The variational symmetries and conservation laws in classical theory of Heisenberg (anti)ferromagnet.

I.G. Bostrem, A.S. Ovchinnikov, and R.F. Egorov

Department of Theoretical Physics, Ural State University,

620083, Lenin Ave. 51, Ekaterinburg, Russia

(Dated: December 7, 2018)

The nonlinear partial differential equations describing the spin dynamics of Heisenberg ferro and antiferromagnet are studied by Lie transformation group method. The generators of the admitted variational Lie symmetry groups are derived and conservation laws for the conserved currents are found via Noether's theorem.

I. INTRODUCTION.

As it has been demonstrated previously in a number of works the symmetry methods are very efficient tool in application to differential equations in physics. A subject of an especial interest is a study of invariance properties (symmetries) of the equations with respect to local Lie groups point transformations of dependent and independent variables. The detailed discussion and applications of the symmetries of differential equations and variational problems may be found in Ref.[1,2] and[3].

The non-linear dynamics of n-dimensional (anti)ferromagnet is described by non-linear second-order partial differential equations (PDE). In our previous work it has been shown that the system of differential equations are the Euler-Lagrange equations of a certain action functional and the corresponding long-wave action has been constructed via Vainberg's theorem (see also Appendix). It is well known that a part of one parameter symmetry groups of these equations turns out to be their variational symmetries as well, i.e. a symmetries of the associated action functional. According to Noether theorem[1] such an invariance of the elementary action is a necessary and sufficient condition of an existence of conservation laws for the smooth solutions of the initial PDE.

The layout of the paper is as follows. By using the Noether identity we define the variational symmetries among the point Lie symmetries of the initial differential equations of a $2D$-dimensional ferromagnet. After they are deduced we establish explicit formulae for the conserved densities and density currents involved in the conservation laws both for a Heisenberg ferromagnet and antiferromagnet. Apparently, the analysis presented may be easily generalized for an arbitrary dimension $n \geq 3$.

II. DYNAMICAL INVARIANTS OF $2D$ FERROMAGNET.

The non-linear partial differential equations in 3 independent variables $(x^1, x^2, x^3) = (t, \vec{r})$ and two dependent variables $u^1 = \theta(t, \vec{r})$ and $u^2 = \varphi(t, \vec{r})$ describing the dynamics of Heisenberg ferromagnet in continuum approximation can be written as follows[4]

$$\mathrm{F}^1 \equiv -S \sin\theta \varphi_t + \beta \left(\Delta\theta - \cos\theta \sin\theta \left(\vec{\nabla}\varphi \right)^2 \right) = 0,$$

$$\mathrm{F}^2 \equiv S \sin\theta \theta_t + \beta \left(2\cos\theta \sin\theta \left(\vec{\nabla}\theta \vec{\nabla}\varphi \right) + \sin^2\theta \Delta\varphi \right) = 0, \tag{1}$$

where $\beta = \frac{JS^2}{\hbar}$. Here and throughout the following notations are taken: α have the range 1, 2 meaning θ and φ, correspondingly, the usual summation convention over a repeated index is employed, $\theta_{j_1 \dots j_k}$ (or $\varphi_{j_1 \dots j_k}$) denote the k-th order partial derivatives of the dependent variables

$$\theta_{(J)} = \theta_{j_1 \dots j_k} = \frac{\partial^k \theta}{\partial x^{j_1} \partial x^{j_2} \dots \partial x^{j_k}}.$$

The J denotes a multi-index $J = (j_1, j_2, \dots, j_k)$ with $j_k = 1, \dots, 3$ $(k \geq 0)$ pointing to the independent variables with respect to which one differentiate.

Consider a local one-parameter Lie group of point transformations acting on a space Ω of the independent and dependent variables involved in the basic equations (1). The infinitesimal generator of such a group is a vector field $\hat{X}$ on Ω

$$\hat{X} = \xi^k \frac{\partial}{\partial x^k} + \eta^1 \frac{\partial}{\partial \theta} + \eta^2 \frac{\partial}{\partial \varphi} \tag{2}$$

whose components ξ^k and η^α are supposed to be functions of class C^∞ on Ω. The infinitesimal invariance criterion of the equations $\mathrm{F} = (\mathrm{F}^1, \mathrm{F}^2)$ under the group G is given by the determining equations

$$\left(\hat{Y}\mathrm{F}\right)_{\mathrm{F}=0} = 0, \tag{3}$$

$$\hat{Y} = \hat{D}_{(J)} \left(W^\alpha\right) \partial_{u^\alpha_{(J)}}, \tag{4}$$

where $\hat{D}_k$ means total differentiation

$$\hat{D}_k = \partial_k + u^\alpha_{k(J)} \partial_{u^\alpha_{(J)}} \tag{5}$$

and $W^\alpha = \eta^\alpha - \xi^i u^\alpha_i$ are the characteristics of the canonical vector field $\hat{Y}$ which is equivalent to $\hat{X}$

$$\hat{Y} = \hat{X} - \xi^k \hat{D}_k. \tag{6}$$

The determining equations (3) are realized as an over-determined system of linear homogeneous partial differential equations with respect to the unknown functions ξ^k and η^α. In the determining equations the $u^\alpha_{(J)}$ up to second order derivatives are the independent variables and these equations must be fulfilled identically over them on the manifold F.

A sufficient condition of an existence of conservation laws is an invariance of the elementary action associated with the Euler equations under a symmetry group of these equations. The starting point is the so-called Noether identity[1]

$$\hat{Y} + \hat{D}_i \cdot \xi^i = W^\alpha \hat{E}^\alpha + \hat{D}_i \hat{N}^i. \tag{7}$$

In Eq.(7) $\hat{N}^i$ are the Noether operators given by the expressions

$$\hat{N}^i = \xi^i + \hat{D}_{i(s)} \left(W^\alpha\right) \left(-1\right)^r \hat{D}_{(r)} \partial_{u^\alpha_{(s)(r)}} \tag{8}$$

and

$$\hat{E}^\alpha = \left(-1\right)^s \hat{D}_{(s)} \partial_{u^\alpha_{(s)}} \tag{9}$$

are the Euler-Lagrange operators. The variational symmetries on the Euler-Lagrange equations may be found via the Noether identity

$$\hat{Y}(L) + \hat{D}_i \cdot \left(\xi^i L\right) = 0 \tag{10}$$

with the Lagrangian density L[5]

$$L = L(\theta, \varphi_t, \theta_k, \varphi_k) = S \left(\cos\theta - 1\right) \varphi_t - \beta \left[\frac{1}{2} \sin^2\theta \left(\vec{\nabla}\varphi\right)^2 + \frac{1}{2} \left(\vec{\nabla}\theta\right)^2\right]. \tag{11}$$

The dot in Eq.(7) means the differentiation rule $\hat{D}_i \cdot \xi^i \left(L\right) \equiv L\hat{D}_i \left(\xi^i\right) + \xi^i \hat{D}_i \left(L\right)$.

Substituting (11) into (10) one obtain

$$\hat{Y}(L) = W^1 \partial_\theta L + \hat{D}_t(W^2) \partial_{\varphi_t} L + \hat{D}_k(W^1) \partial_{\theta_k} L + \hat{D}_k(W^2) \partial_{\varphi_k} L, \tag{12}$$

where

$$\partial_\theta L = -S \sin\theta \varphi_t - \frac{\beta}{2} \sin 2\theta \left(\vec{\nabla}\varphi\right)^2, \ \partial_{\varphi_t} L = S \left(\cos\theta - 1\right),$$
$$\partial_{\theta_k} L = -\beta\theta_k, \ \partial_{\varphi_k} L = -\beta \sin^2\theta \varphi_k. \tag{13}$$

After a little manipulation one get the following equation for the unknowns ξ^k and η^α

$$\begin{aligned}
&\eta^1 \left(-S \sin\theta\varphi_t - \frac{\beta}{2} \sin 2\theta \left(\vec{\nabla}\varphi\right)^2\right) \\
&+S \left(\cos\theta - 1\right) \left(\hat{D}_t \left(\eta^2\right) - \varphi_t \hat{D}_t \left(\xi^t\right) - \varphi_k \hat{D}_t \left(\xi^k\right)\right) \\
&-\beta\theta_k \left(\hat{D}_k \left(\eta^1\right) - \theta_t \hat{D}_k \left(\xi^t\right) - \theta_l \hat{D}_k \left(\xi^l\right)\right) \\
&-\beta \sin^2\theta\varphi_k \left(\hat{D}_k \left(\eta^2\right) - \varphi_t \hat{D}_k \left(\xi^t\right) - \varphi_l \hat{D}_k \left(\xi^l\right)\right) \\
&+ \left(S \left(\cos\theta - 1\right) \varphi_t - \frac{\beta}{2} \sin^2\theta \left(\vec{\nabla}\varphi\right)^2 - \frac{\beta}{2} \left(\vec{\nabla}\theta\right)^2\right) \left(\hat{D}_t \left(\xi^t\right) + \hat{D}_k \left(\xi^k\right)\right) = 0.
\end{aligned} \tag{14}$$

Hereinafter, we keep the upper index k just for the space coordinates, the corresponding time component will be written explicitly. The coefficients at the derivatives of the functions θ and φ constitute an over-determined system of linear homogeneous partial differential

TABLE I: Equations for the uknowns ξ^k and η^α.

Independent variable	Equation
0	$\eta^2_t = 0$
φ_t	$(\cos\theta - 1)\left(\eta^2_\varphi + div\vec{\xi}\right) = \eta^1 \sin\theta$
φ_k	$-S\,(\cos\theta - 1)\,\xi^k_t - \beta\sin^2\theta\eta^2_k = 0 \Longrightarrow\ \xi^k_t = \eta^2_k = 0$
θ_k	$\eta^1_k = 0$
$\left(\vec{\nabla}\varphi\right)^2$	$\xi^k_k = \eta^1\cot\theta + \frac{1}{2}\left(\xi^t_t + div\vec{\xi}\right) + \eta^2_\varphi$
$\varphi_k\varphi_l(k \neq l)$	$\sin^2\theta\left(\xi^l_k + \xi^k_l\right) = 0 \Longrightarrow \xi^l_k + \xi^k_l = 0$
$\left(\vec{\nabla}\theta\right)^2$	$\xi^k_k = \eta^1_\theta + \frac{1}{2}\left(\xi^t_t + div\vec{\xi}\right)$
$\theta_k\theta_l(k \neq l)$	$\xi^l_k + \xi^k_l = 0$
$\theta_k\varphi_k$	$\eta^1_\varphi = 0$
$\left(\vec{\nabla}\varphi\right)^2\varphi_k$	$\xi^k_\varphi = 0$

equations with respect to the unknowns ξ^i and η^α. For example, extracting the terms at the independent variable θ_t and set equal them to zero

$$S\left(\eta^2_\theta - \varphi_k\xi^k_\theta\right)(\cos\theta - 1) + \beta\theta_k\left(\xi^t_k + \theta_k\xi^t_\theta + \varphi_k\xi^t_\varphi\right)$$
$$+\,\beta\left(-\frac{1}{2}\sin^2\theta\left(\vec{\nabla}\varphi\right)^2 - \frac{1}{2}\left(\vec{\nabla}\theta\right)^2\right)\xi^t_\theta = 0 \qquad (15)$$

one obtain the conditions

$$\eta^2_\theta = \xi^k_\theta = \xi^t_k = \xi^t_\varphi = \xi^t_\theta = 0 \qquad (16)$$

that immediately imply $\xi^t = \xi^t(t)$. The results of the full analysis of the Eq. (14) are summarized in Table 1. Finally, we get $\xi^t = \xi^t(t)$, $\xi^k = \xi^k(\vec{r})$, $\eta^1 = \eta^1(t,\theta)$, $\eta^2 = \eta^2(\varphi)$. The uknowns ξ^k satisfy to Killing's equation $\xi^l_k + \xi^k_l = \sigma\delta_{lk}$. As is well known, it has a solution for an arbitrary σ just for $n \geq 3$. However, for the system considered there is a solution in $2D$ case. Indeed, from the Table 1 one have

$$\sigma = 2\xi^k_k = 2\eta^1_\theta + \xi^t_t + div\vec{\xi},$$
$$\sigma = 2\xi^k_k = 2\eta^1\cot\theta + \xi^t_t + div\vec{\xi} + 2\eta^2_\varphi,$$

TABLE II: Conservation laws.

	Conservation law
1	$\hat{D}_t\left(\frac{\beta}{2}\sin^2\theta\left(\vec{\nabla}\varphi\right)^2+\frac{\beta}{2}\left(\vec{\nabla}\theta\right)^2\right)=\hat{D}_k\left(\beta\theta_t\theta_k+\beta\varphi_t\varphi_k\sin^2\theta\right)$
2	$\hat{D}_t\left(S\left(\cos\theta-1\right)\varphi_k\right)=\hat{D}_l\left(T_{kl}\right)$
3	$\hat{D}_t\left(\left[\vec{r}\times S\left(\cos\theta-1\right)\vec{\nabla}\varphi\right]_l\right)=\hat{D}_k\left(\epsilon^{lmn}x^mT_{nk}\right),\ (l=z)$
4	$\hat{D}_t\left(S\left(\cos\theta-1\right)\right)=\hat{D}_k\left(\beta\varphi_k\sin^2\theta\right)$

(no summation over k) that imply

$$\xi^t=\xi^t(t),\ \vec{\xi}(\vec{r})=\vec{a}+\hat{B}\vec{r},\ \hat{B}^T=-\hat{B}=-\begin{pmatrix}0&-b\\b&0\end{pmatrix},$$

$$\eta^1=0,\ \eta^2=\upsilon.$$

The solution found results in the following expressions for the characteristics $W^1=-\xi^t\theta_t-\vec{a}\vec{\bigtriangledown}\theta-b\left[\vec{r}\times\vec{\bigtriangledown}\theta\right]_z$ and $W^2=\upsilon-\xi^t\varphi_t-\vec{a}\vec{\bigtriangledown}\varphi-b\left[\vec{r}\times\vec{\bigtriangledown}\varphi\right]_z$. One can derive corresponding conservation laws through Noether's theorem procedure

$$\hat{D}_i\hat{N}^i\left(L\right)=0 \tag{17}$$

under the condition $\hat{E}^\alpha L=0$. The Noether operators modify the Lagrangian density into a conserved quantity $C^i=\hat{N}^i\left(L\right)$ with a zero total divergence. Using the found variational symmetries we first construct the operators $\hat{N}^i$ from formulae (8) and then calculate from (17) the corresponding conservation laws. The results are brought together in the Table 2. The physical interpretation of the obtained relations is obvious.

1) The symmetry group of translations in time $t\to t+a$ yields to the energy conservation law with the energy density $W=\frac{JS^2}{2}\sin^2\theta\left(\vec{\nabla}\varphi\right)^2+\frac{JS^2}{2}\left(\vec{\nabla}\theta\right)^2$ and the energy density current $\vec{T}=-(JS^2\theta_t\vec{\nabla}\theta+JS^2\varphi_t\vec{\nabla}\varphi\sin^2\theta)$.

2) This law connects the momentum $\vec{P}=\hbar S\left(1-\cos\theta\right)\vec{\nabla}\varphi$ and the canonical energy-momentum tensor

$$T_{kl}=\hbar L\delta_{kl}+\hbar\beta\theta_k\theta_l+\hbar\beta\varphi_k\varphi_l\sin^2\theta \tag{18}$$

and reflects the variational symmetry under space translations $x_l\to x_l+a_l$.

3) The conservation of angular momenta $\vec{L}=\left[\vec{r}\times\vec{P}\right]$ through the angular momenta tensor under the space rotation around z-axis.

4) The conservation law corresponding to the transformations $\varphi \to \varphi + \chi$ connects the scalar variable $N = S\left(1 - \cos\theta\right)$ with the current $\vec{j} = \beta \vec{\nabla} \varphi \sin^2\theta$.

It is useful to apply the obtained laws to some types of solutions of Eqs. (1). We restrict consideration by finite-amplitude spin-waves, dynamical soliton and skyrmion.

1) The finite-amplitude spin-waves $\theta = \theta_0 = const$ and $\varphi = \vec{k}\vec{r} - \omega t$[6] with the dispersion $\hbar\omega = JS\vec{k}^2 \cos\theta_0$. The last conservation law may be interpreted as a conservation of the magnon density $N = S\left(1 - \cos\theta_0\right)$. The corresponding magnon density current is $\vec{j} = \frac{JS^2}{\hbar}\vec{k}\sin^2\theta_0$. The space components of the energy-momentum tensor are

$$\hat{T}_{kl} = \begin{bmatrix} \begin{array}{c} S\left(1-\cos\theta_0\right)\hbar\omega \\ +\frac{JS^2}{2}\sin^2\theta_0\left(k_x^2 - k_y^2\right) \end{array} & JS^2\sin^2\theta_0 k_x k_y \\ JS^2\sin^2\theta_0 k_x k_y & \begin{array}{c} S\left(1-\cos\theta_0\right)\hbar\omega \\ -\frac{JS^2}{2}\sin^2\theta_0\left(k_x^2 - k_y^2\right) \end{array} \end{bmatrix}.$$

The momentum $\vec{P} = S\left(1 - \cos\theta_0\right)\hbar\vec{k}$ is a times of the magnon density and the elementary momentum $\hbar\vec{k}$. The energy density $W = \frac{1}{2}JS^2k^2\sin^2\theta_0 \approx N\hbar\omega$ $(\theta_0 \ll 1)$ has the corresponding energy density current $\vec{T} = JS^2\vec{k}\omega\sin^2\theta_0$. We note a non-additive character of energy density at finite values θ_0.

2) For a dynamical soliton with axial symmetry one have to use the following parametrization $\theta = \theta(r)$ and $\varphi = \omega t$[6]. The magnon density $N = S\left(1 - \cos\theta(r)\right)$ has the zero magnon density current $\vec{j} = 0$. The energy-momentum tensor is

$$\hat{T}_{kl} = \begin{bmatrix} \begin{array}{c} -S\left(1-\cos\theta\left(r\right)\right)\hbar\omega \\ +\frac{JS^2}{2}\cos\left(2\omega t\right)\left(\frac{d\theta}{dr}\right)^2 \end{array} & \frac{JS^2}{2}\sin\left(2\omega t\right)\left(\frac{d\theta}{dr}\right)^2 \\ \frac{JS^2}{2}\sin\left(2\omega t\right)\left(\frac{d\theta}{dr}\right)^2 & \begin{array}{c} -S\left(1-\cos\theta\left(r\right)\right)\hbar\omega \\ -\frac{JS^2}{2}\cos\left(2\omega t\right)\left(\frac{d\theta}{dr}\right)^2 \end{array} \end{bmatrix}.$$

The momentum $\vec{P}$ is zero, the energy density $W = \frac{JS^2}{2}\left(\frac{d\theta}{dr}\right)^2$ and the energy density current $\vec{T} = 0$.

3) For the skyrmion the following parametrization $\theta = 2\tan^{-1}\left(\frac{R}{r}\right)$ and $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$[7] is used (for simplicity we consider an unit topological charge). Then, $N = 2S\frac{R^2}{r^2+R^2}$, $j_r = 0$, $j_\varphi = \frac{JS^2}{\hbar}\frac{4R^2r}{\left(R^2+r^2\right)^2}$. The momentum in polar coordinates of xy-plane $P_r = 0$, $P_\varphi = \frac{2\hbar S}{r}\frac{R^2}{r^2+R^2}$, however, $\hat{T}_{kl} = 0$. The energy density $W = 4JS^2\frac{R^2}{\left(R^2+r^2\right)^2} = \frac{JL_z^2}{\hbar^2R^2}$ and the energy density current $\vec{T} = 0$.

III. DYNAMICAL INVARIANTS OF $2D$ TWO-SUBLATTICE MAGNET.

In this section Lie transformation group methods will be applied to the partial differential equations describing the dynamics of $2D$ generic antiferromagnet (two-sublattice magnet). The established variational Lie symmetries for ferromagnet will be employed to derive antiferromagnet conservation laws revealing its important features.

The action of $2D$ two-sublattice magnet is[5]

$$\begin{aligned} L \;=\; & S\sum_{i=1}^{2}\left(\cos\theta_i-1\right)\varphi_{it}-\beta\left(-2\sin\theta_1\sin\theta_2\cos\left(\varphi_1-\varphi_2\right)-2\cos\theta_1\cos\theta_2\right.\\ & +\cos\theta_1\cos\theta_2\cos\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\theta_1\vec{\nabla}\theta_2\right)-\sin\theta_1\cos\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\varphi_1\vec{\nabla}\theta_2\right)\\ & +\cos\theta_1\sin\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\theta_1\vec{\nabla}\varphi_2\right)+\sin\theta_1\sin\theta_2\cos\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\varphi_1\vec{\nabla}\varphi_2\right)\\ & \left.+\sin\theta_1\sin\theta_2\left(\vec{\nabla}\theta_1\vec{\nabla}\theta_2\right)\right). \end{aligned} \tag{19}$$

The time symmetry group $t\rightarrow t+a$ under which the elementary action is invariant has the infinitesimal symmetry generator $\hat{X}=\frac{\partial}{\partial t}$ and Noether's operators $\hat{N}^t=1-\sum\limits_{i=1}^{2}\theta_{it}\frac{\partial}{\partial\theta_{it}}-\sum\limits_{i=1}^{2}\varphi_{it}\frac{\partial}{\partial\varphi_{it}}$, $\hat{N}^k=-\sum\limits_{i=1}^{2}\theta_{it}\frac{\partial}{\partial\theta ik}-\sum\limits_{i=1}^{2}\varphi_{it}\frac{\partial}{\partial\varphi_{ik}}$. By introducing the functions

$$\begin{aligned} C^t \;=\; & \hat{N}^t\left(L\right)=-\beta\left(-2\sin\theta_1\sin\theta_2\cos\left(\varphi_1-\varphi_2\right)-2\cos\theta_1\cos\theta_2\right.\\ & +\cos\theta_1\cos\theta_2\cos\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\theta_1\vec{\nabla}\theta_2\right)-\sin\theta_1\cos\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\varphi_1\vec{\nabla}\theta_2\right)\\ & +\cos\theta_1\sin\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\theta_1\vec{\nabla}\varphi_2\right)+\sin\theta_1\sin\theta_2\cos\left(\varphi_1-\varphi_2\right)\left(\vec{\nabla}\varphi_1\vec{\nabla}\varphi_2\right)\\ & \left.+\sin\theta_1\sin\theta_2\left(\vec{\nabla}\theta_1\vec{\nabla}\theta_2\right)\right) \end{aligned} \tag{20}$$

and

$$\begin{aligned} C^k \;=\; & \hat{N}^k\left(L\right)=\beta\left(\cos\theta_1\cos\theta_2\cos\left(\varphi_1-\varphi_2\right)+\sin\theta_1\sin\theta_2\right)\left(\theta_{1t}\theta_{2k}+\theta_{2t}\theta_{1k}\right)\\ & +\beta\sin\theta_1\sin\theta_2\cos\left(\varphi_1-\varphi_2\right)\left(\varphi_{1t}\varphi_{2k}+\varphi_{2t}\varphi_{1k}\right)\\ & +\beta\cos\theta_1\sin\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\theta_{1t}\varphi_{2k}+\theta_{1k}\varphi_{2t}\right)\\ & -\beta\sin\theta_1\cos\theta_2\sin\left(\varphi_1-\varphi_2\right)\left(\theta_{2t}\varphi_{1k}+\theta_{2k}\varphi_{1t}\right) \end{aligned} \tag{21}$$

one get the energy conservation law $\hat{D}_t\left(C^t\right)+\hat{D}_k\left(C^k\right)=0$. By the same manner one can obtain the generators and Noether's operators of the translation group $x_k\rightarrow x_k+a_k$
$\hat{X}=\frac{\partial}{\partial x^k}$, $\hat{N}^t=-\sum\limits_{i=1}^{2}\theta_{ik}\frac{\partial}{\partial\theta it}-\sum\limits_{i=1}^{2}\varphi_{ik}\frac{\partial}{\partial\varphi_{it}}$, $\hat{N}^l=\delta_{kl}-\sum\limits_{i=1}^{2}\theta_{jk}\frac{\partial}{\partial\theta_{jl}}-\sum\limits_{i=1}^{2}\varphi_{jk}\frac{\partial}{\partial\varphi_{jl}}$.
The momentum conservation law is

$$\sum_{i=1}^{2}\hat{D}_t\left(S\left(\cos\theta_i-1\right)\varphi_{ik}\right)=\hat{D}_l\left(T_{kl}\right), \tag{22}$$

with the tensor

$$T_{kl} = L\delta_{kl} - \sum_{j=1}^{2} \theta_{jk}\partial_{\theta_{jk}} L - \sum_{j=1}^{2} \varphi_{jk}\partial_{\varphi_{jk}} L. \tag{23}$$

The conservation of angular momenta is

$$\sum_{j=1}^{2} \hat{D}_t \left(\left[\vec{r} \times S\left(\cos\theta_j - 1\right) \vec{\nabla}\varphi_j \right]_z \right) = \hat{D}_k \left(\epsilon^{znl} x^n T_{lk} \right), \tag{24}$$

$$\hat{X} = \epsilon^{znl} x^n \frac{\partial}{\partial x^l},$$

$$\hat{N}^t = -\epsilon^{znl} x^n \theta_l^\alpha \frac{\partial}{\partial \theta_t^\alpha} - \epsilon^{znl} x^n \varphi_l^\alpha \frac{\partial}{\partial \varphi_t^\alpha},$$

$$\hat{N}^k = \epsilon^{znk} x^n - \epsilon^{znl} x^n \theta_l^\alpha \frac{\partial}{\partial \theta_k^\alpha} - \epsilon^{znl} x^n \varphi_l^\alpha \frac{\partial}{\partial \varphi_k^\alpha}.$$

The action is also invariant under a simultaneous rotation in both sublattices $\varphi_1^{'} = \varphi_1 + \chi$, $\varphi_2^{'} = \varphi_2 + \chi$. The generator and Noether operators are $\hat{X} = \sum\limits_{j=1}^{2} \frac{\partial}{\partial \varphi_j}$, $\hat{N}^t = \sum\limits_{j=1}^{2} \frac{\partial}{\partial \varphi_{jt}}$, $\hat{N}^k = \sum\limits_{j=1}^{2} \frac{\partial}{\partial \varphi_{jk}}$ and they imply the conservation law

$$\begin{aligned} \sum_{j=1}^{2} \hat{D}_t \left(S \cos\theta_j \right) \; &= \; \beta \hat{D}_k \left(\sin\theta_1 \sin\theta_2 \cos\left(\varphi_1 - \varphi_2\right) \left(\varphi_{1k} + \varphi_{2k}\right) \right. \\ & \left. - \sin\left(\varphi_1 - \varphi_2\right) \left(\theta_{2k} \sin\theta_1 \cos\theta_2 - \theta_{1k} \sin\theta_2 \cos\theta_1\right) \right). \end{aligned} \tag{25}$$

IV. CONCLUSIONS.

Lie transformation group methods have been applied to the partial differential equations describing the dynamics of ferro and antiferromagnets and the variational symmetries of the action functional on the Euler-Lagrange equations are established. Each such symmetry gives rise to a conservation law with a conserved current. The explicit expressions for the conserved currents are found.

Appendix.

Let's take the following notations:

1) the matrix of Frechet derivatives $\hat{f}_\beta^\alpha = \sum\limits_s \frac{\partial f^\alpha}{\partial u_{(s)}^\beta} \hat{D}_{(s)}$, where the sum is taken over all multi-indices $s = (s_1, s_2, \ldots, s_k)$ with $s_k = 1, \ldots, 3$ $(k \geq 0)$;

2) the adjoint operator of Frechet derivatives[2]

$$\left(\hat{f}_{\alpha}^{\beta}\right)^{*}=\sum_{s}\left(-\hat{D}\right)_{(s)}\cdot\frac{\partial f^{\alpha}}{\partial u_{(s)}^{\beta}}.$$

One have to found a Lagrangian density $L\left(x,u,u^{'},\ldots\right)$ that the initial set of differential equations are the Euler-Lagrange equations of the following action

$$S[u]=\int L\left(x,u,u^{'},\ldots\right)dx\equiv\int\limits_{t_2}^{t_1}dt\int dV L\left(x,u,u^{'},\ldots\right).$$

From the ratio

$$\hat{E}^{\alpha}L=f^{\alpha}+\int\limits_{0}^{1}\left[\hat{f}^{\dagger}-\hat{f}\right]_{\beta}^{\alpha}\left[\lambda u^{\beta}\left(x\right)\right]d\lambda$$

is seen that that sufficient condition of the solution of the problem is the self-adjoint condition of the $\hat{f}$-operator

$$\frac{\partial f^{\alpha}}{\partial u_{(s)}^{\beta}}=\left(-1\right)^{s+r}C_{s+r}^{s}\hat{D}_{(r)}\frac{\partial f^{\beta}}{\partial u_{(s)(r)}^{\alpha}}$$

or in the matrix notations $\hat{f}_{\beta}^{\alpha}=\left(\hat{f}^{\dagger}\right)_{\beta}^{\alpha}=\left(\hat{f}_{\alpha}^{\beta}\right)^{*}$. The straight calculation yields to

$$\hat{f}=\begin{pmatrix} -S\varphi_t\cos\theta-\beta\left(\vec{\nabla}\varphi\right)^2\cos 2\theta+\beta\Delta & -S\sin\theta\partial_t-\beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right) \\ \begin{matrix} S\theta_t\cos\theta+2\beta\cos 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\theta\right)+\beta\sin 2\theta\Delta\varphi \\ +S\sin\theta\partial_t+\beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right)\end{matrix} & \beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right)+\beta\sin^2\theta\Delta \end{pmatrix},$$

and

$$\hat{f}^{*}=\begin{pmatrix} -S\varphi_t\cos\theta-\beta\left(\vec{\nabla}\varphi\right)^2\cos 2\theta+\beta\Delta & \begin{matrix} S\theta_t\cos\theta+2\beta\cos 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\theta\right)+\beta\sin 2\theta\Delta\varphi \\ +S\sin\theta\partial_t+\beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right)\end{matrix} \\ -S\sin\theta\partial_t-\beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right) & \beta\sin 2\theta\left(\vec{\nabla}\varphi\vec{\nabla}\right)+\beta\sin^2\theta\Delta \end{pmatrix}.$$

Obviously, the claimed condition is fulfilled. The Lagrangian density can be constructed via the homotopy formula[2]

$$L[u]=\sum_{\alpha}u^{\alpha}\left(x\right)\int\limits_{0}^{1}f^{\alpha}\left(x,\lambda u,\lambda u^{'},\ldots\right)d\lambda.$$

Using the explicit form of differential equations it results in

$$\begin{aligned} L \;=\; & -S\left(\theta\varphi_t-\varphi\theta_t\right)\int\limits_{0}^{1}d\lambda\,\lambda\sin\lambda\theta+\beta\theta\Delta\theta\int\limits_{0}^{1}d\lambda\,\lambda \\ & +\beta\left(\varphi\left(\vec{\nabla}\varphi\vec{\nabla}\theta\right)-\frac{\theta}{2}\left(\vec{\nabla}\varphi\right)^2\right)\int\limits_{0}^{1}d\lambda\,\lambda^2\sin 2\lambda\theta+\beta\varphi\Delta\varphi\int\limits_{0}^{1}d\lambda\,\lambda\sin^2\lambda\theta, \end{aligned}$$

or after some manipulations in

$$L = S(\cos\theta - 1)\varphi_t - \frac{\beta}{2}\sin^2\theta\left(\vec{\nabla}\varphi\right)^2 - \frac{\beta}{2}\left(\vec{\nabla}\theta\right)^2$$
$$+ S\hat{D}_t\left[\left(1 - \frac{\sin\theta}{\theta}\right)\varphi\right] + \hat{D}_k\left[\frac{\beta}{2}\theta\vec{\nabla}\theta + \frac{\beta}{\theta^2}C_0\varphi\vec{\nabla}\varphi\right],$$

where

$$C_0 = \frac{1}{4}\left[\theta(\theta - \sin 2\theta) + \sin^2\theta\right].$$

The last two terms present total derivatives and may be dropped.

Acknowledgments

This work was supported by the grant NREC-005 of US CRDF (Civilian Research & Development Foundation).

[1] N.H. Ibragimov, *Transformation Groups applied to Mathematical Physics* (Reidel, Boston, 1985).

[2] P. Olver, *Applications of Lie groups to differential equations* (Springer-Verlag, 1986).

[3] L.V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, 1982).

[4] R. Balakrishnan, A.R. Bishop, *Phys. Rev. B* **40** (1989) 9194.

[5] I.G. Bostrem, A.S. Ovchinnikov, R.F. Egorov, *Phys. Lett. A* **279** (2001) 33.

[6] A.M. Kosevich, B.A. Ivanov, A.S. Kovalev, *Phys. Rep.* **194** (1990) 119.

[7] A.A. Belavin, A.M. Polyakov, *JETP Letters* **22** (1975) 245.