# SPECIAL ELEMENTS IN THE LATTICE OF OVERCOMMUTATIVE SEMIGROUP VARIETIES REVISITED 

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#### Abstract

We completely determine all distributive, codistributive, standard, costandard, and neutral elements in the lattice of overcommutative semigroup varieties, thus correcting a gap contained in [5].


## 1. Introduction

The class of all semigroup varieties forms a lattice under the following naturally defined operations: for varieties $\mathcal{X}$ and $\mathcal{Y}$, their join $\mathcal{X} \vee \mathcal{Y}$ is the variety generated by the set-theoretical union of $\mathcal{X}$ and $\mathcal{Y}$ (as classes of semigroups), while their meet $\mathcal{X} \wedge \mathcal{Y}$ coincides with the class-theoretical intersection of $\mathcal{X}$ and $\mathcal{Y}$. This lattice has been intensively studied for about four decades. A systematic overview of the material accumulated here is given in the recent survey [4].

It is a common knowledge that the lattice SEM of all semigroup varieties is divided into two large sublattices with essentially different properties: the coideal OC of all overcommutative varieties (that is, varieties containing the variety of all commutative semigroups) and the ideal of all periodic varieties (that is, varieties consisting of periodic semigroups).

The global structure of the lattice OC has been revealed by Volkov in [14]. It is proved there that this lattice decomposes into a subdirect product of its certain intervals and each of these intervals is anti-isomorphic to the congruence lattice of a certain unary algebra of a special type (namely, of a so-called $G$-set; a basic information about $G$-sets see in [3], for instance). The exact formulation of this result may be found also in [4, Theorem 5.1]. We do not reproduce this formulation here because we do not use it below.

There are several articles where special elements of different types in the lattice SEM have been examined (see $[2,6-13,15]$ ). We refer an interested reader to [4, Section 14] for an overview of the most part of results obtained in these articles.

Recall that an element $x$ of a lattice $\langle L ; \vee, \wedge\rangle$ is called distributive if

$$
\forall y, z \in L: \quad x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z) ;
$$

[^0]standard if
$$
\forall y, z \in L: \quad(x \vee y) \wedge z=(x \wedge z) \vee(y \wedge z)
$$
neutral if, for all $y, z \in L$, the sublattice of $L$ generated by $x, y$, and $z$ is distributive. Codistributive [costandard] elements are defined dually to distributive [respectively standard] ones. An extensive information about elements of all these five types in abstract lattices may be found in [1, Section III.2], for instance. Note that any [co]standard element is [co]distributive, and an element is neutral if and only if it is standard and costandard simultaneously (see [1, Theorem III.2.5], for instance). On the other hand, a [co]distributive element may be not [co]standard, while a [co]standard element may be not neutral.

A complete description of neutral elements in the lattice SEM has been given in [15, Proposition 4.1] (see also [4, Theorem 14.2]). In [12], all distributive elements in SEM are completely determined. In [11], quite a strong necessary condition for semigroup varieties to be a codistributive element in SEM is obtained. In particular, all varieties with each of these three properties (except the trivial extreme case of the variety $\mathcal{S E} \mathcal{M}$ of all semigroups) turn out to be periodic varieties.

So, an examination of special elements of all the mentioned types in the lattice SEM gives no any information concerning the lattice OC. Aiming to obtain some new knowledge about this lattice, it is natural to investigate its special elements.

Such investigations have been started by the second author in [5]. Five types of special elements (namely, distributive, codistributive, standard, costandard, and neutral elements) in the lattice OC have been considered there. Unfortunately, it turns out that considerations in [5] contain a gap, and the main result of this article is incorrect. Namely, it was proved in [5] that, for an overcommutative semigroup variety, the properties of being a distributive element of OC, of being a codistributive element of OC, of being a standard element of OC, of being a costandard element of OC, and of being a neutral element of OC are equivalent. This result of [5] is true. But, besides that, the main result of [5] contains a list of all overcommutative varieties that possess the five mentioned properties. Unfortunately, this list turns out to be non-complete. All varieties from the list really have all the mentioned properties, but there are many other such varieties. The objective of this article is to give a correct description of distributive, codistributive, standard, costandard, and neutral elements in the lattice OC.

The article is structured as follows. In Section 2, we introduce a necessary notation and formulate the main result of the article (Theorem 2.2). In Section 3, we prove several auxiliary facts. Sections 4 and 5 are devoted to the proof of Theorem 2.2. In Section 6, we show that this theorem can not be improved, in a sense. Finally, in Section 7, we formulate some open problems.

## 2. Preliminaries and summary

We denote by $F$ the free semigroup over a countably infinite alphabet $\left\{x_{1}, x_{2}\right.$, $\left.\ldots, x_{n}, \ldots\right\}$. As usual, elements of $F$ are called words. By $F^{1}$ we denote the semigroup $F$ with the empty word ajoined. The symbol $\equiv$ stands for the
equality relation on $F$ and $F^{1}$. If $u$ is a word, then $\ell(u)$ denotes the length of $u$, $\ell_{i}(u)$ is the number of occurrences of the letter $x_{i}$ in $u, c(u)$ stands for the set of all letters occurring in $u$, and $n(u)=|c(u)|$ is the number of letters occurring in $u$. An identity $u \approx v$ is called balanced if $\ell_{i}(u)=\ell_{i}(v)$ for all $i$. It is a common knowledge that if an overcommutative variety satisfies some identity then this identity is balanced.

Let $m$ and $n$ be integers with $2 \leq m \leq n$. A partition of the number $n$ into $m$ parts is a sequence of positive integers $\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right)$ such that

$$
\ell_{1} \geq \ell_{2} \geq \cdots \geq \ell_{m} \quad \text { and } \quad \sum_{i=1}^{m} \ell_{i}=n
$$

The numbers $\ell_{1}, \ell_{2}, \ldots, \ell_{m}$ are called components of the partition $\lambda$. We denote by $\Lambda_{n, m}$ the set of all partitions of the number $n$ into $m$ parts and by $\Lambda$ the union of the sets $\Lambda_{n, m}$ for all natural numbers $m$ and $n$ with $2 \leq m \leq n$. If $\lambda \in \Lambda_{n, m}$ then we denote the numbers $n$ and $m$ by $n(\lambda)$ and $m(\lambda)$ respectively.

If $u$ is a word then we denote by part $(u)$ the partition of the number $\ell(u)$ into $n(u)$ parts consisting of integers $\ell_{i}(u)$ for all $i$ such that $x_{i} \in c(u)$ (the numbers $\ell_{i}(u)$ are placed in $\operatorname{part}(u)$ in non-increasing order). If $u \approx v$ is a balanced identity then, obviously, $\ell(u)=\ell(v), n(u)=n(v)$, and part $(u)=\operatorname{part}(v)$. We call the partition $\operatorname{part}(u)$ a partition of the identity $u \approx v$. We denote the numbers $\ell(u)=\ell(v)$ and $n(u)=n(v)$ by $\ell(u \approx v)$ and $n(u \approx v)$ respectively, and the partition $\operatorname{part}(u)=\operatorname{part}(v)$ by part $(u \approx v)$.

Let $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right) \in \Lambda_{n, m}$. We denote by $W_{n, m, \lambda}$, or simply $W_{\lambda}$, the set of all words $u$ such that $\ell(u)=n, c(u)=\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}, \ell_{i}(u) \geq \ell_{i+1}(u)$ for all $i=1,2, \ldots, m-1$, and $\operatorname{part}(u)=\lambda$. It is evident that every balanced identity $u \approx v$ with $\ell(u \approx v)=n, n(u \approx v)=m$, and part $(u \approx v)=\lambda$ is equivalent to some identity $s \approx t$ where $s, t \in W_{n, m, \lambda}$.

We call sets of the kind $W_{n, m, \lambda}$ transversals. We say that an overcommutative variety $\mathcal{V}$ reduces [collapses] a transversal $W_{n, m, \lambda}$ if $\mathcal{V}$ satisfies some non-trivial identity [all identities] of the kind $u \approx v$ with $u, v \in W_{n, m, \lambda}$. An overcommutative variety $\mathcal{V}$ is said to be greedy if it collapses any transversal it reduces. The following assertion has been proved in [5].

Proposition 2.1. An overcommutative semigroup variety is a distributive [codistributive, standard, costandard, neutral] element of the lattice OC if and only if it is greedy.

This assertion was not formulated in [5] explicitly but it directly follows from the proof of Theorem 2 in [5] (and the corresponding part of the proof in [5] is correct).

It is an appropriate place here to indicate the error made in [5]. Let $m$ and $n$ be positive integers with $2 \leq m \leq n$ and $\lambda \in \Lambda_{n, m}$. A semigroup variety given
by an identity system $\Sigma$ is denoted by var $\Sigma$. We put

$$
\begin{aligned}
& \mathcal{X}_{n}=\operatorname{var}\{u \approx v \mid \text { the identity } u \approx v \text { is balanced and } \ell(u \approx v) \geq n\}, \\
& \mathcal{X}_{n, m}=\mathcal{X}_{n+1} \wedge \operatorname{var}\{u \approx v \mid \quad \text { the identity } u \approx v \text { is balanced, } \ell(u \approx v)=n, \\
& \quad \text { and } n(u \approx v) \leq m\}, \\
& \mathcal{X}_{n, 1}=\mathcal{X}_{n+1}, \\
& \mathcal{X}_{n, m, \lambda}=\mathcal{X}_{n, m-1} \wedge \operatorname{var}\left\{u \approx v \mid u, v \in W_{n, m, \lambda}\right\} .
\end{aligned}
$$

It is claimed in [5] without any proof that an overcommutative variety is greedy if and only if it coincides with one of the varieties $\mathcal{S E M}, \mathcal{X}_{n}, \mathcal{X}_{n, m}$ or $\mathcal{X}_{n, m, \lambda}$. Combining this claim with Proposition 2.1, we obtain the main result of [5]: the varieties $\mathcal{S E} \mathcal{M}, \mathcal{X}_{n}, \mathcal{X}_{n, m}, \mathcal{X}_{n, m, \lambda}$, and only they are [co]distributive, [co]standard, and neutral elements in OC. In actual fact, it is true that all these varieties are elements of the mentioned types in OC. But the list of [co]distributive, [co]standard, and neutral elements in OC is not exhausted by the varieties $\mathcal{S E M}, \mathcal{X}_{n}, \mathcal{X}_{n, m}$, and $\mathcal{X}_{n, m, \lambda}$. There are many other varieties with such a property. Exactly this fact has been so unfortunately overseen in [5].

For a partition $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right) \in \Lambda_{n, m}$, we define numbers $q(\lambda), r(\lambda)$, and $s(\lambda)$ by the following way:
$q(\lambda)$ is the number of $\ell_{i}$ 's with $\ell_{i}=1$ (if $\ell_{m}>1$ then $q(\lambda)=0$ );
$r(\lambda)$ is the sum of all $\ell_{i}$ 's with $\ell_{i}>1$ (if $\ell_{1}=1$ then $r(\lambda)=0$ );
$s(\lambda)=\max \{r(\lambda)-q(\lambda)-\delta, 0\}$
where

$$
\delta= \begin{cases}0 & \text { whenever } n=3, m=2, \text { and } \lambda=(2,1) \\ 1 & \text { otherwise }\end{cases}
$$

If $k$ is a non-negative integer then $\lambda^{k}$ stands for the following partition of $n+k$ into $m+k$ parts:

$$
\lambda^{k}=(\ell_{1}, \ell_{2}, \ldots, \ell_{m}, \underbrace{1, \ldots, 1}_{k \text { times }})
$$

(in particular, $\lambda^{0}=\lambda$ ).
For a partition $\lambda \in \Lambda_{n, m}$, we put

$$
\mathcal{W}_{n, m, \lambda}=\operatorname{var}\left\{u \approx v \mid u, v \in W_{n, m, \lambda}\right\} \quad \text { and } \quad \mathcal{S}_{\lambda}=\bigwedge_{i=0}^{s(\lambda)} \mathcal{W}_{n+i, m+i, \lambda^{i}}
$$

Sometimes we will write $\mathcal{W}_{\lambda}$ rather than $\mathcal{W}_{n, m, \lambda}$.
The main result of the article is the following
Theorem 2.2. For an overcommutative semigroup variety $\mathcal{V}$, the following are equivalent:
(i) $\mathcal{V}$ is a distributive element of the lattice $\mathbf{O C}$;
(ii) $\mathcal{V}$ is a codistributive element of the lattice $\mathbf{O C}$;
(iii) $\mathcal{V}$ is a standard element of the lattice $\mathbf{O C}$;
(iv) $\mathcal{V}$ is a costandard element of the lattice $\mathbf{O C}$;
(v) $\mathcal{V}$ is a neutral element of the lattice $\mathbf{O C}$;
(vi) either $\mathcal{V}=\mathcal{S E M}$ or $\mathcal{V}=\bigwedge_{i=1}^{k} \mathcal{S}_{\lambda_{i}}$ for some partitions $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k} \in \Lambda$.

The following claim was formulated in [5] as a corollary of the main result of that article. Theorem 2.2 shows that the claim is correct.

Corollary 2.3. The set of all $[$ co distributive elements of the lattice $\mathbf{O C}$ is countably infinite.

This corollary is of some interest because the set of all overcommutative semigroup varieties is well known to be uncountably infinite. On the other hand, it is interesting to note that the set of all neutral elements in the lattice OC is infinite, while the set of all neutral elements in the lattice SEM consists of 5 varieties only [ 15 , Proposition 4.1].

In view of Proposition 2.1, Theorem 2.2 is equivalent to the following
Proposition 2.4. An overcommutative semigroup variety $\mathcal{V}$ satisfies the condition (vi) of Theorem 2.2 if and only if it is greedy.

It is this claim that will be verified in Sections 4 and 5 (in fact, we prove the 'only if' and 'if' parts of Proposition 2.4 in Sections 4 and 5 respectively). To prepare this proof, we introduce some order relation on the set $\Lambda$ and consider some properties of this relation in Section 3.

## 3. An order relation on the set $\Lambda$

Let $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right) \in \Lambda_{n, m}$ where $m \geq 3$ and $1 \leq i<j \leq m$. We denote by $U_{i, j}(\lambda)$ the partition of the number $n$ into $m-1$ parts with the components $\ell_{1}, \ell_{2}, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_{j-1}, \ell_{j+1}, \ldots, \ell_{m}$, and $\ell_{i}+\ell_{j}$ (these components are written in $U_{i, j}(\lambda)$ in non-increasing order). We will say that the partition $U_{i, j}(\lambda)$ is obtained from $\lambda$ by the union of components $\ell_{i}$ and $\ell_{j}$. The partitioin obtained from $\lambda$ by a finite (may be empty) set $S$ of unions of components is denoted by $U_{S}(\lambda)$; in particular, $U_{\varnothing}(\lambda)=\lambda$.

We introduce a binary relation $\preceq$ on the set $\Lambda$ by the following rule:

$$
\lambda \preceq \mu \text { if and only if } \mu=U_{S}\left(\lambda^{k}\right) \text { for some } S \text { and } k \text {. }
$$

The principal property of the relation $\preceq$ is given by the following
Lemma 3.1. The relation $\preceq$ is a partial order on the set $\Lambda$.
Proof. Reflexivity of $\preceq$ is evident because $\lambda=U_{\varnothing}\left(\lambda^{0}\right)$. The claim that $\preceq$ is transitive also is evident because if $\mu=U_{S}\left(\lambda^{k}\right)$ and $\nu=U_{T}\left(\mu^{\ell}\right)$ then $\nu=$ $U_{S \cup T}\left(\lambda^{k+\ell}\right)$. To prove that $\preceq$ is antisymmetric, we suppose that $\lambda \preceq \mu$ and $\mu \preceq \lambda$ for some $\lambda, \mu \in \Lambda$. Then $\mu=U_{S}\left(\lambda^{k}\right)$ and $\lambda=U_{T}\left(\mu^{\ell}\right)$ for some $S, T, k$, and $\ell$. Let $n(\lambda)=n$ and $n(\mu)=q$. Then $q=n+k$ and $n=q+\ell$. Therefore, $q=q+k+\ell$, whence $k+\ell=0$. This means that $k=\ell=0$. Thus, $\mu=U_{S}(\lambda)$ and $\lambda=U_{T}(\mu)$. If $S \neq \varnothing$ then $r<m$. But $m \leq r$ because $\lambda$ is obtained from $\mu$ by unions of components. Therefore, $S=\varnothing$ and $\mu=U_{\varnothing}(\lambda)=\lambda$.

Now we are going to show that the partial order $\preceq$ has some nice properties that will be played the crucial role in Section 5 . The first such property is given by the following

Lemma 3.2. The partially ordered set $\langle\Lambda ; \preceq\rangle$ satisfies the descending chain condition.

Proof. Let $\lambda, \mu \in \Lambda$ and $\lambda \preceq \mu$. Put $n(\lambda)=n$ and $n(\mu)=q$. Then $q \leq n$. Evidently, the set

$$
\bigcup_{\substack{q \leq n \\ 2 \leq r \leq q}} \Lambda_{q, r}
$$

is finite. Thus, there exists finitely many partitions $\mu$ with $\mu \preceq \lambda$ only. This immediately implies the desirable conclusion.

We define one more binary relation $\unlhd$ on the set $\Lambda$ by the following rule. Let $\lambda, \nu \in \Lambda, \lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right)$, and $\nu=\left(n_{1}, n_{2}, \ldots, n_{k}\right)$. Then $\lambda \unlhd \nu$ if and only if $m \leq k$ and $\ell_{i} \leq n_{i}$ for all $i=1,2, \ldots, m$. It is evident that $\unlhd$ is a partial order on $\Lambda$. The following claim shows a relationship between orders $\preceq$ and $\unlhd$.

Lemma 3.3. Let $\lambda, \nu \in \Lambda$. If $\lambda \unlhd \nu$ then $\lambda \preceq \nu$.
Proof. Let $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right)$ and $\nu=\left(n_{1}, n_{2}, \ldots, n_{k}\right)$. Then $m \leq k$ and $\ell_{i} \leq n_{i}$ for all $i=1,2, \ldots, m$. Put $s=n(\nu)-n(\lambda)$. It is evident that $s \geq 0$. If $s=0$ then $\lambda=\nu$ and we are done. Let now $s>0$. By the trivial induction, it suffices to consider the case $s=1$. Then either $k=m+1, \ell_{i}=n_{i}$ for all $i=1,2, \ldots, m$, and $n_{k}=1$ or $k=m, n_{i}=\ell_{i}+1$ for some $i \in\{1,2, \ldots, m\}$ and $n_{j}=\ell_{j}$ for all $j \neq i$. It is evident that $\nu=U_{\varnothing}\left(\lambda^{1}\right)$ in the former case, while $\nu=U_{i, m+1}\left(\lambda^{1}\right)$ in the latter one. Thus, $\lambda \preceq \nu$ in any case.

The second important property of the relation $\preceq$ is given by the following
Lemma 3.4. The partially ordered set $\langle\Lambda ; \preceq\rangle$ does not contain infinite antichains.

Proof. Arguing by contradiction, suppose that $\Lambda$ contains an infinite antichain $A_{0}$. Put $m_{1}=\min \left\{m(\lambda) \mid \lambda \in A_{0}\right\}$. Let us fix a partition $\lambda_{1}=$ $\left(\ell_{1}^{1}, \ell_{2}^{1}, \ldots, \ell_{m_{1}}^{1}\right) \in A_{0}$. If $\nu=\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ is an arbitrary partition from $A_{0}$ then $\lambda_{1} \npreceq \nu$, whence $\lambda_{1} \npreceq \nu$ by Lemma 3.3. Since $m_{1} \leq k$, this means that $\ell_{i}^{1}>n_{i}$ for some $i \in\left\{1,2, \ldots, m_{1}\right\}$. The set $A_{0}$ is infinite, while the index $i$ runs over the finite set $\left\{1,2, \ldots, m_{1}\right\}$. Hence there is an index $i_{1} \leq m_{1}$ such that $n_{i_{1}}<\ell_{i_{1}}^{1}$ for an infinite set of partitions $A_{1} \subseteq A_{0}$. Put $j_{1}=\ell_{i_{1}}^{1}$.

Put $m_{2}=\min \left\{m(\lambda) \mid \lambda \in A_{1}\right\}$. Let us fix a partition $\lambda_{2}=\left(\ell_{1}^{2}, \ell_{2}^{2}, \ldots, \ell_{m_{2}}^{2}\right) \in$ $A_{1}$. The same arguments as in the previous paragraph show that there is a number $i_{2} \leq m_{2}$ and an infinite set $A_{2} \subseteq A_{1}$ such that $n_{i_{2}}<\ell_{i_{2}}^{2}$ for every $\nu=\left(n_{1}, n_{2}, \ldots, n_{k}\right) \in A_{2}$. Put $j_{2}=\ell_{i_{2}}^{2}$.

Continuing this process, we construct a sequence of infinite sets of partitions $A_{0} \supseteq A_{1} \supseteq A_{2} \supseteq \cdots$, a sequence of partitions $\left\{\lambda_{s}=\left(\ell_{1}^{s}, \ell_{2}^{s}, \ldots, \ell_{m_{s}}^{s}\right) \mid s \in \mathbb{N}\right\}$, and two sequences of numbers $\left\{i_{s} \mid s \in \mathbb{N}\right\}$ and $\left\{j_{s} \mid s \in \mathbb{N}\right\}$ such that, for any $s \in \mathbb{N}$, the following holds: $\lambda_{s} \in A_{s-1}, i_{s} \leq m_{s}, j_{s}=\ell_{i_{s}}^{s}$, and $n_{i_{s}}<j_{s}$ for any $\nu=\left(n_{1}, n_{2}, \ldots, n_{k}\right) \in A_{s}$. The choice of the partitions $\lambda_{1}, \lambda_{2}, \ldots$ guarantees that if $p>q$ then

$$
\ell_{i_{q}}^{p}<\ell_{i_{q}}^{q}=j_{q} .
$$

In particular, if $p>q$ and $i_{p}=i_{q}$ then

$$
j_{p}=\ell_{i_{p}}^{p}=\ell_{i_{q}}^{p}<j_{q} .
$$

This means that all pairs of the kind $\left(i_{s}, j_{s}\right)$ are different. Furthermore, if $i_{p} \geq i_{q}$ then $\ell_{i_{p}}^{p} \leq \ell_{i_{q}}^{p}$ because all partitions $\lambda_{1}, \lambda_{2}, \ldots$ are non-increasing sequences of numbers. Therefore, if $p>q$ and $i_{p} \geq i_{q}$ then

$$
j_{p}=\ell_{i_{p}}^{p} \leq \ell_{i_{q}}^{p}<j_{q} .
$$

Put $i_{r}=\min \left\{i_{s} \mid s \in \mathbb{N}\right\}, j_{t}=\min \left\{j_{s} \mid s \in \mathbb{N}\right\}$, and $h=\max \{r, t\}$. If $s>h$ then $i_{s} \geq i_{r}$ and $j_{s} \geq j_{t}$, whence $j_{s}<j_{r}$ and $i_{s}<i_{t}$. We see that both the sequences $\left\{i_{s} \mid s \in \mathbb{N}\right\}$ and $\left\{j_{s} \mid s \in \mathbb{N}\right\}$ are bounded. But this is impossible because all pairs of the kind $\left(i_{s}, j_{s}\right)$ are different. The contradiction completes the proof.

## 4. Proof of Proposition 2.4: Necessity

Here we aims to verify that if an overcommutative variety satisfies the condition (vi) of Theorem 2.2 then it is greedy. We start with some new notation and several auxiliary facts.

For arbitrary words $w_{1}, w_{2}$ and an identity system $\Sigma$, we write $w_{1} \xrightarrow{\Sigma} w_{2}$ if there exist $a, b \in F^{1}, s, t \in F$, and an endomorphism $\zeta$ on $F$ such that $w_{1} \equiv a \zeta(s) b, w_{2} \equiv a \zeta(t) b$, and the identity $s \approx t$ belongs to $\Sigma$. It is a common knowledge that an identity $u \approx v$ follows from a system $\Sigma$ if and only if there exists a sequence of words $w_{0}, w_{1}, \ldots, w_{\ell}$ such that

$$
\begin{equation*}
u \equiv w_{0} \xrightarrow{\Sigma} w_{1} \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} w_{\ell} \equiv v . \tag{4.1}
\end{equation*}
$$

This sequence is called a deduction of the identity $u \approx v$ from $\Sigma$. Note that if $\Sigma$ consists of balanced identities then $\ell\left(w_{i}\right)=\ell(u \approx v), n\left(w_{i}\right)=n(u \approx v)$, and $\operatorname{part}\left(w_{i}\right)=\operatorname{part}(u \approx v)$ for all $i=0,1, \ldots, \ell$.
Lemma 4.1. Let $u$ be a word and $\xi$ an endomorphism on $F$ such that $\ell(\xi(u))=$ $\ell(u)$. Then $\operatorname{part}(u) \preceq \operatorname{part}(\xi(u))$.
Proof. Put $\lambda=\operatorname{part}(u)$. It is clear that $\xi(x)$ is a letter for every letter $x$. The requirement conclusion follows from the following evident observation: $\operatorname{part}(\xi(u))=U_{S}\left(\lambda^{0}\right)$ where $S$ is a finite (may be empty) set of unions of components of $\lambda$ corresponding to letters from $c(u)$ with the same image under $\xi$.

Lemma 4.2. Let $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right) \in \Lambda_{n, m}$. If a non-trivial identity $u \approx v$ holds in the variety $S_{\lambda}$ then $\lambda \preceq \operatorname{part}(u \approx v)$.
Proof. We put

$$
\Sigma=\left\{f \approx g \mid \text { there is } i \in\{0,1, \ldots, s(\lambda)\} \text { with } f, g \in W_{n+i, m+i, \lambda^{i}}\right\} .
$$

Thus, $S_{\lambda}=\operatorname{var} \Sigma$. Let (4.1) be a deduction of the identity $u \approx v$ from $\Sigma$. Note that $\ell \geq 1$ because the identity $u \approx v$ is non-trivial. We have $w_{0} \equiv a \zeta(s) b$ and $w_{1} \equiv a \zeta(t) b$ for some homomorphism $\zeta$ on $F$, some $a, b \in F^{1}$, and some $s, t \in W_{n+i, m+i, \lambda^{i}}$ where $i \in\{0,1, \ldots, s(\lambda)\}$.

If $\ell(a \zeta(s) b)=\ell(s)$ then the words $a$ and $b$ are empty and $\ell(\zeta(s))=\ell(s)$. Here we may apply Lemma 4.1 and conclude that

$$
\begin{aligned}
\lambda \preceq U_{\varnothing}\left(\lambda^{i}\right) & =\lambda^{i}=\operatorname{part}(s) \preceq \operatorname{part}(\zeta(s))= \\
& =\operatorname{part}(a \zeta(s) b)=\operatorname{part}\left(w_{0}\right)=\operatorname{part}(u \approx v) .
\end{aligned}
$$

Therefore, $\lambda \preceq \operatorname{part}(u \approx v)$, and we are done.
Suppose now that $\ell(a \zeta(s) b)>\ell(s)$. For each $j=1,2, \ldots, m+i$, we denote by $y_{j}$ the first letter of the word $\zeta\left(x_{j}\right)$. Thus, $\zeta\left(x_{j}\right) \equiv y_{j} u_{j}$ for some $u_{j} \in F^{1}$. We have

$$
\begin{aligned}
\operatorname{part}(a \zeta(s) b) & =\operatorname{part}\left(a\left(\zeta\left(x_{1}\right)\right)^{\ell_{1}} \cdots\left(\zeta\left(x_{m}\right)\right)^{\ell_{m}} \zeta\left(x_{m+1}\right) \cdots \zeta\left(x_{m+i}\right) b\right)= \\
& =\operatorname{part}\left(a\left(y_{1} u_{1}\right)^{\ell_{1}} \cdots\left(y_{m} u_{m}\right)^{\ell_{m}} y_{m+1} u_{m+1} \cdots y_{m+i} u_{m+i} b\right)= \\
& =\operatorname{part}\left(y_{1}^{\ell_{1}} \cdots y_{m}^{\ell_{m}} y_{m+1} \cdots y_{m+i} u_{1}^{\ell_{1}} \cdots u_{m}^{\ell_{m}} u_{m+1} \cdots u_{m+i} a b\right)
\end{aligned}
$$

Let $k=\ell(a \zeta(s) b)-\ell(s)$. Put

$$
\begin{aligned}
c_{1} & \equiv y_{1}^{\ell_{1}} \cdots y_{m}^{\ell_{m}} y_{m+1} \cdots y_{m+i} \\
c_{2} & \equiv u_{1}^{\ell_{1}} u_{2}^{\ell_{2}} \cdots u_{m}^{\ell_{m}} u_{m+1} \cdots u_{m+i} a b \\
c & \equiv c_{1} c_{2} \equiv y_{1}^{\ell_{1}} \cdots y_{m}^{\ell_{m}} y_{m+1} \cdots y_{m+i} u_{1}^{\ell_{1}} \cdots u_{m}^{\ell_{m}} u_{m+1} \cdots u_{m+i} a b \\
d & \equiv x_{1}^{\ell_{1}} \cdots x_{m}^{\ell_{m}} x_{m+1} \cdots x_{m+i+k}
\end{aligned}
$$

Since $\operatorname{part}(c)=\operatorname{part}(a \zeta(s) b)$ and $\operatorname{part}\left(c_{1}\right)=\operatorname{part}(s)$, we have $\ell(c)=\ell(a \zeta(s) b)$ and $\ell\left(c_{1}\right)=\ell(s)$. Besides that, $\ell(c)=\ell\left(c_{1}\right)+\ell\left(c_{2}\right)$. Therefore,

$$
\ell\left(c_{2}\right)=\ell(c)-\ell\left(c_{1}\right)=\ell(a \zeta(s) b)-\ell(s)=k
$$

It is convenient for us to rewrite the word $c_{2}$ in the form $c_{2} \equiv z_{1} z_{2} \ldots z_{k}$ where $z_{1}, z_{2}, \ldots, z_{k}$ are (not necessarily different) letters. Let $\xi$ be an endomorphism on $F$ such that

$$
\xi\left(x_{j}\right) \equiv \begin{cases}y_{j} & \text { whenever } 1 \leq j \leq m+i \\ z_{j-m-i} & \text { whenever } m+i+1 \leq j \leq m+i+k\end{cases}
$$

Then $c \equiv \xi(d)$. It is clear that

$$
\ell(d)=\ell\left(c_{1}\right)+k=\ell(s)+\ell(a \zeta(s) b)-\ell(s)=\ell(a \zeta(s) b)=\ell(c)=\ell(\xi(d)) .
$$

Now we may apply Lemma 4.1 and conclude that

$$
\begin{aligned}
\lambda \preceq U_{\varnothing}\left(\lambda^{k}\right) & =\lambda^{k}=\operatorname{part}(d) \preceq \operatorname{part}(\xi(d))=\operatorname{part}(c)= \\
& =\operatorname{part}(a \zeta(s) b)=\operatorname{part}\left(w_{0}\right)=\operatorname{part}(u \approx v) .
\end{aligned}
$$

Therefore, $\lambda \preceq \operatorname{part}(u \approx v)$, and we are done.
Lemma 4.3. Let $\lambda, \mu \in \Lambda$. If $\mu=U_{S}(\lambda)$ for some finite set $S$ of unions of components then $\mathcal{W}_{\lambda} \subseteq \mathcal{W}_{\mu}$.
Proof. Let $\lambda=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{m}\right) \in \Lambda_{n, m}$. By the trivial induction, it suffices to consider the case when $\mu=U_{i, j}(\lambda)$ for some $i$ and $j$. We have to verify that if $u, v \in W_{n, m-1, \mu}$ then the identity $u \approx v$ holds in $\mathcal{W}_{\lambda}$. Since part $(u \approx v)=$ $U_{i, j}(\lambda)$, there is a letter $x_{k}$ with $\ell_{k}(u)=\ell_{k}(v)=\ell_{i}+\ell_{j}$. Let $x_{p}$ and $x_{q}$ be some letters such that $x_{p}, x_{q} \notin c(u)$. One can change the first $\ell_{i}$ occurences of $x_{k}$ in
$u$ and in $v$ by $x_{p}$, while the last $\ell_{j}$ occurences of $x_{k}$ in $u$ and in $v$ by $x_{q}$. We obtain some identity $s \approx t$ with part $(s \approx t)=\lambda$. Hence the variety $\mathcal{W}_{n, m, \lambda}$ satisfies $s \approx t$. If we substitute $x_{k}$ for $x_{p}$ and $x_{q}$ in $s \approx t$ then we return to the identity $u \approx v$. Therefore, $u \approx v$ follows from $s \approx t$, whence $u \approx v$ holds in $\mathcal{W}_{n, m, \lambda}$.

Recall that a letter $x_{i}$ is called simple in a word $u$ if $\ell_{i}(u)=1$.
Lemma 4.4. If $\lambda \in \Lambda_{n, m}$ and $s(\lambda)=0$ then $\mathcal{W}_{n, m, \lambda} \subseteq \mathcal{W}_{n+k, m+k, \lambda^{k}}$ for any positive integer $k$.

Proof. The definition of the number $s(\lambda)$ immediately implies that if $s(\lambda)=0$ then $s\left(\lambda^{k}\right)=0$ as well for any $k>0$. This observation implies that, by the trivial induction, it suffices to consider the case $k=1$.

First of all, we note that $\lambda \neq(2,1)$ because $s(\lambda)=1$ otherwise. For brevity, put $q=q(\lambda), r=r(\lambda), s=s(\lambda)$, and $t=m-q$. Since $\lambda \neq(2,1)$, we have $\delta=1$ and therefore, $s=\max \{r-q-1,0\}$. The equality $s=0$ implies now that $r-q-1 \leq 0$, that is

$$
\begin{equation*}
r \leq q+1 \tag{4.2}
\end{equation*}
$$

Suppose that $q \leq 1$. Then $r \leq 2$. Let $\lambda=\left(\ell_{1} \cdot \ell_{2}, \ldots, \ell_{m}\right)$. The definition of the number $r(\lambda)$ and the inequality $r \leq 2$ imply that either $\ell_{1}=\ell_{2}=\cdots=\ell_{m}=1$ or $\ell_{1}=2$ and $\ell_{2}=\cdots=\ell_{m}=1$. Since $\lambda \neq(2,1)$, this implies that $q \geq 2$. We have a contradiction with the inequality $q \leq 1$. Therefore, $q \geq 2$. This implies that every word from the transversal $W_{n+1, m+1, \lambda^{1}}$ has at least three simple letters.

We need to verify that any identity of the kind $u \approx v$ with $u, v \in W_{n+1, m+1, \lambda^{1}}$ holds in $W_{n, m, \lambda}$. It suffices to check that if $u \in W_{n+1, m+1, \lambda^{1}}$ then the variety $W_{n, m, \lambda}$ satisfies the identity

$$
\begin{equation*}
u \approx x_{1}^{\ell_{1}} \cdots x_{t}^{\ell_{t}} x_{t+1} \cdots x_{m+1} . \tag{4.3}
\end{equation*}
$$

At the rest part of the proof of this lemma, the words 'a simple letter' mean 'a simple in $u$ letter'. One can note that one of the following three claims hold:

1) the word $u$ ends with a simple letter;
2) the word $u$ starts with a simple letter;
3) the word $u$ contains a subword of the kind $x_{i} x_{j}$ where $x_{i}$ and $x_{j}$ are simple letters.
Indeed, if all these three claims fail then

$$
u \equiv w_{1} y_{1} w_{2} y_{2} \cdots w_{q+1} y_{q+1} w_{q+2}
$$

where $y_{1}, y_{2}, \ldots, y_{q+1}$ are simple letters, while $w_{1}, w_{2}, \ldots, w_{q+2}$ are non-empty words such that the word $w \equiv w_{1} w_{2} \cdots w_{q+2}$ does not contain simple letters.
Then $r=\ell(w)=\sum_{i=1}^{q+2} \ell\left(w_{i}\right) \geq q+2$, contradicting the inequality (4.2).
Now we consider three cases corresponding to the claims 1)-3).
Case 1: $u \equiv w x_{i}$ for some word $w$ and some simple letter $x_{i}$. The identity

$$
\begin{equation*}
w \approx x_{1}^{\ell_{1}} \cdots x_{t}^{\ell_{t}} x_{t+1} \cdots x_{i-1} x_{i+1} \cdots x_{m+1} \tag{4.4}
\end{equation*}
$$

has the partition $\lambda$, whence it holds in $\mathcal{W}_{n, m, \lambda}$. Multiplying (4.4) by $x_{i}$ from the right, we have the identity

$$
\begin{equation*}
u \approx x_{1}^{\ell_{1}} \cdots x_{t}^{\ell_{t}} x_{t+1} \cdots x_{i-1} x_{i+1} \cdots x_{m+1} x_{i} \tag{4.5}
\end{equation*}
$$

that also holds in $\mathcal{W}_{n, m, \lambda}$. If $i=m+1$ then (4.5) coincides with (4.3) and we are done. Let now $i \leq m$. Put

$$
j= \begin{cases}m-1 & \text { whenever } i=m \\ m & \text { otherwise }\end{cases}
$$

Since $u$ contains at least three simple letters, the letter $x_{j}$ is simple. The identity (4.5) has the form

$$
\begin{equation*}
u \approx a x_{j} x_{m+1} x_{i} \tag{4.6}
\end{equation*}
$$

for some $a \in F^{1}$. Let $x_{p}$ be a letter with $x_{p} \notin c(u)$. The identity

$$
\begin{equation*}
a x_{p} x_{i} \approx a x_{i} x_{p} \tag{4.7}
\end{equation*}
$$

has the partition $\lambda$, whence it holds in $\mathcal{W}_{n, m, \lambda}$. Substituting $x_{j} x_{m+1}$ for $x_{p}$ in (4.7), we obtain the identity

$$
\begin{equation*}
a x_{j} x_{m+1} x_{i} \approx a x_{i} x_{j} x_{m+1} \tag{4.8}
\end{equation*}
$$

that holds in $\mathcal{W}_{n, m, \lambda}$. The identity

$$
\begin{equation*}
a x_{i} x_{j} \approx x_{1}^{\ell_{1}} \cdots x_{t}^{\ell_{t}} x_{t+1} \cdots x_{m} \tag{4.9}
\end{equation*}
$$

has the partition $\lambda$, whence it holds in $\mathcal{W}_{n, m, \lambda}$ too. Multiplying (4.9) on $x_{m+1}$ from the right, we obtain the identity

$$
\begin{equation*}
a x_{i} x_{j} x_{m+1} \approx x_{1}^{\ell_{1}} \cdots x_{t}^{\ell_{t}} x_{t+1} \cdots x_{m+1} \tag{4.10}
\end{equation*}
$$

that holds in $\mathcal{W}_{n, m, \lambda}$ as well. Combining the identities (4.6), (4.8), and (4.10), we obtain the identity (4.3).

Case 2: $u \equiv x_{i} w$ for some simple letter $x_{i}$ and some word $w$. The word $u$ contains at least three simple letters. Therefore, there is a simple letter $x_{j} \in c(w)$. Thus, $w \equiv a x_{j} b$ for some $a, b \in F^{1}$. The identity

$$
\begin{equation*}
a x_{j} b \approx a b x_{j} \tag{4.11}
\end{equation*}
$$

has the partition $\lambda$, whence it holds in $\mathcal{W}_{n, m, \lambda}$. Multiplying (4.11) on $x_{i}$ from the left, we obtain the identity $u \approx x_{i} a b x_{j}$ that also holds in $\mathcal{W}_{n, m, \lambda}$. We come to the situation considered in Case 1 .

Case 3: $u \equiv a x_{i} x_{j} b$ for some $a, b \in F^{1}$ and some simple letters $x_{i}$ and $x_{j}$. Let $x_{p}$ be a letter with $x_{p} \notin c(u)$. The identity

$$
\begin{equation*}
a x_{p} b \approx a b x_{p} \tag{4.12}
\end{equation*}
$$

has the partition $\lambda$, whence it holds in $\mathcal{W}_{n, m, \lambda}$. Substituting $x_{i} x_{j}$ for $x_{p}$ in (4.12), we obtain the identity $u \approx a b x_{i} x_{j}$ that holds in $\mathcal{W}_{n, m, \lambda}$. We come to the situation considered in Case 1 again.
Corollary 4.5. If $\lambda \in \Lambda$ then $\mathcal{S}_{\lambda} \subseteq \mathcal{W}_{\lambda^{k}}$ for any $k \geq 0$.
Proof. If $k \leq s(\lambda)$ then the desired inclusion holds by the definition of the variety $\mathcal{S}_{\lambda}$. Let now $k>s(\lambda)$. It is easy to see that $s\left(\lambda^{s(\lambda)}\right)=0$. Now we may apply Lemma 4.4 and conclude that $\mathcal{S}_{\lambda} \subseteq \mathcal{W}_{\lambda^{s(\lambda)}} \subseteq \mathcal{W}_{\lambda^{k}}$.

Proposition 4.6. If $\lambda \in \Lambda$ then the variety $\mathcal{S}_{\lambda}$ is greedy.
Proof. Suppose that $\mu \in \Lambda$ and the variety $\mathcal{S}_{\lambda}$ reduces the transversal $W_{\mu}$, that is $\mathcal{S}_{\lambda}$ satisfies some non-trivial identity $u \approx v$ with $u, v \in W_{\mu}$. Lemma 4.2 implies that $\lambda \preceq \mu$, that is $\mu=U_{S}\left(\lambda^{k}\right)$ for some $S$ and $k$. Applying Corollary 4.5 and Lemma 4.3, we have $\mathcal{S}_{\lambda} \subseteq \mathcal{W}_{\lambda^{k}} \subseteq \mathcal{W}_{\mu}$. Thus, if $s, t \in W_{\mu}$ then the identity $s \approx t$ holds in $\mathcal{S}_{\lambda}$. This means that $\mathcal{S}_{\lambda}$ collapses $W_{\mu}$. We see that the variety $\mathcal{S}_{\lambda}$ collapses any transversal it reduces, that is $\mathcal{S}_{\lambda}$ is greedy.

Now we are well prepared to prove the 'only if' part of Proposition 2.4. Let an overcommutative variety $\mathcal{V}$ satisfy the condition (vi) of Theorem 2.2 . We need to verify that $\mathcal{V}$ is greedy. It is evident that the variety $\mathcal{S E M}$ is greedy because it does not reduce any transversal. Let now $\mathcal{V}=\bigwedge_{i=1}^{k} \mathcal{S}_{\lambda_{i}}$ for some partitions $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$. By Proposition 4.6, the varieties $\mathcal{S}_{\lambda_{1}}{ }^{i=1} \mathcal{S}_{\lambda_{2}}, \ldots, \mathcal{S}_{\lambda_{k}}$ are greedy. Proposition 2.1 implies now that all these varieties are neutral elements of the lattice OC. It is well known that the set of all neutral elements of a lattice $L$ forms a sublattice of $L$ (see [1, Theorem III.2.9], for instance). Therefore, $\mathcal{V}$ is a neutral element of OC. Now we may apply Proposition 2.1 again and conclude that $\mathcal{V}$ is greedy.

## 5. Proof of Proposition 2.4: Sufficiency

Here we are going to verify that a greedy overcommutative variety satisfies the condition (vi) of Theorem 2.2, thus completing the proof of Proposition 2.4 and therefore, of Theorem 2.2. We start with a few easy observations.

Lemma 5.1. If an overcommutative semigroup variety $\mathcal{V}$ reduces (in particular, collapses) a transversal $W_{\lambda}$ then $\mathcal{V}$ reduces transversals $W_{\lambda^{k}}$ for all $k \geq 0$.
Proof. The case $k=0$ is obvious because $W_{\lambda^{0}}=W_{\lambda}$. Let now $k>0$. Suppose that $\mathcal{V}$ satisfies a non-trivial identity of the kind $u \approx v$ with $u, v \in W_{\lambda}$. Let $y_{1}, \ldots, y_{k}$ be letters with $y_{1}, \ldots, y_{k} \notin c(u)$. The identity $u y_{1} \cdots y_{k} \approx v y_{1} \cdots y_{k}$ is non-trivial and holds in $\mathcal{V}$ because it follows from $u \approx v$. Since

$$
\operatorname{part}\left(u y_{1} \cdots y_{k} \approx v y_{1} \cdots y_{k}\right)=\lambda^{k}
$$

$\mathcal{V}$ reduces $W_{\lambda^{k}}$.
Lemma 5.2. Let $\mathcal{V}$ be a greedy variety. If a non-trivial identity $u \approx v$ holds in $\mathcal{V}$ and $\operatorname{part}(u \approx v)=\lambda$ then $\mathcal{V} \subseteq \mathcal{S}_{\lambda}$.

Proof. By Lemma 5.1, $\mathcal{V}$ reduces transversals $W_{\lambda^{k}}$ for all $k=0,1, \ldots, s(\lambda)$. Being greedy, $\mathcal{V}$ collapses all these transversals. Therefore, $\mathcal{V} \subseteq \mathcal{S}_{\lambda}$.

Corollary 5.3. Let $\lambda, \mu \in \Lambda$. Then $\mathcal{S}_{\lambda} \subseteq \mathcal{S}_{\mu}$ if and only if $\lambda \preceq \mu$.
Proof. Necessity. Suppose that $\mathcal{S}_{\lambda} \subseteq \mathcal{S}_{\mu}$. Let $u \approx v$ be an identity with part $(u \approx$ $v)=\mu$. Then $u \approx v$ holds in $\mathcal{S}_{\mu}$, whence it holds in $\mathcal{S}_{\lambda}$. Now Lemma 4.2 applies with the conclusion that $\lambda \preceq \mu$.

Sufficiency. Let $\lambda \preceq \mu$ and $u \approx v$ an identity with $\operatorname{part}(u \approx v)=\lambda$. Since $\lambda \preceq \mu$, there is an identity $s \approx t$ such that $u \approx v$ implies $s \approx t$ and $\operatorname{part}(s \approx t)=\mu$. The variety $\mathcal{S}_{\lambda}$ satisfies the identity $u \approx v$. Hence $s \approx t$ holds
in $\mathcal{S}_{\lambda}$ as well. According to Proposition 4.6, the variety $\mathcal{S}_{\lambda}$ is greedy. Now Lemma 5.2 succsessfully applies with the conclusion that $\mathcal{S}_{\lambda} \subseteq \mathcal{S}_{\mu}$.

Now we are ready to prove the 'if' part of Proposition 2.4. Let $\mathcal{V}$ be a greedy variety and $\mathcal{V} \neq \mathcal{S E} \mathcal{M}$. The last unequality means that $\mathcal{V}$ satisfies some nontrivial identity $u \approx v$. Put $\lambda=\operatorname{part}(u \approx v)$. Lemma 5.2 shows that $\mathcal{V} \subseteq \mathcal{S}_{\lambda}$. Thus, the set $\Gamma=\left\{\lambda \in \Lambda \mid \mathcal{V} \subseteq \mathcal{S}_{\lambda}\right\}$ is non-empty. Put $\mathcal{X}=\bigwedge_{\lambda \in \Gamma} \mathcal{S}_{\lambda}$. Clearly, $\mathcal{V} \subseteq \mathcal{X}$. Suppose that $\mathcal{V} \neq \mathcal{X}$. Then there is an identity $u \approx v$ that holds in $\mathcal{V}$ but fails in $\mathcal{X}$. Let $\mu=\operatorname{part}(u \approx v)$. By Lemma $5.2, \mathcal{V} \subseteq \mathcal{S}_{\mu}$. This means that $\mu \in \Gamma$, whence $\mathcal{X} \subseteq \mathcal{S}_{\mu}$. Since $\mathcal{W}_{\mu}$ satisfies $u \approx v$ and $\mathcal{X} \subseteq \mathcal{S}_{\mu} \subseteq \mathcal{W}_{\mu}$, we have that the identity $u \approx v$ holds in $\mathcal{X}$. A contradiction shows that $\mathcal{V}=\mathcal{X}$.

Lemma 3.2 and Corollary 5.3 imply together that $\mathcal{V}=\bigwedge_{\lambda \in \Gamma^{\prime}} \mathcal{S}_{\lambda}$ where $\Gamma^{\prime}$ is the set of all minimal elements of the partially ordered set $\langle\Gamma ; \preceq\rangle$. Since $\Gamma^{\prime}$ forms an anti-chain in $\langle\Lambda ; \preceq\rangle$, Lemma 3.4 implies that the set $\Gamma^{\prime}$ is finite. Thus, $\mathcal{V}$ satisfies the condition (vi) of Theorem 2.2.

Proposition 2.4 and Theorem 2.2 are proved.

## 6. Additional remarks

Here we are going to show that the description of the varieties under consideration given by Theorem 2.2 may not be improved, in a sense. Theorem 2.2 shows that the varieties of the kind $\mathcal{S}_{\lambda}$ play the crucial role in the description of varieties we consider in this article. Recall that the variety $\mathcal{S}_{\lambda}$ is defined as the intersection of the varieties $\mathcal{W}_{\lambda^{i}}$ where $i$ runs over the set $\{0,1, \ldots, s(\lambda)\}$. A natural question arises, whether or not the number $s(\lambda)$ may be changed on some lesser number here.

For any $\lambda \in \Lambda$ and $k \in\{0,1, \ldots, s(\lambda)\}$, we put

$$
\mathcal{S}_{\lambda}^{k}=\bigwedge_{i=0}^{k} \mathcal{W}_{\lambda^{i}}
$$

In particular, $\mathcal{S}_{\lambda}^{0}=\mathcal{W}_{\lambda}$ and $\mathcal{S}_{\lambda}^{s(\lambda)}=\mathcal{S}_{\lambda}$. The crucial property of the variety $\mathcal{S}_{\lambda}$ is given by Proposition 4.6: this variety is greedy. The following statement together with Lemma 5.1 show that varieties $\mathcal{S}_{\lambda}^{k}$ with $k<s(\lambda)$ does not have this property. Thus, the question posed in the previous paragraph is answered in negative.

Proposition 6.1. Let $\lambda \in \Lambda, s(\lambda)>0$, and $0 \leq k<s(\lambda)$. Then the variety $\mathcal{S}_{\lambda}^{k}$ does not collapse the transversals $W_{\lambda^{k+1}}, W_{\lambda^{k+2}}, \ldots, W_{\lambda^{s(\lambda)}}$.
Proof. Let $i \in\{k+1, \ldots, s(\lambda)\}$. Suppose that $\mathcal{S}_{\lambda}^{k}$ collapses the transversal $W_{\lambda^{i}}$. Further considerations are divided into two cases.

Case 1: $\lambda \neq(2,1)$. The definition of the number $s(\lambda)$ and the inequality $s(\lambda)>0$ imply that $s(\lambda)=r(\lambda)-q(\lambda)-1$ here. Since $s(\lambda) \geq i$, we have $r(\lambda)-q(\lambda)-1 \geq i$. Evident equalities $r\left(\lambda^{i}\right)=r(\lambda)$ and $q\left(\lambda^{i}\right)=q(\lambda)+i$ then imply that $r\left(\lambda^{i}\right) \geq q\left(\lambda^{i}\right)+1$. Hence the transversal $W_{\lambda^{i}}$ contains a word $u$ of the kind

$$
u \equiv w_{1} y_{1} w_{2} y_{2} \cdots w_{q+i} y_{q+i} w_{q+i+1}
$$

where $y_{1}, y_{2}, \ldots, y_{q+i}$ are simple in $u$ letters, while $w_{1}, w_{2}, \ldots, w_{q+i+1}$ are nonempty words such that the word $w_{1} w_{2} \cdots w_{q+i+1}$ does not contain simple in $u$ letters. Let $v \in W_{\lambda^{i}}$ and $v \not \equiv u$. Since $\mathcal{S}_{\lambda}^{k}$ collapses $W_{\lambda^{i}}$, the identity $u \approx v$ holds in $S_{\lambda}^{k}$. Therefore, this identity follows from the identity system

$$
\Sigma=\left\{g \approx h \mid \text { there is } j \in\{0,1, \ldots, k\} \text { with } g, h \in W_{\lambda^{j}}\right\} .
$$

Let (4.1) be a deduction of $u \approx v$ from $\Sigma$. We have $u \equiv a \zeta(s) b$ and $w_{1} \equiv a \zeta(t) b$ for some homomorphism $\zeta$ on $F$, some $a, b \in F^{1}$, and some $s, t \in W_{\lambda^{j}}$ where $j \in\{0,1, \ldots, k\}$. Furthermore,

$$
\begin{equation*}
r(\operatorname{part}(s))=r\left(\lambda^{j}\right)=r\left(\lambda^{i}\right)=r(\operatorname{part}(u))=r(\operatorname{part}(a \zeta(s) b)) . \tag{6.1}
\end{equation*}
$$

On the other hand, it is evident that

$$
\begin{equation*}
r(\operatorname{part}(s)) \leq r(\operatorname{part}(\zeta(s))) \leq r(\operatorname{part}(a \zeta(s) b)) \tag{6.2}
\end{equation*}
$$

Combining (6.1) and (6.2), we have

$$
\begin{equation*}
r(\operatorname{part}(s))=r(\operatorname{part}(\zeta(s)))=r(\operatorname{part}(a \zeta(s) b)) \tag{6.3}
\end{equation*}
$$

This implies that the subword $\zeta(s)$ of the word $a \zeta(s) b$ contains all occurrences of non-simple in $a \zeta(s) b$ letters, whence all letters from $c(a b)$ are simple in $u$. But the word $a \zeta(s) b \equiv u$ starts and ends with non-simple in $u$ letters. Therefore, the words $a$ and $b$ are empty. Thus, $u \equiv \zeta(s)$. If either there is a non-simple in $s$ letter $x$ with $\ell(\zeta(x))>1$ or there is a simple in $s$ letter $y$ such that the word $\zeta(y)$ contains some non-simple in $u$ letter then $r(\operatorname{part}(\zeta(s)))>r(\operatorname{part}(s))$, contradicting (6.3). Hence $\zeta$ maps every non-simple in $s$ letter to a letter and maps every simple in $s$ letter to a word consisting of simple letters. If $y$ is a simple in $s$ letter then $\zeta(y)$ is a subword of $u$. But $u$ does not contain subwords consisting of simple letters except subwords of length 1 , that is letters. Thus, $\zeta$ maps a simple in $s$ letter to a simple in $\zeta(s) \equiv u$ letter. We conclude that $\ell(u)=\ell(\zeta(s))=\ell(s)$. But $\ell(s)=n+j$ and $\ell(u)=n+i$ where $n=\ell(\lambda)$. Therefore, $j=i$. But this is impossible because $i \geq k+1$, while $j \leq k$.

Case 2: $\lambda=(2,1)$. Here $r(\lambda)=2, q(\lambda)=1$, and $\delta=0$, whence $s(\lambda)=1$. Therefore, $k=0$ and $i=1$. This means that $\mathcal{S}_{\lambda}^{k}=\mathcal{W}_{\lambda}$ and $\lambda^{i}=(2,1,1)$. Let $u \equiv x^{2} y z$ and $v \equiv x^{2} z y$. Suppose that the identity $u \approx v$ holds in $\mathcal{W}_{\lambda}$. Then it follows from the identity system

$$
\Sigma=\left\{x^{2} y \approx x y x \approx y x^{2}\right\}
$$

Let (4.1) be a deduction of $u \approx v$ from $\Sigma$. Then there is $j \in\{0,1, \ldots, \ell\}$ such that the first occurrence of $z$ in the word $w_{j}$ precedes the first occurrence of $y$ in $w_{j}$. Let $j$ be the least number with such a property. It is evident that $j>0$. Thus, the following holds:

$$
\begin{align*}
& w_{j-1} \in\left\{x^{2} y z, x y x z, x y z x, y x^{2} z, y x z x, y z x^{2}\right\}  \tag{6.4}\\
& w_{j} \in\left\{x^{2} z y, x z x y, x z y x, z x^{2} y, z x y x, z y x^{2}\right\} . \tag{6.5}
\end{align*}
$$

Furthermore, $w_{j-1} \equiv a \zeta(s) b$ and $w_{j} \equiv a \zeta(t) b$ for some homomorphism $\zeta$ on $F$, some $a, b \in F^{1}$, and some $s, t \in\left\{x^{2} y, x y x, y x^{2}\right\}$. Repeating mutatis mutandi arguments from Case 1), we obtain that $r\left(\operatorname{part}\left(w_{j-1}\right)\right)=r\left(\operatorname{part}\left(w_{j}\right)\right)=2$ and deduct from these equalities that $x \notin c(a b), \zeta(x)$ is a letter, and $\zeta(y) \in$
$\{y, z, y z, z y\}$. If $\zeta(x) \equiv e$ then $e$ is a non-simple in $w_{j-1}$ letter. In view of (6.4), $\zeta(x) \equiv x$.
If $\zeta(y) \equiv y z$ then the word $w_{j}$ contains the subword $y z$, contradicting (6.5). Analogously, if $\zeta(y) \equiv z y$ then the word $w_{j-1}$ contains the subword $z y$, contradicting (6.4).

Suppose now that $\zeta(y) \equiv y$. Then $\zeta(s) \equiv s$ and $\zeta(t) \equiv t$. Since $\zeta(s)$ is a subword in $w_{j-1}$, this means that one of the words $x^{2} y, x y x$ or $y x^{2}$ is a subword in $w_{j-1}$. In view of (6.4), this means that $w_{j-1}$ coincides with one of the words $x^{2} y z, x y x z$ or $y x^{2} z$. Thus, the word $a$ is empty and therefore, $w_{j} \equiv \zeta(t) b \equiv t b$. Since $z \notin c(t)$, we have that the first occurrence of $y$ in $w_{j}$ precedes the first occurrence of $z$ in $w_{j}$. But this contradicts the choice of the number $j$.

Finally, let $\zeta(y) \equiv z$. Then $\zeta(s) \in\left\{x^{2} z, x z x, z x^{2}\right\}$, whence $w_{j-1}$ coincides with one of the words $a x^{2} z b, a x z x b$ or $a z x^{2} b$. In view of (6.4), this means that $w_{j-1} \in\left\{y x^{2} z, y x z x, y z x^{2}\right\}$. Therefore, $a \equiv y$. Thus, the word $w_{j}$ starts with the letter $y$. As in the previous paragraph, we have that the first occurrence of $y$ in $w_{j}$ precedes the first occurrence of $z$ in $w_{j}$ that contradicts the choice of $j$.

We prove that the variety $\mathcal{W}_{\lambda}=\mathcal{S}_{\lambda}^{k}$ does not satisfy the identity $x^{2} y z \approx x^{2} z y$. Since part $\left(x^{2} y z \approx x^{2} z y\right)=(2,1,1)=\lambda^{i}$, we have that $\mathcal{S}_{\lambda}^{k}$ does not collapse the transversal $W_{\lambda^{i}}$.

One can return to the definition of the variety $\mathcal{S}_{\lambda}$. It may be written in the form

$$
\begin{equation*}
\mathcal{S}_{\lambda}=\bigwedge_{\mu \in \Gamma} \mathcal{W}_{\mu} \tag{6.6}
\end{equation*}
$$

where $\Gamma=\left\{\lambda^{k} \mid k=0,1, \ldots, s(\lambda)\right\}$. The following assertion shows that the set $\left\{\lambda^{k} \mid k=0,1, \ldots, s(\lambda)\right\}$ is the least set of partitions $\Gamma$ such that the equality (6.6) holds.
Corollary 6.2. If the equality (6.6) holds for some $\Gamma \subseteq \Lambda$ then $\lambda^{k} \in \Gamma$ for all $k=0,1, \ldots, s(\lambda)$.

Proof. Suppose that $\lambda^{k} \notin \Gamma$ for some $k \in\{0,1, \ldots, s(\lambda)\}$. Let $u, v \in W_{\lambda^{k}}$. The definition of the variety $\mathcal{S}_{\lambda}$ implies that the identity $u \approx v$ holds in $\mathcal{S}_{\lambda}$. Therefore, this identity follows from the identity system

$$
\Sigma=\left\{g \approx h \mid \text { there is } \mu \in \Gamma \text { such that } g, h \in W_{\mu}\right\} .
$$

As usual, let (4.1) be a deduction of $u \approx v$ from $\Sigma$. Let $i \in\{0,1, \ldots, \ell-1\}$. Then the identity $w_{i} \approx w_{i+1}$ follows from an identity of the kind $s \approx t$ where $s, t \in W_{\mu}$ for some $\mu \in \Gamma$. The identity $s \approx t$ holds in the variety $\mathcal{S}_{\mu}$. Therefore, $u \approx v$ holds in $\mathcal{S}_{\mu}$ too. Then Lemma 4.2 implies that $\mu \preceq \operatorname{part}(u \approx v)=$ $\lambda^{k}$. Furthemore, the identity $s \approx t$ holds in the variety $\mathcal{S}_{\lambda}$ because $\mathcal{S}_{\lambda} \subseteq \mathcal{W}_{\mu}$. Applying Lemma 4.2 again, we have $\lambda \preceq \mu$. Therefore, $\mu=U_{S}\left(\lambda^{j}\right)$ and $\lambda^{k}=U_{T}\left(\mu^{q}\right)$ for some finite (may be empty) sets of partitions $S$ and $T$ and some non-negative integers $j$ and $q$.

Let $n(\lambda)=n$. Then $n(\mu)=n+j$, while $n\left(\lambda^{k}\right)$ equals both $n+k$ (that is evident) and $n+j+q$ (because $n\left(\lambda^{k}\right)=n(\mu)+q$ ). Therefore, $n+k=$ $n+j+q$, whence $q=k-j$. Thus, $\mu=U_{S}\left(\lambda^{j}\right)$ and $\lambda^{k}=U_{T}\left(\mu^{k-j}\right)$. Hence
$\lambda^{k}=U_{S \cup T}\left(\lambda^{k}\right)$ and therefore, $S=T=\varnothing$. In particular, this means that $\mu=U_{\varnothing}\left(\lambda^{j}\right)=\lambda^{j}$. We see that $\lambda^{j}=\mu \preceq \lambda^{k}$, whence $j \leq k$. But $j \neq k$ because $\lambda^{k} \notin \Gamma$, while $\lambda^{j}=\mu \in \Gamma$. Hence $j<k$. Besides that, the equality $\mu=\lambda^{j}$ implies that the identity $w_{i} \approx w_{i+1}$ holds in the variety $\mathcal{W}_{\lambda^{j}}$. Since $j \leq k-1$, we have $\mathcal{S}_{\lambda}^{k-1} \subseteq \mathcal{W}_{\lambda^{j}}$. Thus, $w_{i} \approx w_{i+1}$ holds in $\mathcal{S}_{\lambda}^{k-1}$. This is the case for all $i=0,1, \ldots, \ell-1$. Therefore, the identity $u \approx v$ holds in $\mathcal{S}_{\lambda}^{k-1}$ too. This is valid for all $u, v \in W_{\lambda^{k}}$. Hence the variety $\mathcal{S}_{\lambda}^{k-1}$ collapses the transversal $W_{\lambda^{k}}$, contradicting Proposition 6.1.

## 7. Open problems

Recall that an element $x$ of a lattice $L$ is called modular if

$$
\forall y, z \in L: \quad y \leq z \longrightarrow(x \vee y) \wedge z=(x \wedge z) \vee y,
$$

and upper-modular if

$$
\forall y, z \in L: \quad y \leq x \longrightarrow x \wedge(y \vee z)=(x \wedge z) \vee y .
$$

Lower-modular elements are defined dually to upper-modular ones.
Problem 7.1. Describe
a) modular;
b) upper-modular;
c) lower-modular
elements of the lattice $\mathbf{O C}$.
As we have already mentioned in Section 1, neutral elements of the lattice SEM are completely determined in [15], while distributive elements of this lattice are completely described in [12]. It is interesting to note that Proposition 2.1 plays an important role in the proof of the result of [12].

Problem 7.2. Describe
a) codistributive;
b) standard;
c) costandard
elements of the lattice SEM.
Some particular results concerning Problem 7.2a) are obtained in [11]. The following two examples show that, in contrast with the overcommutative case, the lattice SEM contains distributive but not codistributive elements and codistributive but not distributive ones. In particular, there are [co]distributive but not neutral elements of SEM.

Example 7.3. The variety $\mathcal{N}=\operatorname{var}\left\{x^{2} y=x y x=y x^{2}=0\right\}$ is a distributive element of the lattice SEM by [12, Theorem 1.1]. But this variety is not a codistributive (and moreover not a neutral) element of SEM by [11, Theorem 1.1].
Example 7.4. The varieties $\mathcal{A}_{p}=\operatorname{var}\left\{x^{p} y=y, x y=y x\right\}$ with any prime $p$, $\mathcal{L Z}=\operatorname{var}\{x y=x\}$, and $\mathcal{R Z}=\operatorname{var}\{x y=y\}$ are codistributive elements of the
lattice SEM. This follows from the well known facts that these varieties are atoms of SEM and SEM satisfies the condition

$$
\forall x, y, z: \quad x \wedge z=y \wedge z=0 \longrightarrow(x \vee y) \wedge z=0
$$

(see $\left[4\right.$, Section 1], for instance). But $\mathcal{A}_{p}, \mathcal{L Z}$, and $\mathcal{R Z}$ are not distributive (and moreover not neutral) elements of SEM by [12, Theorem 1.1].

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