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Numerically pricing double barrier options in a time-fractional Black–Scholes model

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ABSTRACT

The numerical solution of the time fractional Black–Scholes model (TFBSM) of order $0 < \alpha < 1$ governing European options is studied. Zhang et al. (2016) derived a numerical scheme of second-order in space. We improve their results by constructing a scheme of fourth-order in space while keeping $2-\alpha$ in time. The solvability, stability and convergence of the proposed numerical scheme are proved using a Fourier analysis. The results are demonstrated on two examples.

1. Introduction

Options are one of the most traded financial products. Pricing them has received a lot of attention and dates back to the Black–Scholes (B–S) model, proposed in 1973 by Black and Scholes [1] and Merton [2]. Though very popular it has some shortcomings like missing the “volatility smile” [3] observed in real markets. The use of fractional derivatives and integrals is booming as it provides a powerful tool for incorporating history due to its non-local nature [4]. Also distributed order fractional equations [5] are emerging, where the fractional order is a continuous pallet. Among the numerical methods available for solving fractional differential equations we mention [6]; finite difference methods, finite element methods, finite volume methods, spectral methods, and meshless methods.

With the discovery of the fractal structure of a stochastic process, fractional calculus has found its way to stochastic models and financial theory. Wyss [7] priced a European call option by a time fractional B–S model. A single parameter and a bi-parameter fractional Black–Scholes–Merton differential equation was derived by Liang et al. [8] under the assumption that the stock price dynamics follows a fractional Ito process. Also numerical methods for the time-fractional Fokker–Planck equation [9] are receiving more attention.

In this paper we continue the work of Chen et al. [10] and Zhang et al. [11]. We assume that the underlying still follows the classical Brownian motion as in the B–S model, but consider the change in the option price as a fractal transmission system. As a result, the spatial-fractional derivative in the governing equation disappears, but the time-fractional derivative remains, see (1). Chen et al. [10] derive a series solution for the price of a double barrier option by using the eigenfunction expansion method together with the Laplace transform. Zhang et al. [11] construct a discrete implicit numerical scheme

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with a spatially second-order accuracy and a temporally $2-\alpha$ order accuracy. (In [12] the same is done in the case tempered fractional derivatives are used.) We will improve the spatial accuracy of [11] to fourth-order.

Let $C(S,t)$ be the time-t price of a European double barrier option with underlying S. More specific we consider for $0<\alpha\leq 1$ [10,11]:

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C(S,t)}{\partial S^2}+(r-D)S\frac{\partial C(S,t)}{\partial S}=rC(S,t),\quad (S,t)\in(B_d,B_u)\times(0,T), \tag{1a}$$

with the following boundary (barrier) and final conditions

$$C(B_d,t)=P(t),\qquad C(B_u,t)=Q(t),\quad 0<t<T, \tag{1b}$$

$$C(S,T)=V(S),\qquad B_d<S<B_u, \tag{1c}$$

where r is the risk free rate, D the dividend rate and $\sigma\geq 0$ is the volatility of the returns. The functions P and Q are the rebates paid when the corresponding barrier is hit. The terminal payoff of the option is $V(S)$. For example, a European double barrier knock-out call option has $P=0=Q$ and $V(S)=(S-K)_+$ where K is the strike and $(\cdot)_+=\max\{\cdot,0\}$.

The fractional derivative in (1a) is a modified right Riemann–Liouville derivative defined as

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{\partial}{\partial t}\displaystyle\int_t^T\frac{C(S,\xi)-C(S,T)}{(\xi-t)^\alpha}d\xi & 0<\alpha<1\\[2mm] \dfrac{\partial C(S,t)}{\partial t} & \alpha=1.\end{cases}$$

We transform the problem to an initial value problem by using the time to maturity $\tau:=T-t$. Note that for $0<\alpha<1$ one has

$$-\frac{\partial^\alpha C(S,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^\tau\frac{C(S,T-\eta)-C(S,T)}{(\tau-\eta)^\alpha}d\eta:={}_0D_\tau^\alpha C(S,T-\tau).$$

When we put $x=\ln S$, $U(x,\tau)=C(e^x,T-\tau)$, we find

$${}_0D_\tau^\alpha U(x,\tau)=a\frac{\partial^2U(x,\tau)}{\partial x^2}+b\frac{\partial U(x,\tau)}{\partial x}-cU(x,\tau),\quad (x,\tau)\in(0,+\infty)\times(0,T), \tag{2a}$$

where $a=\frac{1}{2}\sigma^2>0$, $b=r-a-D$, $c=r$, and with the following boundary (barrier) and initial conditions

$$U(b_d,\tau)=p(\tau),\qquad U(b_u,\tau)=q(\tau),\qquad 0<\tau<T, \tag{2b}$$

$$U(x,0)=v(x),\qquad b_d<x<b_u. \tag{2c}$$

In fact, the fractional derivative ${}_0D_\tau^\alpha$ coincides with the Caputo fractional derivative for $0<\alpha\leq 1$, that is

$${}_0D_\tau^\alpha U(x,\tau)=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^\tau\frac{U(x,\eta)-U(x,0)}{(\tau-\eta)^\alpha}d\eta=\frac{1}{\Gamma(1-\alpha)}\int_0^\tau\frac{\partial U(x,\eta)}{\partial\eta}\frac{1}{(\tau-\eta)^\alpha}d\eta={}_0^CD_\tau^\alpha U(x,\tau),$$

when U is continuous time differentiable.

We will develop a numerical scheme for the more general problem

$$a\frac{\partial^2U(x,\tau)}{\partial x^2}+b\frac{\partial U(x,\tau)}{\partial x}={}_0D_\tau^\alpha U(x,\tau)+cU(x,\tau)-f(x,\tau),\quad (x,\tau)\in(0,+\infty)\times(0,T). \tag{3}$$

The outline of this paper is arranged in the following way: we introduce a step by step construction of the difference scheme in the following section. Next, in the third section, the unique solvability, convergence and un-conditional stability for the difference scheme are analyzed carefully. In the fourth section, numerical examples are given to illustrate the accuracy of the presented scheme and to support our theoretical results. Finally, the paper ends with a conclusion and some remarks for future work.

2. Construction of the difference scheme

A numerical solution based on a compact difference scheme is derived. Before we continue, some notations are fixed. Take two positive integers M and N, let $h=\frac{b_u-b_d}{M}$, $\Delta\tau=\frac{T}{N}$ and denote $x_i=b_d+i\,h$ for $i=0,\dots,M$, and $\tau_k=k\,\Delta\tau$, for $k=0,\dots,N$.

Based on Taylor expansion of $V\in\mathcal{C}^4(b_d,b_u)$ we have

$$\frac{\partial V(x_i,\tau_k)}{\partial x}=\underbrace{\frac{V(x_{i+1},\tau_k)-V(x_{i-1},\tau_k)}{2h}}_{:=\delta_xV(x_i,\tau_k)}-\frac{h^2}{6}\frac{\partial^3V(x_i,\tau_k)}{\partial x^3}+O(h^4), \tag{4a}$$

and

$$\frac{\partial^2 V(x_i,\tau_k)}{\partial x^2} = \underbrace{\frac{V(x_{i+1},\tau_k)-2V(x_i,\tau_k)+V(x_{i-1},\tau_k)}{h^2}}_{:=\delta_x^2 V(x_i,\tau_k)} - \frac{h^2}{12}\frac{\partial^4 V(x_i,\tau_k)}{\partial x^4} + O(h^4). \tag{4b}$$

Using (4) in (3) we arrive in a grid point (x_i, τ_k) at

$$a\delta_x^2 U(x_i,\tau_k) + b\delta_x U(x_i,\tau_k) - R_i^k = g(x_i,\tau_k), \tag{5}$$

with

$$R_i^k = \frac{h^2}{12}\left(a\frac{\partial^4 U(x_i,\tau_k)}{\partial x^4} + 2b\frac{\partial^3 U(x_i,\tau_k)}{\partial x^3}\right) + O(h^4), \quad g(x,\tau) = {}_0D_\tau^\alpha U(x,\tau) + cU(x,\tau) - f(x,\tau).$$

Note that from (3)–(4) it follows that

$$\frac{\partial^3 U(x_i,\tau_k)}{\partial x^3} = \frac{1}{a}\left(\delta_x g(x_i,\tau_k) - b\delta_x^2 U(x_i,\tau_k)\right) + O(h^2) \tag{6}$$

and invoking the latter

$$\frac{\partial^4 U(x_i,\tau_k)}{\partial x^4} = \frac{1}{a}\left(\delta_x^2 g(x_i,\tau_k) - \frac{b}{a}\left(\delta_x g(x_i,\tau_k) - b\delta_x^2 U(x_i,\tau_k)\right)\right) + O(h^2). \tag{7}$$

When we substitute (6)–(7) in R_i^k and consequently in (5) we obtain

$$R_i^k = \frac{h^2}{12}\left(\delta_x^2 g(x_i,\tau_k) + \frac{b}{a}\delta_x g(x_i,\tau_k) - \frac{b^2}{a}\delta_x^2 U(x_i,\tau_k)\right) + O(h^4),$$

and

$$\left(a + \frac{h^2b^2}{12a}\right)\delta_x^2 U(x_i,\tau_k) + b\delta_x U(x_i,\tau_k) = \frac{h^2}{12}\left(\delta_x^2 g(x_i,\tau_k) + \frac{b}{a}\delta_x g(x_i,\tau_k)\right) + g(x_i,\tau_k) + O(h^4), \tag{8}$$

with $g(x,\tau) = {}_0D_\tau^\alpha U(x,\tau) + cU(x,\tau) - f(x,\tau)$.

Next, it is clear that to obtain a numerical scheme, we need to approximate the Caputo derivative in g. This is based on a result of Sun and Wu [13].

Lemma 2.1. *Let $u \in \mathcal{C}^2[0, t_k]$ and $\alpha \in (0,1)$ then*

$$_0D_t^\alpha u(x_i,\tau_k) = \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(c_0^\alpha u(x_i,\tau_k) - \sum_{j=1}^{k-1}(c_{k-j-1}^\alpha - c_{k-j}^\alpha)u(x_i,\tau_j) - c_{k-1}^\alpha u(x_i,\tau_0)\right) + O(\Delta\tau^{2-\alpha}), \tag{9}$$

where $c_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$, and in fact, $1 = c_0^\alpha > c_1^\alpha > \cdots > c_j^\alpha \to 0$ as $j \to +\infty$.

Proof. See [13,14]. □

We evaluate (8) at (x_i, τ_k) with the help of (9) to obtain

$$\begin{aligned}
&\left(a + \frac{h^2b^2}{12a}\right)\delta_x^2 U(x_i,\tau_k) + b\delta_x U(x_i,\tau_k) \\
&= \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(c_0^\alpha u(x_i,\tau_k) - \sum_{j=1}^{k-1}(c_{k-j-1}^\alpha - c_{k-j}^\alpha)u(x_i,\tau_j) - c_{k-1}^\alpha u(x_i,\tau_0)\right) + cU(x_i,\tau_k) - f(x_i,\tau_k) \\
&\quad + \frac{h^2}{12}\left(\delta_x^2 + \frac{b}{a}\delta_x\right)\left[\frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(c_0^\alpha u(x_i,\tau_k) - \sum_{j=1}^{k-1}(c_{k-j-1}^\alpha - c_{k-j}^\alpha)u(x_i,\tau_j) - c_{k-1}^\alpha u(x_i,\tau_0)\right)\right. \\
&\quad \left. + cU(x_i,\tau_k) - f(x_i,\tau_k)\right] + R_i^k,
\end{aligned} \tag{10}$$

where the estimate

$$\left|R_i^k\right| \le C(\Delta\tau^{2-\alpha} + h^4), \tag{11}$$

holds. Denote $U(x_i, t_k) = U_i^k$, $\mu = \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$ and $\mu_1 = \frac{1}{h^2}\left(a + \frac{b^2h^2}{12a}\right)$. When we omit R_i^k and doing some rearrangements, we get the our final difference scheme

$$\begin{aligned}&\left[\mu_1 - \frac{\mu+c}{12} - \frac{b}{2h} + \frac{bh}{24a}\left(\mu+c\right)\right]U_{i-1}^k + \left[-2\mu_1 - \frac{10}{12}(\mu+c)\right]U_i^k + \left[\mu_1 - \frac{\mu+c}{12} + \frac{b}{2h} - \frac{bh}{24a}\left(\mu+c\right)\right]U_{i+1}^k \\ &= \left[-f_i^k - \frac{1}{12}(f_{i-1}^k - 2f_i^k + f_{i+1}^k) - \frac{bh}{24a}(f_{i+1}^k - f_{i-1}^k) - \mu\left(\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})U_i^j + c_{k-1}^{\alpha}U_i^0\right)\right] \\ &\quad -\frac{\mu}{12}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(U_{i-1}^j - 2U_i^j + U_{i+1}^j) + c_{k-1}^{\alpha}(U_{i-1}^0 - 2U_i^0 + U_{i+1}^0)\right] \\ &\quad -\frac{b\mu h}{24a}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(U_{i+1}^j - U_{i-1}^j) + c_{k-1}^{\alpha}(U_{i+1}^0 - U_{i-1}^0)\right],\ i = 1, 2, \ldots, M-1,\ k = 1, 2, \ldots, N. \end{aligned} \tag{12}$$

3. Theoretical analysis of the difference scheme

In this section, we provide the uniqueness, stability and convergence theorems for the proposed difference scheme.

Theorem 1 (*Solvability*). *The compact difference scheme (12) is uniquely solvable.*

Proof. The compact difference scheme (12) can be written in a more concise form

$$\mathsf{A}\mathbf{U}^k = \mathbf{b}_{k-1},$$

where the right hand side $\mathbf{b}_{k-1}$ depends only on the history $\mathbf{U}^{k-1}, \mathbf{U}^{k-2}, \ldots, \mathbf{U}^0$. The tridiagonal coefficient matrix $\mathsf{A} = (a_{ij})$ is strictly diagonally dominant because $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$, where

$$|a_{ii}| = 2\mu_1 + \frac{10}{12}(\mu+c), \qquad \sum_{j\neq i}|a_{ij}| = 2\mu_1 + \frac{2}{12}(\mu+c).$$

Therefore, the coefficient matrix is nonsingular and hence invertible. The theorem is now readily proved by strong induction. □

The stability analysis of the proposed difference scheme (12) will be discussed in terms of a Fourier analysis as in [11] which in turn goes back to also [15,16]. Let $\hat{U}_i^k$ be an approximate solution of (12) and define $\epsilon_i^k = U_i^k - \hat{U}_i^k$ for $i = 0, 1, \ldots, M$, $k = 0, 1, \ldots, N$. The roundoff error equation in terms of ϵ_i^k can be obtained from (12), namely

$$\begin{aligned}&\left[\mu_1 - \frac{\mu+c}{12} - \frac{b}{2h} + \frac{bh}{24a}\left(\mu+c\right)\right]\epsilon_{i-1}^k + \left[-2\mu_1 - \frac{10}{12}(\mu+c)\right]\epsilon_i^k + \left[\mu_1 - \frac{\mu+c}{12} + \frac{b}{2h} - \frac{bh}{24a}\left(\mu+c\right)\right]\epsilon_{i+1}^k \\ &= -\mu\left(\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})\epsilon_i^j + c_{k-1}^{\alpha}\epsilon_i^0\right) - \frac{\mu}{12}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(\epsilon_{i-1}^j - 2\epsilon_i^j + \epsilon_{i+1}^j) + c_{k-1}^{\alpha}(\epsilon_{i-1}^0 - 2\epsilon_i^0 + \epsilon_{i+1}^0)\right] \\ &\quad -\frac{b\mu h}{24a}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(\epsilon_{i+1}^j - \epsilon_{i-1}^j) + c_{k-1}^{\alpha}(\epsilon_{i+1}^0 - \epsilon_{i-1}^0)\right], \end{aligned} \tag{13}$$

and $\epsilon_0^k = \epsilon_M^k = 0$. The grid function

$$\epsilon^k(x) := \begin{cases} \epsilon_i^k & x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2} \\ 0 & B_d \leq x \leq B_d + \frac{h}{2} \text{ or } B_u - \frac{h}{2} < x \leq B_u, \end{cases}$$

can be expanded in a Fourier series

$$\epsilon^k(x) = \sum_{l=-\infty}^{+\infty} d_k(l)\exp(\mathrm{i}\,2\pi lx/L), \quad k = 1, 2, \ldots, N, \quad d_k(l) = \frac{1}{L}\int_0^L \epsilon^k(x)\exp(-\mathrm{i}\,2\pi lx/L)dx,$$

where $L = B_u - B_d$ and $\mathrm{i}^2 = -1$. Let $\epsilon^k = \left(\epsilon_1^k, \epsilon_2^k, \ldots, \epsilon_{M-1}^k\right) \in \mathbb{C}^{M-1}$, with the following norm

$$\left\|\epsilon^k\right\|_2 = \left(\sum_{i=1}^{M-1} h\left|\epsilon_i^k\right|^2\right)^{1/2} = \left[\int_0^L \left|\epsilon^k(x)\right|^2 dx\right]^{1/2}.$$

The application of the Parseval identity leads to

$$\left\|\epsilon^k\right\|_2^2 = \sum_{l=-\infty}^{+\infty} |d_k(l)|^2. \tag{14}$$

Based on the above analysis, we suppose that the solution of (12) has the following form

$$\epsilon_j^k = d_k \exp(\mathrm{i}\,\sigma jh), \quad \sigma = \frac{2\pi l}{L}.$$

Substituting the above formula into (13) we arrive at

$$d_k = \frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)}\left(\sum_{j=1}^{k-1}\left(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha}\right)d_j + c_{k-1}^{\alpha}d_0\right). \tag{15}$$

Lemma 3.1. *The following estimate holds*

$$\left|\frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)}\right| \le 1. \tag{16}$$

Proof. We know that $h \le 1$, $\Delta\tau \le 1$ and $0 < \alpha < 1$ so $\mu > 0$. Also if $r < a$ then $\mu_1 > \frac{b}{a}$. Inequality (16) holds iff

$$\left|-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)\right| \le \left|\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)\right|,$$

or equivalent

$$(-4\mu_1 S)^2 - 8\mu_1 S\left(-\mu + \frac{\mu}{3}S\right) - 4\mu_1 S\left(-c + \frac{c}{3}S\right) + \left(-\frac{b}{6a}\mu - \frac{b}{12a}c + \frac{b}{h^2}\right)\sin^2(\sigma h) \ge 0,$$

with $S := \sin^2\left(\frac{\sigma h}{2}\right)$. Since

$$-\frac{b}{6a}\mu\sin^2(\sigma h) = \frac{2b}{3a}\mu S^2 - \frac{2b}{3a}\mu S \quad \text{and} \quad -\frac{b}{12a}c\sin^2(\sigma h) = \frac{2b}{6a}cS^2 - \frac{2b}{6a}cS,$$

inequality (16) holds iff

$$(4\mu_1 S)^2 + 8\mu_1 S\left(\mu - \frac{\mu}{3}S\right) + (4\mu_1 S)\left(c - \frac{c}{3}S\right) + \left(\frac{2b}{3a}\mu S^2 - \frac{2b}{3a}\mu S\right) + \left(\frac{2b}{6a}cS^2 - \frac{2b}{6a}cS\right) + \frac{b^2}{h^2}\sin^2(\sigma h) \ge 0,$$

or equivalent

$$(4\mu_1 S)^2 + \left(\frac{8}{3}\mu\mu_1 S - \frac{8}{3}\mu\mu_1 S^2\right) + \left(\frac{2}{3}\mu\mu_1 S - \frac{2b}{3a}\mu S\right) + \frac{2b}{3a}\mu S^2 + \frac{14}{3}\mu\mu_1 S + \left(\frac{4}{3}c\mu_1 S - \frac{4}{3}c\mu_1 S^2\right)$$
$$+ \left(\frac{2}{6}\mu_1 cS - \frac{2b}{6a}cS\right) + \frac{2b}{6a}cS^2 + \frac{14}{6}\mu_1 cS + \frac{b^2}{h^2}\sin^2(\sigma h) \ge 0.$$

All parts of the previous inequality are positive. Indeed,

$$\left(\frac{8}{3}\mu\mu_1 S - \frac{8}{3}\mu\mu_1 S^2\right) = \frac{8}{3}\mu\mu_1 S\cos^2\frac{\sigma h}{2} \ge 0, \qquad \left(\frac{4}{3}c\mu_1 S - \frac{4}{3}c\mu_1 S^2\right) = \frac{4}{3}c\mu_1 S\cos^2\frac{\sigma h}{2} \ge 0,$$

and

$$\left(\frac{2}{3}\mu\mu_1 S - \frac{2b}{3a}\mu S\right) = \frac{2}{3}\mu S\left(\mu_1 - \frac{b}{a}\right), \qquad \left(\frac{2}{6}\mu_1 cS - \frac{2b}{6a}cS\right) = \frac{2}{6}\mu S\left(\mu_1 - \frac{b}{a}\right) \ge 0. \quad \square$$

Lemma 3.2. *Suppose that d_k, $k = 1, 2, \ldots, N$, are the solutions of* Eq. (15), *we have*

$$|d_k| \le |d_0|. \tag{17}$$

Proof. Strong mathematical induction is used to prove (17). For $k = 1$ in (15), we have

$$d_1 = \frac{\left(-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)\right) c_0^\alpha d_0}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)}, \quad c_0^\alpha = 1.$$

Suppose that $|d_n| \le |d_0|$, for $n = 1, 2, \ldots, k-1$. Based on (15), we can write

$$|d_k| \le \left| \frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)} \right| \left(\sum_{j=1}^{k-1} \left(c_{k-j-1}^\alpha - c_{k-j}^\alpha\right)|d_0| + c_{k-1}^\alpha |d_0| \right).$$

Invoking estimate (16) we obtain

$$|d_k| \le \left(c_0^\alpha - c_{k-1}^\alpha\right)|d_0| + c_{k-1}^\alpha |d_0|,$$

which coincides with the aim of the theorem. □

Theorem 2 (*Stability*)**.** *The compact difference scheme (12) is unconditionally stable.*

Proof. From Lemmata 3.1 and 3.2, we deduce

$$\left\|\epsilon^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} |d_k(l)|^2 \le \sum_{l=-\infty}^{\infty} |d_0(l)|^2 = \left\|\epsilon^0\right\|_2^2.$$

Hence, $\left\|\epsilon^k\right\|_2 \le \left\|\epsilon^0\right\|_2$ and the unconditional stability of the proposed scheme is achieved. □

To conclude, we proof that the difference scheme (12) converges with time accuracy of order $2-\alpha$ and spatial order of four. We define the grid functions

$$e^k(x) = \begin{cases} e_i^k & x_i - \frac{h}{2} < x \le x_i + \frac{h}{2}, \quad i = 1, 2, \ldots, M-1 \\ 0 & B_d \le x \le B_d + \frac{h}{2} \quad \text{or} \quad B_u - \frac{h}{2} < x \le B_u \end{cases}$$

and

$$R^k(x) = \begin{cases} R_i^k & x_i - \frac{h}{2} < x \le x_i + \frac{h}{2}, \quad i = 1, 2, \ldots, M-1 \\ 0, & B_d \le x \le B_d + \frac{h}{2} \quad \text{or} \quad B_u - \frac{h}{2} < x \le B_u. \end{cases}$$

As above, we can write the following series expansions

$$e^k(x) = \sum_{l=-\infty}^{\infty} \eta_k(l) \exp(\mathrm{i}\, 2\pi l x/L), \quad k = 0, 1, \ldots, N,$$
$$R^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l) \exp(\mathrm{i}\, 2\pi l x/L), \quad k = 0, 1, \ldots, N.$$

Define the following

$$e_i^k = u(x_i, t_k) - U_i^k, \quad k = 0, 1, \ldots, N, \; j = 0, 1, \ldots, M,$$

$$e^k = \left[e_1^k, e_2^k, \ldots, e_{M-1}^k\right], \qquad R^k = \left[R_1^k, R_2^k, \ldots, R_{M-1}^k\right], \quad k = 1, 2, \ldots, N,$$

and introduce the norms

$$\left\|e^k\right\|_2 = \left(\sum_{i=1}^{M-1} h\left|e_i^k\right|^2\right)^{1/2} = \left[\int_0^L \left|e^k(x)\right|^2 dx\right]^{1/2}, \quad k = 0, 1, \ldots, N,$$
$$\left\|R^k\right\|_2 = \left(\sum_{i=1}^{M-1} h\left|R_i^k\right|^2\right)^{1/2} = \left[\int_0^L \left|R^k(x)\right|^2 dx\right]^{1/2}, \quad k = 0, 1, \ldots, N. \tag{18}$$

An application of the Parseval identity leads to

$$\|e^k\|_2^2 = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2, \quad k = 0, 1, \ldots, N, \tag{19}$$

$$\|R^k\|_2^2 = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2, \quad k = 0, 1, \ldots, N. \tag{20}$$

Also, (11) gives

$$\left|R_i^k\right| \le C\left(\Delta\tau^{2-\alpha} + h^4\right). \tag{21}$$

Subtracting (12) from (10), we get

$$\begin{aligned}
&\left[\mu_1 - \frac{\mu+c}{12} - \frac{b}{2h} + \frac{bh}{24a}\Big(\mu+c\Big)\right] e_{i-1}^k + \left[-2\mu_1 - \frac{10}{12}(\mu+c)\right] e_i^k \\
&\quad + \left[\mu_1 - \frac{\mu+c}{12} + \frac{b}{2h} - \frac{bh}{24a}\Big(\mu+c\Big)\right] e_{i+1}^k \\
&\quad = \left[-\mu\left(\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})e_i^j\right)\right] - \frac{\mu}{12}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(e_{i-1}^j - 2e_i^j + e_{i+1}^j) + c_{k-1}^{\alpha}(e_{i-1}^0 - 2e_i^0 + e_{i+1}^0)\right] \\
&\quad\quad - \frac{b\mu h}{24a}\left[\sum_{j=1}^{k-1}(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha})(e_{i+1}^j - e_{i-1}^j) + c_{k-1}^{\alpha}(e_{i+1}^0 - e_{i-1}^0)\right] + R_i^k,
\end{aligned} \tag{22}$$

with

$$e_0^k = e_M^k = 0, \quad k = 1, 2, \ldots, N-1, \tag{23}$$

$$e_i^0 = 0, \quad i = 1, 2, \ldots, M-1. \tag{24}$$

Based on the above analysis, we assume that e_i^k and R_i^k are written as follows

$$e_i^k = \eta_k \exp(\mathrm{i}\,\sigma i h), \qquad R_i^k = \xi_k \exp(\mathrm{i}\,\sigma i h), \quad \sigma = \frac{2\pi l}{L}.$$

Substituting the above relations into (22) gives

$$\begin{aligned}
\eta_k = &\frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)}\left(\sum_{j=1}^{k-1}\left(c_{k-j-1}^{\alpha} - c_{k-j}^{\alpha}\right)\eta_j\right) \\
&+ \frac{\xi_k}{\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu+c)\right)\sin(\sigma h) - (\mu+c)},
\end{aligned} \tag{25}$$

and noticing that $e^0 = 0$ we have $\eta_0 = 0$.

From (18) and (21), we obtain

$$\left\|R^k\right\|_2 \le \sqrt{Mh}C_1\left(\Delta\tau^{3-\alpha} + h^4\right) = \sqrt{L}C_1\left(\Delta\tau^{2-\alpha} + h^4\right). \tag{26}$$

Due to the convergence of the series on the right hand side of (20), there exists a positive constant c_2 such that

$$|\xi_k| \equiv |\xi_k(n)| \le c_2\Delta\tau\,|\xi_1(n)|, \quad k = 1, 2, \ldots, N. \tag{27}$$

Similar to the proof of Lemma 3.1 we can prove the estimate:

Lemma 3.3. *The following relation holds*

$$\frac{1}{\left(4\mu_1\sin^2\frac{\sigma h}{2} + (\mu+c) - \frac{\mu+c}{3}\sin^2\frac{\sigma h}{2}\right)^2 + \left(\frac{b}{h}\sin(\sigma h) - \frac{bh}{12a}(\mu+c)\sin(\sigma h)\right)^2} \le 9. \tag{28}$$

Lemma 3.4. *Suppose that η_k, $k = 1, 2, \ldots, N$, are the solutions of (25), then there exists a positive constant C_2 such that*

$$|\eta_k| \le C_2(1 + 3\Delta\tau)^k\,|\xi_1|.$$

Proof. Strong mathematical induction will be used. From (25), (27) and (28), we get

$$|\eta_1|^2 \le \frac{|\xi_1|^2}{\left(4\mu_1\sin^2\frac{\sigma h}{2} + (\mu + c) - \frac{\mu+c}{3}\sin^2\frac{\sigma h}{2}\right)^2 + \left(\frac{b}{h}\sin(\sigma h) - \frac{bh}{12a}(\mu + c)\sin(\sigma h)\right)^2} \le 9\Delta\tau^2 C_2^2|\xi_1|^2,$$

hence, $|\eta_1| \le 3\Delta\tau C_2\,|\xi_1| \le (1 + 3\Delta\tau)C_2\,|\xi_1|$. Now, let

$$|\eta_n| \le (1 + 3\Delta\tau)^n C_2\,|\xi_1|\,, \quad n = 1, 2, \ldots, k - 1.$$

Using (16), (27) and (28) in (25) we obtain

$$\begin{aligned}|\eta_k| \le\; & \frac{\left|-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\,\mathrm{i}}{12a}\sin(\sigma h)\right|}{\left|\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu + c)\right)\sin(\sigma h) - (\mu + c)\right|}\left|\sum_{j=1}^{k-1}\left(c^\alpha_{k-j-1} - c^\alpha_{k-j}\right)\eta_j\right| \\ & + \frac{|\xi_k|}{\left|\left(-4\mu_1 + \frac{\mu+c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\,\mathrm{i}}{h} - \frac{bh\,\mathrm{i}}{12a}(\mu + c)\right)\sin(\sigma h) - (\mu + c)\right|} \\ \le\; & \sum_{j=1}^{k-1}\left(c^\alpha_{k-j-1} - c^\alpha_{k-j}\right)(1 + 3\Delta\tau)^j C_2\,|\xi_1| + 3C_2\Delta\tau\,|\xi_1|\,.\end{aligned}$$

The proof is concluded by observing that

$$\begin{aligned}|\eta_k| &\le (1 + 3\Delta\tau)^{k-1}C_2\,|\xi_1|\sum_{j=1}^{k-1}\left(c^\alpha_{k-j-1} - c^\alpha_{k-j}\right) + 3C_2\Delta\tau\,|\xi_1|\,, \\ &\le (1 + 3\Delta\tau)^{k-1}C_2\,|\xi_1|\left(1 - c^\alpha_{k-1}\right) + (1 + 3\Delta\tau)\,C_2\,|\xi_1| \le (1 + 3\Delta\tau)^k C_2\,|\xi_1|\,. \quad \square\end{aligned}$$

Theorem 3 (*Convergence*). *Assume that $u(x,t)$ is the exact solution of (3), then the compact difference scheme (12) is L_2-convergent with convergence order $O(\Delta\tau^{2-\alpha} + h^4)$.*

Proof. Consider Lemma 3.4 and combine (19), (20) and (26), to obtain

$$\left\|e^k\right\|_2 \le (1 + 3\Delta\tau)^k C_2\left\|R^1\right\|_2 \le C_1\sqrt{L}C_2\exp(3k\Delta\tau)\left(\Delta\tau^{2-\alpha} + h^4\right).$$

Since $k\Delta\tau \le T$, we obtain $\left\|e^k\right\|_2 \le C\left(\Delta\tau^{2-\alpha} + h^4\right)$, where $C = c_1c_2\sqrt{L}\exp(3T)$. $\square$

4. Numerical experiments

To demonstrate the accuracy of the solution and the order of convergence of our proposed difference scheme, we introduce two examples exhibiting an exact solution.

Let $u_i^k = u(k\Delta\tau, h,)$ be the solution of the constructed difference scheme (12) with the step size $\Delta\tau$ in time and h in space. Define the maximum norm error by

$$E(\Delta\tau, h) = \max_{\substack{0\le i\le M \\ 0\le k\le N}}\left\|U_i^k - u_i^k\right\|_\infty,$$

and the following error rates, $\mathrm{rate}_{\Delta\tau} = \log_2\left(\frac{E(2\Delta\tau,h)}{E(\Delta\tau,h)}\right)$, $\mathrm{rate}_h = \log_2\left(\frac{E(\tau,2h)}{E(\tau,h)}\right)$. We reconsider two numerical examples which appeared in [11] and compare.

4.1. Example 1

Consider the following time fractional model

$$D^\alpha_\tau U(x,\tau) = a\frac{\partial^2 U(x,\tau)}{\partial x^2} + b\frac{\partial U(x,\tau)}{\partial x} - cU(x,\tau) + f(x,\tau),$$

with the following initial and boundary conditions

$$U(x,0) = x^2(1 - x), \quad U(0,\tau) = U(1,\tau) = 0$$

such that the source term

$$f(x,\tau) = \left(\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)}x^2(1 - x) - (\tau + 1)^2\left[a(2 - 6x) + b(2x - 3x^2) - cx^2(1 - x)\right]\right)$$

Table 1
Errors and convergence orders of the numerical solution for Example 1.

(a) $M = 150$			(b) $N = 1000$		
τ	$E(\Delta\tau, h)$	$\text{rate}_{\Delta\tau}$	h	$E(\Delta\tau, h)$	rate_h
$\frac{1}{10}$	0.0035		$\frac{1}{4}$	0.0028	
$\frac{1}{20}$	0.00144	1.28	$\frac{1}{8}$	0.00019	3.875
$\frac{1}{40}$	0.00059	1.29	$\frac{1}{16}$	0.000013	3.889
$\frac{1}{80}$	0.00024	1.295	$\frac{1}{32}$	8.33475×10^{-7}	3.95
$\frac{1}{160}$	0.000095	1.315	$\frac{1}{64}$	4.76035×10^{-8}	4.13
$\frac{1}{320}$	0.000038	1.32	$\frac{1}{128}$	2.38336×10^{-9}	4.32

Table 2
Errors and convergence orders of the numerical solution for Example 2.

(a) $M = 150$			(b) $N = 1000$		
τ	$E(\Delta\tau, h)$	$\text{rate}_{\Delta\tau}$	h	$E(\Delta\tau, h)$	rate_h
$\frac{1}{10}$	0.0052		$\frac{1}{4}$	0.0125	
$\frac{1}{20}$	0.00207	1.33	$\frac{1}{8}$	0.00079	3.98
$\frac{1}{40}$	0.00083	1.315	$\frac{1}{16}$	0.00005	3.995
$\frac{1}{80}$	0.00033	1.34	$\frac{1}{32}$	3.0412×10^{-6}	4.03
$\frac{1}{160}$	0.00013	1.36	$\frac{1}{64}$	1.79822×10^{-7}	4.08
$\frac{1}{320}$	0.00005	1.38	$\frac{1}{128}$	9.6493×10^{-9}	4.22

is chosen such that the exact solution of this problem is

$$U(x, \tau) = (\tau + 1)^2 x^2 (1 - x^2).$$

The parameters values are chosen as $r = 0.05$, $\sigma = 0.25$, $\alpha = 0.7$, $a = \frac{\sigma^2}{2}$, $b = r - a$, $c = r$ and $T = 1$. The results are shown in Table 1.

4.2. Example 2

The second example has nonhomogeneous boundary conditions;

$$D_\tau^\alpha U(x, \tau) = a\frac{\partial^2 U(x, \tau)}{\partial x^2} + b\frac{\partial U(x, \tau)}{\partial x} - cU(x, \tau) + f(x, \tau),$$

with the following initial and boundary conditions

$$U(x, 0) = x^3 + x^2 + 1, \qquad U(0, \tau) = (\tau + 1)^2, \qquad U(1, \tau) = 3(\tau + 1)^2$$

such that the source term

$$f(x, \tau) = \left(\frac{2\tau^{2-\alpha}}{\Gamma(3 - \alpha)} + \frac{2\tau^{1-\alpha}}{\Gamma(2 - \alpha)}(x^3 + x^2 + 1) - (t + 1)^2 \left[a(6x + 2) + b(3x^2 + 2x) - c(x^3 + x^2 + 1)\right] \right)$$

is chosen such that the exact solution of this problem is

$$U(x, \tau) = (\tau + 1)^2 (x^3 + x^2 + 1).$$

The parameters values are chosen as $r = 0.5$, $\alpha = 0.7$, $a = 1$, $b = r - a$, $c = r$ and $T = 1$. The results are shown in Table 2.

Both examples support the theoretical results established in Theorem 3, that is, an order of convergence in time of $2 - 0.7 = 1.3$ and of order 4 in space.

5. Conclusions

The time fractional B–S model is a generalization of the classical B–S model. An implicit numerical scheme with a temporal accuracy of order $2 - \alpha$ and spatial accuracy of fourth order is constructed to approximate the time fractional B–S model. It is proved that the implicit numerical scheme is unconditional stable and convergent by using the Fourier analysis method. Two numerical examples with exact solutions are chosen in order to illustrate the accuracy and convergence order of the numerical method. The numerical technique presented in this paper can be extended to other fractional models for pricing different European options.

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