}$, $\mathsf{typ}(\alpha,\mathbf{1})=\mathbf{j}$ is denoted by $\mathscr{I}[d;\mathbf{k},\mathbf{i},\mathbf{j}]$.

We hope that the methods developed in [6,7] and in this paper help to solve the following.

Problem 8. *Does there exist a finite set with uncountably many polynomial Mal'tsev clones?*

References

[1] I. Ágoston, L. Demetrovic, L. Hannák, On the number of clones containing all constants, in: Lectures in Universal Algebra, Colloquia Mathematica Societatis János Bolyai, vol. 43, North-Holland, Amsterdam, New York, 1983, pp. 21–25.

[2] K.A. Baker, A. Pixley, Polynomial interpolation and the Chinese remainder theorem for algebraic systems, Math. Z. 143 (1975) 165–174.

[3] R. Freese, R. McKenzie, Commutator Theory for Congruence Modular Varieties, London Mathematical Society, Lecture Notes, vol. 125, Cambridge University Press, Cambridge, 1987.

[4] D. Hobby, R. McKenzie, The Structure of Finite Algebras, American Mathematical Society, Contemporary Mathematics, vol. 76, Amer. Math. Soc., Providence, RI, 1988.

[5] A. Hulanicki, S. Świerczkowski, Number of algebras with a given set of elements, Bull. Acad. Sci. Polon., Ser. Sci., Math. Phys. Astron. 8 (1960) 283–284.

[6] P.M. Idziak, Clones with Malcev operations, Int. J. Algebra Comput. 9 (1999) 213–226.

[7] P.M. Idziak, K. Słomczyńska, Polynomially rich algebras, J. Pure Appl. Algebra 156 (2001) 33–68.

[8] E.W. Kiss, Three remarks on the modular commutator, Algebra Univ. 29 (1992) 455–476.

[9] A.A. Krokhin, K.L. Safin, E.V. Sukhanov, On the structure of the lattice of closed classes of polynomials Diskretnaja Math. 9 (1997) 24–40 (in Russian).

[10] M. Porebska, Interpolation for fragments of intermediate logics, Rep. Math. Logic 21 (1987) 9–14.

[11] E.L. Post, The Two-valued Iterative Systems of Mathematical Logic, Annals of Mathematical Studies, vol. 5, Princeton University Press, Princeton, NJ, 1941.

[12] Yu. Yanov, A.A. Muchnik, On existence of k-valued closed classes without finite basis, Dokl. AN SSSR 127 (1959) 44–46 (in Russian).