ON REGULAR κ-BOUNDED SPACES ADMITTING ONLY CONSTANT CONTINUOUS MAPPINGS INTO T_1 SPACES OF PSEUDO-CHARACTER $\leq \kappa$

SERHII BARDYLA AND ALEXANDER OSIPOV

Abstract. In this paper for each cardinal κ we construct an infinite κ-bounded (and hence countably compact) regular space R_κ such that for any T_1 space Y of pseudo-character $\leq \kappa$, each continuous function $f : R_\kappa \to Y$ is constant. This result resolves two problems posted by Tzannes in Open Problems from Topology Proceedings [13] and extends results of Ciesielski and Wojciechowski [4] and Herrlich [8].

We shall follow the terminology of [6, 12]. Throughout of this paper all cardinals are assumed to be infinite.

Regular spaces on which every continuous real-valued function (or, more generally, spaces on which every continuous function into a given space Y) is constant are of particular interest in general topology. Such spaces were constructed and investigated in [1, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15, 16, 17]. For instance, a well-known result of Herrlich [8] states the following:

Theorem 1 ([8, Theorem]). *Let Y be a topological space. The following conditions are equivalent:*

- *Y is a T_1-space;*
- *there exists a regular space X (having at least two points), such that every continuous map from X to Y is constant.*

Also, Ciesielski and Wojciechowski in [4] proved the following:

Theorem 2 ([4, Theorem 7]). *For any uncountable cardinal κ there exists a regular space Y of cardinality κ such that any continuous function from Y into any Hausdorff space Z with a countable pseudo-character is constant.*

However, all known examples of regular spaces on which every continuous real-valued function is constant are far from being countably compact. In [16] Tzannes constructed a Hausdorff countably compact space T on which every continuous real-valued function is constant. Nevertheless, the space T is strongly non-Urysohn. In particular, no pair of distinct points of T have disjoint closed neighborhoods. In [13] Tzannes posed the following two problems:

Problem 1 ([13, Problem C65]). *Does there exist a regular (first countable, separable) countably compact space on which every continuous real-valued function is constant?*

Problem 2 ([13, Problem C66]). *Does there exist, for every Hausdorff space R, a regular (first countable, separable) countably compact space on which every continuous function into R is constant?*

Let κ be a cardinal. A topological space X is called *κ-bounded* if the the closure of each subset $A \subset X$ of cardinality $\leq \kappa$ is compact. It is clear that each κ-bounded space is countably compact and each κ-bounded space of density $\leq \kappa$ is compact.

The pseudo-character $\psi(X)$ of a space X is the smallest cardinal λ such that each point is the intersection of a family of cardinality $\leq \lambda$ of sets which are open in X.

In this paper for each cardinal κ we construct an infinite κ-bounded regular space R_κ such that for each T_1 space Y of pseudo-character $\leq \kappa$, each continuous function $f : R_\kappa \to Y$ is constant. This result resolves Problems 1, 2 and extends Theorems 1, 2.

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1. Wallman κ-bounded extension

For a subset A of a topological space X by $\mathrm{cl}_X(A)$ (or simply $\overline{A}$) we denote the closure of A in X.

We recall [6, §3.6] that the Wallman extension $W(X)$ of a T_1 space X consists of closed ultrafilters, i.e., families $\mathcal{F}$ of closed subsets of X satisfying the following conditions:

- $\emptyset \notin \mathcal{F}$;
- $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$;
- a closed set $F \subset X$ belongs to $\mathcal{F}$ if $F \cap A \neq \emptyset$ for every $A \in \mathcal{F}$.

For any $A \subset X$ put

$$\langle A \rangle = \{\mathcal{F} \in W(X) \mid \text{ there exists } F \in \mathcal{F} \text{ such that } F \subset A\}.$$

The Wallman extension $W(X)$ of X carries the topology generated by the base consisting of the sets $\langle U \rangle$ where U runs over open subsets of X.

By Theorem [6, 3.6.21], the Wallman extension $W(X)$ is T_1 and compact. By Theorem [6, 3.6.22] a T_1-space X is normal if and only if $W(X)$ is Hausdorff.

Consider the map $j_X : X \to W(X)$ assigning to each point $x \in X$ the principal closed ultrafilter consisting of all closed sets $F \subset X$ containing the point x. It is clear that the image $j_X(X)$ is dense in $W(X)$. Since the space X is T_1, Theorem 3.6.21 from [6] provides that the map $j_X : X \to W(X)$ is a topological embedding. So, X can be identified with the subspace $j_X(X)$ of $W(X)$.

Lemma 1. *Let $\mathcal{F} \in W(X)$ and $A \subset X$. Then $\mathcal{F} \in \mathrm{cl}_{W(X)}(A)$ if and only if $\mathrm{cl}_X(A) \in \mathcal{F}$.*

Proof. ($\Rightarrow$) To derive a contradiction assume that $\mathcal{F} \in \mathrm{cl}_{W(X)}(A)$, but $\mathrm{cl}_X(A) \notin \mathcal{F}$. Using the maximality of $\mathcal{F}$ we can find an element $F \in \mathcal{F}$ such that $F \subset X \setminus \mathrm{cl}_X(A)$. Then $\langle X \setminus \mathrm{cl}_X(A)\rangle$ is an open neighborhood of $\mathcal{F}$ such that $\langle X \setminus \mathrm{cl}_X(A)\rangle \cap A = \emptyset$ which implies a contradiction.

($\Leftarrow$) Assume that $\mathrm{cl}_X(A) \in \mathcal{F}$ and fix any open neighborhood $\langle U \rangle$ of $\mathcal{F}$. Then there exists an element $H \in \mathcal{F}$ such that $H \subset U$. Observe that $\mathrm{cl}_X(A) \cap H \in \mathcal{F}$ and $\mathrm{cl}_X(A) \cap H \subset U$. Hence $A \cap \langle U \rangle \neq \emptyset$ witnessing that $\mathcal{F} \in \mathrm{cl}_{W(X)}(A)$. □

Given a cardinal κ, in the Wallman extension $W(X)$ of a T_1-space X, consider the subspace

$$W_\kappa X = \bigcup\{\mathrm{cl}_{W(X)}(C) \mid C \subset X \text{ and } |C| \leq \kappa\}$$

of $W(X)$. The space $W_\kappa(X)$ is called the *Wallman κ-bounded extension* of X. The Wallman κ-bounded extension was introduced and investigated in [2]. In particular, there it was proved the following:

Proposition 1 ([2, Proposition 3.2])**.** *For any T_1 space X, the space $W_\kappa(X)$ is κ-bounded.*

For a topological space X by 2^X we denote the family of all closed subsets of X. Let $\mathcal{C}$ be any subfamily of 2^X. Following [2], a topological space X is called *totally $\mathcal{C}$-normal* if for any disjoint sets $A \in \mathcal{C}$ and $B \in 2^X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Let κ be a cardinal. A topological space X is called *totally $\overline{\kappa}$-normal* if it is totally $\mathcal{C}$-normal for the family $\mathcal{C}$ of closed subsets of the closures of subsets of cardinality $\leq \kappa$ in X.

Proposition 2 ([2, Proposition 2.9])**.** *Each κ-bounded regular space X is totally $\overline{\kappa}$-normal.*

Proposition 3. *The Wallman κ-bounded extension $W_\kappa(X)$ of X is regular iff X is totally $\overline{\kappa}$-normal.*

Proof. ($\Rightarrow$) Assume that the space $W_\kappa(X)$ is regular. By Proposition 1, the space $W_\kappa(X)$ is κ-bounded. To show that the space X is totally $\overline{\kappa}$-normal, take any subset $C \subset X$ of cardinality $|C| \leq \kappa$ and two disjoint closed subsets F, E of X such that $F \subset \mathrm{cl}_X(C)$. Lemma 1 implies that $\mathrm{cl}_{W_\kappa(X)}(F) \cap \mathrm{cl}_{W_\kappa(X)}(E) = \emptyset$. Since $\mathrm{cl}_{W_\kappa(X)}(F) \subset \mathrm{cl}_{W_\kappa(X)}(C)$ and $|C| \leq \kappa$, the set $\mathrm{cl}_{W_\kappa(X)}(F)$ is compact. Since $W_\kappa(X)$ is regular the sets $\mathrm{cl}_{W_\kappa(X)}(F)$ and $\mathrm{cl}_{W_\kappa(X)}(E)$ have disjoint open neighborhoods U_1 and U_2, respectively, in $W_\kappa(X)$. Put $V_1 = U_1 \cap X$ and $V_2 = U_2 \cap X$. Then V_1 and V_2 are disjoint open neighborhoods (in X) of F and E, respectively. Hence X is totally $\overline{\kappa}$-normal.

($\Leftarrow$) Assume that X is totally $\overline{\kappa}$-normal. Given any closed ultrafilter $\mathcal{F} \in W_\kappa(X)$ and open neighborhood $\langle U \rangle$ of $\mathcal{F}$ find a closed set $F \in \mathcal{F}$ such that $F \subset U$. With no loss of generality we can assume that there exists a subset $C \subset X$ such that $|C| \leq \kappa$ and $F \subset \mathrm{cl}_X(C)$. By the total $\overline{\kappa}$-normality of X, there exists an open subset V of X such that $F \subset V \subset \mathrm{cl}_X(V) \subset U$. Then Lemma 1 implies that

$$\mathcal{F} \in \langle V \rangle \subset \mathrm{cl}_{W_\kappa(X)}(\langle V \rangle) = \langle \mathrm{cl}_X(V) \rangle \subset \langle U \rangle$$

witnessing that the space $W_\kappa(X)$ is regular. □

2. Herrlich extension

In this section we briefly recall a part of famous construction due to Herrlich [8] and establish a few important properties of it.

Let X be a topological space and ρ be an equivalence relation on X. Then for any $x \in X$ by $[x]$ we denote the equivalence class of the relation ρ which contains x and for any subset $A \subset X$ put $[A] = \{[x] \mid x \in A\}$. Also, we agree to denote $[x]$ ($[A]$, resp.) simply as x (A, resp.) if $[x] = \{x\}$ ($[x] = \{x\}$ for each $x \in A$, resp.).

For any distinct elements a, b of a topological space X by $\mathrm{Const}(X)_{a,b}$ we denote the class of T_1 spaces such that $f(a) = f(b)$ for any continuous map $f : X \to Y \in \mathrm{Const}(X)_{a,b}$. Following [8], for any distinct points a, b of a space X it can be constructed a space $H_{a,b}(X)$ such that for any $Y \in \mathrm{Const}(X)_{a,b}$, each continuous function $h : H_{a,b}(X) \to Y$ is constant. It can be done in two steps.

Step 1. For each topological space Z by $P(Z)$ we denote the set $Z \times X$ endowed with the topology τ which satisfies the following conditions:

- if $(z, x) \in U \in \tau$, then there exists an open set $V \subset X$ such that $x \in V$ and $\{z\} \times V \subset U$;
- if $(z, a) \in U \in \tau$, then there exists an open set $W \subset Z$ such that $z \in W$ and $W \times \{a\} \subset U$.

By $X(Z)$ we denote the quotient space $P(Z)/\rho$ where ρ is the smallest equivalence relation satisfying $(z_1, b)\rho(z_2, b)$ for any $z_1, z_2 \in Z$. Observe that for any $z \in Z$ the vertical fiber $[\{z\} \times X] \subset X(Z)$ is homeomorphic to X. Since the map $h(z) = (z, a)$ is a canonical embedding of Z into $X(Z)$, we can identify Z with the subspace $Z \times \{a\} \subset X(Z)$. Observe that for each $Y \in \mathrm{Const}(X)_{a,b}$, for any continuous map $f : X(Z) \to Y$, $f((z, a)) = f([z, b])$ where z is an arbitrary element of Z. Hence for any $Y \in \mathrm{Const}(X)_{a,b}$ each continuous map $f : X(Z) \to Y$ is constant on Z.

Step 2. Put $H_1 = X$ and for each $n \in \mathbb{N}$ let $H_{n+1} = X(H_n)$. For any $n \in \mathbb{N}$ by b_n we denote the element $[(z, b)] \in H_n$ where z is any element of H_{n-1}. Recall that we identify H_n with a subspace $H_n \times \{a\}$ of H_{n+1} which implies that $H_n \subset H_{n+1}$ for each $n \in \mathbb{N}$. Finally, by $H_{a,b}(X)$ we denote the set $\cup_{n\in\mathbb{N}} H_n$ endowed with the topology τ which satisfies the following condition: a subset $U \subset H_{a,b}(X)$ is open in $(H_{a,b}(X), \tau)$ iff the set $U \cap H_n$ is open in H_n for each $n \in \mathbb{N}$. The space $H_{a,b}(X)$ is called a *Herrlich extension* of X. Fix any $Y \in \mathrm{Const}(X)_{a,b}$. To see that each continuous function $h : H_{a,b}(X) \to Y$ is constant take any distinct points $x, y \in H_{a,b}(X)$ and observe that there exists $n \in \mathbb{N}$ such that $\{x, y\} \subset H_n$. Recall that any continuous function $g : H_{n+1} \to Y$ is constant on H_n. Hence the restriction of h on the set H_{n+1} is constant on H_n witnessing that $h(x) = h(y)$.

Remark 1. *The definition of the topology on $H_{a,b}(X)$ (see also* [8]*) implies the following:*

- *H_n is a closed subset of $H_{a,b}(X)$ for each $n \in \mathbb{N}$;*
- *a subset $A \subset H_{a,b}(X)$ is closed iff $A \cap H_n$ is closed in H_n for each $n \in \mathbb{N}$;*
- *if X is regular, then so is $H_{a,b}(X)$.*

Let κ be a cardinal. A space X is called *κ-accumulative* if $|\overline{A}| \leq \kappa$ for each subset $A \subset X$ of cardinality $\leq \kappa$.

Proposition 4. *For a cardinal κ the following statements hold:*

1) *if X is κ-accumulative, then so is $H_{a,b}(X)$;*
2) *if X is κ-accumulative and totally $\overline{\kappa}$-normal, then $H_{a,b}(X)$ is totally $\overline{\kappa}$-normal.*

Proof. Consider statement 1. Observe that $H_1 = X$ is κ-accumulative by the assumption. Assume that H_{k-1} is κ-accumulative and consider a subset $A \subset H_k$ of cardinality $\leq \kappa$. For each $h \in H_{k-1}$ by X_h we denote the vertical fiber $\{h\} \times (X \setminus \{b\}) \cup \{b_n\} \subset H_k$ which is homeomorphic to X and hence it is κ-accumulative. Put $A_1 = \{h \in H_{k-1} \mid X_h \cap A \setminus \{b_k\} \neq \emptyset\}$. Since $|A| \leq \kappa$ the set

A_1 has cardinality $\leq \kappa$ as well. Put $A_2 = \cup\{\mathrm{cl}_{X_h}(A \cap X_h) \mid h \in A_1\}$. Since for each $h \in H_{n-1}$ the subspace X_h is κ-accumulative, the set A_2 has cardinality $\leq \kappa$. Let $A_3 = \mathrm{cl}_{H_{k-1}}(A_1)$. Since H_{k-1} is κ-accumulative and $|A_1| \leq \kappa$, the set A_3 has cardinality $\leq \kappa$. Finally, the definition of the topology on H_k implies that $\mathrm{cl}_{H_k}(A) \subset A_2 \cup A_3 \cup \{b_k\}$ witnessing that H_k is κ-accumulative. Hence for each $n \in \mathbb{N}$ the space H_n is κ-accumulative.

To see that $H_{a,b}(X)$ is κ-accumulative fix any subset $B \subset H_{a,b}(X)$ of cardinality $\leq \kappa$. By Remark 1, $\mathrm{cl}_{H_{a,b}(X)}(B) = \cup_{n\in\mathbb{N}} \mathrm{cl}_{H_n}(B\cap H_n)$. Since each H_n is κ-accumulative $|\mathrm{cl}_{H_n}(B\cap H_n)| \leq \kappa$. Hence $|\mathrm{cl}_{H_{a,b}(X)}(B)| \leq \kappa$ witnessing that $H_{a,b}(X)$ is κ-accumulative.

Consider statement 2. By statement 1, the space $H_{a,b}(X)$ is κ-accumulative. Then it is clear that for each $n \in \mathbb{N}$ the space H_n is κ-accumulative as well. Observe that $H_1 = X$ is totally $\overline{\kappa}$-normal by the assumption. Assume that H_{k-1} is totally $\overline{\kappa}$-normal and consider a closed subset $A \subset H_k$ of cardinality $\leq \kappa$. Fix any open set $U \subset H_k$ such that $A \subset U$. We will show that there exists an open set $W \subset H_k$ such that $A \subset W \subset \mathrm{cl}_{H_k}(W) \subset U$ which would provide that H_k is totally $\overline{\kappa}$-normal. Since H_{k-1} is totally $\overline{\kappa}$-normal there exists an open subset V of H_{k-1} such that $A \cap H_{k-1} \subset V \subset \mathrm{cl}_{H_{k-1}}(V) \subset U \cap H_{k-1}$. Moreover, since U is open in H_k for each $h \in V$ there exists an open neighborhood $V_h(a)$ of a in X_h such that $\{h\}\times V_h(a) \subset U$. Since X_h is regular we can assume that $\mathrm{cl}_{X_h}(V_h(a)) \subset U \cap X_h$ for each $h \in V$. Put $V^* = \cup\{\{h\}\times V_h(a) \mid h \in V\}$. Let $A_1 = \{h \in H_{k-1} \mid A \cap X_h \neq \emptyset\}$. Since the set X_h is closed in H_k for each $h \in H_{k-1}$, the set $A \cap X_h$ is closed in H_k as well. Since X_h is totally $\overline{\kappa}$-normal for each $h \in A_1$ there exists an open subset W_h of X_h such that $A \cap X_h \subset W_h \subset \mathrm{cl}_{X_h}(W_h) \subset U \cap X_h$. Since X_h is regular we can also assume that for each $h \in A_1$, W_h satisfies the following condition: if $(h, a) \notin A \cap X_h$, then $(h, a) \notin \mathrm{cl}_{X_h}(W_h)$. Let $W = \cup\{W_h \mid h \in A_1\} \cup V^*$. One can check that W is an open subset of H_k and $A \subset W \subset \mathrm{cl}_{H_k}(W) \subset U$ witnessing that H_k is totally $\overline{\kappa}$-normal. Hence H_n is totally $\overline{\kappa}$-normal for each $n \in \mathbb{N}$.

To see that $H_{a,b}(X)$ is totally $\overline{\kappa}$-normal fix any closed subset $B \subset H_{a,b}(X)$ of cardinality $\leq \kappa$ and an open subset $U \subset H_{a,b}(X)$ such that $B \subset U$. By Remark 1, for each $n \in \mathbb{N}$ the set $B_n = B \cap H_n$ is closed in H_n. Next we shall construct a sequence of sets $\{V_n\}_{n\in\mathbb{N}}$ which satisfies the following conditions:

- $B_n \subset V_n \subset \mathrm{cl}_{H_n}(V_n) \subset H_n \cap U$ for each $n \in \mathbb{N}$;
- V_n is an open subset of H_n for each $n \in \mathbb{N}$;
- $V_i \subset V_j$ for each $i \leq j$.

Observe that if we construct such a sequence $\{V_n\}_{n\in\mathbb{N}}$, then the set $V = \cup_{n\in\mathbb{N}} V_n$ will be a desired open set in $H_{a,b}(X)$ which contains B and $\mathrm{cl}_{H_{a,b}(X)}(V) \subset U$, witnessing that $H_{a,b}(X)$ is totally $\overline{\kappa}$-normal.

We shall construct the mentioned above family $\{V_n\}_{n\in\mathbb{N}}$ by the induction. Since H_1 is totally $\overline{\kappa}$-normal there exists an open set $W \subset H_1$ such that $B_1 \subset W \subset \mathrm{cl}_{H_1}(W) \subset U \cap H_1$. Put $V_1 = W$. Assume that we already constructed sets $V_1, \ldots, V_n$. Since $U \cap H_{n+1}$ is an open subset of H_{n+1} and $V_n \subset U \cap H_{n+1}$, the definition of the topology on H_{n+1} and the regularity of X imply that for each $h \in V_n$ there exists an open neighborhood V_h of a in X such that $\{h\}\times \mathrm{cl}_X(V_h) \subset U$. Let $V_n' = \cup\{V_h \mid h \in V_n\}$. Obviously, $V_n \subset V_n'$ and V_n' is an open subset of H_{n+1}. Put $C_{n+1} = B_{n+1} \setminus V_n'$. Recall that H_{n+1} is totally $\overline{\kappa}$-normal. Hence for a closed set C_{n+1} of cardinality $\leq \kappa$ there exists an open subset $W_{n+1} \subset H_{n+1}$ such that $C_{n+1} \subset W_{n+1} \subset \mathrm{cl}_{H_{n+1}}(W_{n+1}) \subset U \cap H_{n+1}$. Finally, put $V_{n+1} = V_n' \cup W_{n+1}$. At this point it is easy to check that so defined family $\{V_n\}_{n\in\mathbb{N}}$ satisfies the above conditions. □

3. Main result

For any ordinals α, β by $[\alpha, \beta)$ ($[\alpha, \beta]$, resp.) we denote the set of all ordinals γ such that $\alpha \leq \gamma < \beta$ ($\alpha \leq \gamma \leq \beta$, resp.). Further if some ordinal α is considered as a topological space, then α is assumed to carry the order topology.

Before formulating the main result of this paper we prove a few auxiliaries lemmas.

Lemma 2. *Let κ be a cardinal, ξ be a regular cardinal $> \kappa^+$ and Y be a T_1 space of pseudo-character $\psi(Y) \leq \kappa$. Then for each continuous map $f : \xi \to Y$ there exist $y \in Y$ and $\mu \in \xi$ such that $[\mu, \xi) \subset f^{-1}(y)$.*

Proof. Let $B = \{y \in Y \mid f^{-1}(y)$ is a cofinal subset of $\xi\}$. We claim that $|B| \leq 1$. Indeed, if there exist distinct elements $y_1, y_2 \in B$, then the sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed and unbounded in ξ providing that $f^{-1}(y_1) \cap f^{-1}(y_2) \neq \emptyset$ which yields a contradiction. Hence there are two cases to consider:

1) the set B is empty;
2) the set B is singleton;

Consider case 1. Put $\alpha_0 = 0$ and for each $\beta \leq \kappa^+$ put $\alpha_\beta = \sup(f^{-1}(f(\alpha_{\beta-1}))) + 1$ if β is a successor ordinal and $\alpha_\beta = \sup\{\alpha_\delta \mid \delta < \beta\}$ if β is a limit ordinal. Since $cf(\xi) > \kappa^+$ the sequence $\{\alpha_\beta \mid \beta \leq \kappa^+\}$ is a subset of ξ. Observe that for each distinct $\beta_1, \beta_2 \in \kappa^+ + 1$, $f(\alpha_{\beta_1}) \neq f(\alpha_{\beta_2})$. Put $y = f(\alpha_{\kappa^+})$ and observe that $cf(\alpha_{\kappa^+}) = \kappa^+$. Since $\psi(Y) \leq \kappa$ there exists a family $\mathcal{U}$ of cardinality $\leq \kappa$ of open neighborhoods of y such that $\cap\mathcal{U} = \{y\}$. Then for each $U \in \mathcal{U}$, $f^{-1}(U)$ is an open neighborhood of α_{κ^+} providing that there exists $\mu_U \in \alpha_{\kappa^+}$ such that $[\mu_U, \alpha_{\kappa^+}] \subset f^{-1}(U)$. Since $cf(\alpha_{\kappa^+}) > \kappa$ the ordinal $\mu = \sup\{\mu_U \mid U \in \mathcal{U}\}$ belongs to α_{κ^+}. Hence $[\mu, \alpha_{\kappa^+}] \subset \cap\{f^{-1}(U) \mid U \in \mathcal{U}\} = f^{-1}(y)$. Then for each ordinal $\alpha_\beta \in [\mu, \alpha_{\kappa^+}]$, $f(\alpha_\beta) = f(\alpha_{\kappa^+})$ which implies a contradiction. Hence case 1 is not possible.

Consider case 2. Let $y \in B$. Observe that $f^{-1}(y)$ is closed and unbounded in ξ. Fix any family $\mathcal{U}$ of cardinality $\leq \kappa$ of open neighborhoods of y such that $\cap\mathcal{U} = \{y\}$. Since for each $U \in \mathcal{U}$, $f^{-1}(U)$ is an open set which contains $f^{-1}(y)$, for each $\alpha \in f^{-1}(y)$ there exists an ordinal $\mu^U_\alpha \in \alpha$ such that $[\mu^U_\alpha, \alpha] \subset f^{-1}(U)$. For each $U \in \mathcal{U}$ define the map $h_U : f^{-1}(y) \to \xi$ by $h_U(\alpha) = \mu^U_\alpha$, $\alpha \in f^{-1}(y)$. Since ξ is a regular cardinal, for each $U \in \mathcal{U}$ the map h_U satisfies conditions of Fodor's Lemma [12, Lemma III.6.14]. Hence for each $U \in \mathcal{U}$ there exists an ordinal $\mu_U \in \xi$ and an unbounded (even stationary) in ξ subset $A \subset f^{-1}(y)$ such that $h_U(\alpha) = \mu_U$ for each $\alpha \in A$. At this point it is clear that $[\mu_U, \xi) \subset f^{-1}(U)$ for any $U \in \mathcal{U}$. Since $cf(\xi) > \kappa$ the ordinal $\mu = \sup\{\mu_U \mid U \in \mathcal{U}\}$ belongs to ξ. Hence $[\mu, \xi) \subset \cap\{f^{-1}(U) \mid U \in \mathcal{U}\} = f^{-1}(y)$. ☐

For any distinct cardinals α, β, by $T_{\alpha,\beta}$ we denote the punctured Tychonoff (α, β)-plank, i.e., the subspace $(\alpha + 1)\times(\beta + 1) \setminus \{(\alpha, \beta)\}$ of the Tychonoff product $(\alpha + 1)\times(\beta + 1)$.

Lemma 3. *Let κ be a cardinal and Y be a T_1 space such that $\psi(Y) \leq \kappa$. Then for each regular cardinals λ, ξ such that $\kappa^+ < \lambda < \xi$ and for each continuous map $f : T_{\lambda,\xi} \to Y$ there exist $y \in Y$, $\alpha \in \lambda$ and $\mu \in \xi$ such that $[\alpha, \lambda]\times[\mu, \xi] \setminus \{(\lambda, \xi)\} \subset f^{-1}(y)$.*

Proof. By Lemma 2, there exist $y \in Y$ and $\alpha \in \lambda$ such that $[\alpha, \lambda)\times\{\xi\} \subset f^{-1}(y)$. Since $\psi(Y) \leq \kappa$ there exists a family $\mathcal{U}$ of cardinality $\leq \kappa$ of open neighborhoods of y such that $\cap\mathcal{U} = \{y\}$. The continuity of f implies that for each $U \in \mathcal{U}$ the set $f^{-1}(U)$ is open and contains the set $[\alpha, \lambda)\times\{\xi\}$. Since $\lambda < cf(\xi)$ for each $U \in \mathcal{U}$ there exists $\mu_U \in \xi$ such that $[\alpha, \lambda)\times[\mu_U, \xi] \subset f^{-1}(U)$. Since $\kappa < cf(\xi)$ the ordinal $\mu = \sup\{\mu_U \mid U \in \mathcal{U}\}$ belongs to ξ. Then $[\alpha, \lambda)\times[\mu, \xi] \subset \cap\{f^{-1}(U) \mid U \in \mathcal{U}\} = f^{-1}(y)$. Since the set $\{y\}$ is closed in Y

$$[\alpha, \lambda]\times[\mu, \xi] \setminus \{(\lambda, \xi)\} \subset \overline{[\alpha, \lambda)\times[\mu, \xi]} \subset \overline{f^{-1}(y)} = f^{-1}(y).$$

☐

Lemma 4. *Let κ be a cardinal and λ be any ordinal $\geq \kappa$. Then λ if κ-accumulative.*

Proof. Fix any subset $X \subset \lambda$ such that $|X| \leq \kappa$. Observe that the map $f : \overline{X} \setminus (X \cup \{\sup X\}) \to X$ defined by $f(\alpha) = \min(X \setminus (\alpha + 1))$ is injective. Hence $|\overline{X}| \leq \kappa$ witnessing that λ is κ-accumulative. ☐

Since the finite Tychonoff product of κ-accumulative spaces is κ-accumulative and a subspace of a κ-accumulative space is κ-accumulative, Lemma 4 implies the following:

Corollary 1. *For any cardinals α, β, κ such that $\min\{\alpha, \beta\} \geq \kappa$ the space $T_{\alpha,\beta}$ is κ-accumulative.*

The following Theorem resolves Problem 1 and Problem 2 and is the main result of this paper.

Theorem 3. *For each cardinal κ there exists a regular infinite κ-bounded space R_κ such that for any T_1 space Y of pseudo-character $\psi(Y)\leq\kappa$ each continuous map $f:R_\kappa\to Y$ is constant.*

Proof. Fix any cardinal κ. By T we denote the punctured Tychonoff plank $T_{\kappa^{++},\kappa^{+++}}$. Observe that T is κ-bounded and κ-accumulative. Let $\mathbb{Z}$ be a discrete set of integers and a,b be distinct points which do not belong to $T{\times}\mathbb{Z}$. By R we denote the set $T{\times}\mathbb{Z}\cup\{a,b\}$ endowed with the topology τ which satisfies the following conditions:

- the Tychonoff product $T{\times}\mathbb{Z}$ is open in R;
- if $a\in U\in\tau$, then there exists $n\in\mathbb{N}$ such that $\{(t,-k)\mid t\in T \text{ and } k>n\}\subset U$;
- if $b\in U\in\tau$, then there exists $n\in\mathbb{N}$ such that $\{(t,k)\mid t\in T \text{ and } k>n\}\subset U$.

One can check that the space R is regular, κ-bounded and κ-accumulative (see Corollary 1). For convenience, we denote the subset $T{\times}\{n\}\subset R$ by T_n for each $n\in\mathbb{Z}$.

On the space R consider the smallest equivalence relation $\sim$ such that

$$(x,\kappa^{+++},2n)\sim(x,\kappa^{+++},2n+1) \text{ and } (\kappa^{++},y,2n)\sim(\kappa^{++},y,2n-1)$$

for any $n\in\mathbb{Z}$, $x\in\kappa^{++}$ and $y\in\kappa^{+++}$.

Let X be the quotient space $R/\sim$ of R by the equivalence relation $\sim$. Being the continuous image of the κ-bounded space R the space X is κ-bounded as well. It is straightforward to check that X is regular and hence, by Proposition 2, X is totally $\overline{\kappa}$-normal. Also, one can check that X is κ-accumulative. We claim that for each T_1 space Y of pseudo-character $\psi(Y)\leq\kappa$ and for each continuous map $f:X\to Y$, $f(a)=f(b)$. Indeed, fix any space Y such that $\psi(Y)\leq\kappa$ and let $f:X\to Y$ be any continuous map. Observe that for any $n\in\mathbb{Z}$ the subspace $[T_n]=\{[x]\mid x\in T_n\}\subset X$ is homeomorphic to T_n. By Lemma 3, for each $n\in\mathbb{Z}$ there exist $y_n\in Y$ and ordinals $\alpha_n\in\kappa^{++}$, $\beta_n\in\kappa^{+++}$ such that

$$\{[(x,\kappa^{+++},n)]\mid x\in[\alpha_n,\kappa^{++})\}\subset f^{-1}(y_n) \text{ and } \{[(\kappa^{++},y,n)]\mid y\in[\beta_n,\kappa^{+++})\}\subset f^{-1}(y_n).$$

Recall that $[(x,\kappa^{+++},2n)]=[(x,\kappa^{+++},2n+1)]$ and $[(\kappa^{++},y,2n)]=[(\kappa^{++},y,2n-1)]$ for any $n\in\mathbb{Z}$, $x\in\kappa^{++}$ and $y\in\kappa^{+++}$ which implies that $y_{2n-1}=y_{2n}=y_{2n+1}$, $n\in\mathbb{Z}$. Put $\alpha=\sup\{\alpha_n\mid n\in\mathbb{Z}\}$ and $\beta=\sup\{\beta_n\mid n\in\mathbb{Z}\}$. Hence there exists a unique $y\in Y$ such that

$$D=\{[(x,\kappa^{+++},n)]\mid x\in[\alpha,\kappa^{++}),n\in\mathbb{Z}\}\cup\{[(\kappa^{++},y,n)]\mid y\in[\beta,\kappa^{+++}),n\in\mathbb{Z}\}\subset f^{-1}(y).$$

Observe that $\{a,b\}\subset\overline{D}\subset\overline{f^{-1}(y)}=f^{-1}(y)$. Hence $f(a)=f(b)$.

Next consider the Herrlich extension $H_{a,b}(X)$ of X. Recall that for any space Y of pseudo-character $\psi(Y)\leq\kappa$ each continuous function $f:H_{a,b}(X)\to Y$ is constant. Since X is totally $\overline{\kappa}$-normal and κ-accumulative, Proposition 4 implies that $H_{a,b}(X)$ is totally $\overline{\kappa}$-normal.

Finally, let R_κ be the Wallman κ-bounded extension $W_\kappa(H_{a,b}(X))$ of $H_{a,b}(X)$. Recall that $H_{a,b}(X)$ is a dense subspace of R_κ. Hence for any T_1 space Y of pseudo-character $\psi(Y)\leq\kappa$ each continuous function $f:R_\kappa\to Y$ is constant on the dense subspace $H_{a,b}(X)\subset R_\kappa$ witnessing that f is constant on the whole R_κ. By Proposition 1, the space R_κ is κ-bounded. Proposition 3 implies that R_κ is regular. □

The next theorem shows that the main result never holds if the space Y is not T_1.

Theorem 4. *Let X be a non-anti-discrete space and Y be a space which is not T_1. Then there exists a continuous non-constant map $f:X\to Y$.*

Proof. Since the space Y is not T_1 it contains points p_1 and p_2 such that $p_2\in\overline{\{p_1\}}$. Since X is not anti-discrete it contains a proper open subset U. For each $a\in U$ put $f(a)=p_1$ and for each $b\in X\setminus U$ put $f(b)=p_2$. It is clear that the map f is continuous and $f(a)\neq f(b)$ for any $a\in U$ and $b\in X\setminus U$. □

Theorem 3 and Theorem 4 provide the following analogue of Theorem 1.

Theorem 5. *A space Y is T_1 if and only if for any cardinal κ there exists an infinite regular κ-bounded space R_κ such that any continuous map $f:R_k\to Y$ is constant.*

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S.Bardyla: University of Vienna, Austria
E-mail address: `sbardyla@yahoo.com`

A. Osipov: Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, and Ural State University of Economics, Yekaterinburg, Russia
E-mail address: `oab@list.ru`