

LIMIT VARIETIES OF APERIODIC MONOIDS WITH COMMUTING IDEMPOTENTS

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ABSTRACT. A variety of algebras is called limit if it is non-finitely based but all its proper subvarieties are finitely based. A monoid is aperiodic if all its subgroups are trivial. We classify all limit varieties of aperiodic monoids with commuting idempotents.

1. INTRODUCTION AND SUMMARY

A variety of algebras is called *finitely based* if it has a finite basis of its identities, otherwise, the variety is said to be *non-finitely based*. Much attention is paid to studying of finitely based and non-finitely based varieties of algebras of various types. In particular, the finitely based and non-finitely based varieties of semigroups and monoids have been the subject of an intensive research (see the surveys [14,15]).

A variety is *hereditarily finitely based* if all its subvarieties are finitely based. A variety is called a *limit variety* if it is non-finitely based but every its proper subvariety is finitely based. The limit varieties play an important role because every non-finitely based variety contains some limit subvariety by Zorn's lemma. It follows that a variety is hereditarily finitely based if and only if it does not contain any limit variety. So, if one manages to classify all limit varieties within some class of varieties then this classification implies a description of all hereditarily finitely based varieties in this class.

We consider varieties of monoids as semigroups equipped with an additional 0-ary operation that fixes the identity element. A monoid is *aperiodic* if all its subgroups are trivial. The article is devoted to study the limit varieties within the class \mathbf{A}_{com} of aperiodic monoids with commuting idempotents. In [3], Jackson found the first two examples of limit monoid varieties \mathbf{L} and \mathbf{M} . It turned out that \mathbf{L} and \mathbf{M} lie in \mathbf{A}_{com} . Lee established that only \mathbf{L} and \mathbf{M} are limit varieties within several classes of monoid varieties [6,7]. In particular, he proved in [7] the uniqueness of the limit varieties \mathbf{L} and \mathbf{M} in an important subclass of \mathbf{A}_{com} , namely, in the class of varieties of aperiodic monoids with central idempotents. Just recently, the third example of a limit variety \mathbf{J} from \mathbf{A}_{com} was found in [1]. In this article, we completely classify all limit varieties within the class \mathbf{A}_{com} .

In order to formulate the main result of the article, we need some definitions and notation. The free monoid over a countably infinite alphabet is denoted by F^1 . As usual, elements of F^1 and elements of the alphabet are called *words* and *letters* respectively. Words and letters are denoted by small Latin letters. However,

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words unlike letters are written in bold. The following construction was used by Perkins [10] to build the first two examples of non-finitely based finite semigroups. For any set of words $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$, let $S(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k)$ denote the Rees quotient monoid of F^1 over the ideal of all words that are not subwords of any word in W . The above-mentioned varieties \mathbf{L} and \mathbf{M} are introduced as the varieties generated by the monoid of such a form. Namely, \mathbf{L} and \mathbf{M} denote the varieties generated by the monoids $S(xzxyty)$ and $S(xyzxty, xtyzxy)$ respectively.

To introduce the remaining limit varieties from \mathbf{A}_{com} , we need some more definitions and notation. Expressions like $\mathbf{u} \approx \mathbf{v}$ are used for identities, whereas $\mathbf{u} = \mathbf{v}$ means that the words \mathbf{u} and \mathbf{v} coincide. As usual, the symbol \mathbb{N} stands for the set of all natural numbers. For an arbitrary $n \in \mathbb{N}$, we denote by S_n the full symmetric group on the set $\{1, 2, \dots, n\}$. The above-mentioned variety \mathbf{J} is given by the identities

$$(1.1) \quad xyx \approx xyx^2,$$

$$(1.2) \quad x^2y^2 \approx y^2x^2,$$

$$(1.3) \quad xyzxy \approx yzxxy,$$

$$(1.4) \quad xyxztx \approx xyxzttx,$$

$$x z_{1\pi} z_{2\pi} \cdots z_{n\pi} x \left(\prod_{i=1}^n t_i z_i \right) \approx x^2 z_{1\pi} z_{2\pi} \cdots z_{n\pi} \left(\prod_{i=1}^n t_i z_i \right)$$

where n ranges over \mathbb{N} and π ranges over S_n . If \mathbf{X} is a monoid variety then we denote by $\overleftarrow{\mathbf{X}}$ the variety *dual to* \mathbf{X} , i.e., the variety consisting of monoids antiisomorphic to monoids from \mathbf{X} .

The main result of the paper is the following

Theorem 1.1. *The only limit subvarieties of the class \mathbf{A}_{com} are \mathbf{L} , \mathbf{M} , \mathbf{J} and $\overleftarrow{\mathbf{J}}$.*

Theorem 1.1 shows that there are only four limit varieties within the class \mathbf{A}_{com} . By contrast, Kozhevnikov proves in [5] that there are continuum many limit varieties of periodic groups. As for the aperiodic monoid varieties, it is known only one example of a limit variety of such a type that does not lie in \mathbf{A}_{com} . This example was provided just recently by Zhang and Luo [17].

The article consists of four sections. Section 2 contains definitions, notation and several known auxiliary results. Section 3 is devoted to the proof of the fact that the monoid variety \mathbf{O} given by the identities (1.2) and

$$(1.5) \quad xzxyty \approx xzyxty$$

is hereditarily finitely based. The proof of Theorem 1.1 is given in Section 4.

2. PRELIMINARIES

A variety of monoids is called *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups). *Band* is a semigroup (monoid) in which every element is an idempotent. If \mathbf{u} and \mathbf{v} are words and ε is an identity then we will write $\mathbf{u} \overset{\varepsilon}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε .

Lemma 2.1. *Let \mathbf{V} be a variety of aperiodic monoids that does not contain $S(xyx)$. Suppose that \mathbf{V} is not hereditarily finitely based. Then \mathbf{V} satisfies either (1.1) or*

$$(2.1) \quad xyx \approx x^2yx.$$

Proof. According to [4, Lemma 5.3], \mathbf{V} satisfies a non-trivial identity of the form $xyx \approx \mathbf{w}$. A completely regular variety of aperiodic monoids is a variety of bands. Since all varieties of band monoids are finitely based [16], \mathbf{V} is non-completely regular. Then $\mathbf{w} = x^p y x^q$ for some p and q such that $p \geq 2$ or $q \geq 2$ by [2, Proposition 2.2 and Corollary 2.6]. By symmetry, we may assume that $q \geq 2$.

Suppose at first that $p = 0$. Then \mathbf{V} satisfies the identity $x^2 \approx x^q$ and, therefore, the identity $xyx \approx yx^2$. It follows from [11] that every variety that satisfies the latest identity is finitely based. We obtain a contradiction with the hypothesis.

Suppose now that $p \geq 1$. Then \mathbf{V} satisfies the identity $x^2 \approx x^{p+q}$ and, therefore, the identity

$$(2.2) \quad x^2 \approx x^3$$

because \mathbf{V} is aperiodic. Then the identities

$$xyx \approx x^p y x^q \stackrel{(2.2)}{\approx} x^p y x^{q+1} \approx xyx^2$$

hold in \mathbf{V} . Thus, (1.1) is satisfied in \mathbf{V} . \square

For an identity system Σ , we denote by $\text{var } \Sigma$ the variety of monoids given by Σ . Let us fix notation for the following two varieties:

$$\begin{aligned} \mathbf{K} &= \text{var}\{(1.1), (1.2), x^2 y \approx x^2 y x\}, \\ \mathbf{Q} &= \text{var}\{(1.1), (1.2), (2.1)\}. \end{aligned}$$

The following claim follows from [2, Proposition 6.1] for the variety \mathbf{K} and from [9, Condition 4 on page 8] for the variety \mathbf{Q} .

Lemma 2.2. *The varieties \mathbf{K} and \mathbf{Q} are hereditarily finitely based.* \square

Put

$$\begin{aligned} \mathbf{E} &= \text{var}\{(1.2), (2.2), yx^2 \approx xyx\}, \\ \mathbf{F} &= \text{var}\{(1.1), (1.2), (1.3), x^2 y \approx x^2 y x\}. \end{aligned}$$

To avoid a confusion below, we note that, in [1, 2], the variety \mathbf{E} is denoted by $\overleftarrow{\mathbf{E}}$ and $\overleftarrow{\mathbf{E}}$ denotes the variety \mathbf{E} , while, in [2], the variety \mathbf{F} is denoted by \mathbf{F}_1 .

Lemma 2.3. *Let \mathbf{V} be a variety of monoids that satisfies the identities (1.1) and (1.2).*

- (i) *If $\mathbf{E} \not\subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{K}$.*
- (ii) *If $\mathbf{F} \not\subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{Q}$.*

Proof. Since (1.1) implies (2.2), \mathbf{V} consists of aperiodic monoids. If \mathbf{V} is completely regular then \mathbf{V} is a variety of bands. Then \mathbf{V} is commutative because it satisfies the identity (1.2). Thus, \mathbf{V} is contained in the variety of all semilattice monoids. In view of [2, Lemma 2.1], $\mathbf{V} \subseteq \mathbf{K} \wedge \mathbf{Q}$. So, we may assume that \mathbf{V} is non-completely regular.

If $\mathbf{E} \not\subseteq \mathbf{V}$ then the identity $x^2 y \approx x^2 y x^2$ holds in \mathbf{V} by [2, the dual to Lemma 4.3]. Then \mathbf{V} satisfies the identities

$$x^2 y \approx x^2 y x^2 \stackrel{(1.1)}{\approx} x^2 y x,$$

whence $\mathbf{V} \subseteq \mathbf{K}$. The claim (i) is proved.

If $\mathbf{F} \not\subseteq \mathbf{V}$ then arguments from the third paragraph of the proof of Lemma 3.2 of [1] imply that the identity $xyx^2 \approx x^2yx^2$ holds in \mathbf{V} . Then \mathbf{V} satisfies the identities

$$x^2yx \stackrel{(1.1)}{\approx} x^2yx^2 \approx xyx^2 \stackrel{(1.1)}{\approx} xyx,$$

whence $\mathbf{V} \subseteq \mathbf{Q}$. The claim (ii) is proved. \square

A letter is called *simple* [*multiple*] *in a word* \mathbf{w} if it occurs in \mathbf{w} once [at least twice]. The set of all simple [multiple] letters in a word \mathbf{w} is denoted by $\text{sim}(\mathbf{w})$ [respectively, $\text{mul}(\mathbf{w})$]. The *content* of a word \mathbf{w} , i.e., the set of all letters occurring in \mathbf{w} , is denoted by $\text{con}(\mathbf{w})$. The number of occurrences of the letter x in \mathbf{w} is denoted by $\text{occ}_x(\mathbf{w})$. For a word \mathbf{w} and letters $x_1, x_2, \dots, x_k \in \text{con}(\mathbf{w})$, let $\mathbf{w}(x_1, x_2, \dots, x_k)$ be the word obtained from \mathbf{w} by deleting all letters except x_1, x_2, \dots, x_k .

Now we are going to define several notions that appear in [1] and [2, Chapter 3]. Let \mathbf{w} be a word and $\text{sim}(\mathbf{w}) = \{t_1, t_2, \dots, t_m\}$. We may assume without loss of generality that $\mathbf{w}(t_1, t_2, \dots, t_m) = t_1 t_2 \cdots t_m$. Then $\mathbf{w} = t_0 \mathbf{w}_0 t_1 \mathbf{w}_1 \cdots t_m \mathbf{w}_m$ where $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are possibly empty words and t_0 is the empty word. The words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are called *blocks* of a word \mathbf{w} , while t_0, t_1, \dots, t_m are said to be *dividers* of \mathbf{w} . The representation of the word \mathbf{w} as a product of alternating dividers and blocks starting with the divider t_0 and ending with the block \mathbf{w}_m is called a *decomposition* of the word \mathbf{w} . For a given word \mathbf{w} , a letter $x \in \text{con}(\mathbf{w})$ and a natural number $i \leq \text{occ}_x(\mathbf{w})$, we denote by $h_i(\mathbf{w}, x)$ the right-most divider of \mathbf{w} that precedes the i th occurrence of x in \mathbf{w} , and by $t(\mathbf{w}, x)$ the right-most divider of \mathbf{w} that precedes the latest occurrence of x in \mathbf{w} .

Lemma 2.4 ([1, Corollary 2.5]). *A non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $\mathbf{F} \vee \mathbf{E}$ if and only if the claims*

$$(2.3) \quad \text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v}) \text{ and } \text{mul}(\mathbf{u}) = \text{mul}(\mathbf{v}),$$

$$(2.4) \quad h_1(\mathbf{u}, x) = h_1(\mathbf{v}, x) \text{ for all } x \in \text{con}(\mathbf{u}),$$

$$(2.5) \quad h_2(\mathbf{u}, x) = h_2(\mathbf{v}, x) \text{ for all } x \in \text{con}(\mathbf{u}),$$

$$(2.6) \quad t(\mathbf{u}, x) = t(\mathbf{v}, x) \text{ for all } x \in \text{con}(\mathbf{u})$$

are true. \square

The following claim is evident.

Lemma 2.5. *Let $\mathbf{u} \approx \mathbf{v}$ be an identity that satisfies the claims (2.3) and (2.4). Suppose that*

$$(2.7) \quad t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_m \mathbf{u}_m$$

is the decomposition of \mathbf{u} . Then the decomposition of \mathbf{v} has the form

$$(2.8) \quad t_0 \mathbf{v}_0 t_1 \mathbf{v}_1 \cdots t_m \mathbf{v}_m$$

for some words $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$. \square

If ρ is an equivalence relation on the free monoid F^1 then we say that a word \mathbf{w} is a ρ -term for a variety \mathbf{V} if $\mathbf{w} \rho \mathbf{w}'$ whenever \mathbf{V} satisfies $\mathbf{w} \approx \mathbf{w}'$. The following construction from [12] is a generalisation of the construction $S(W)$. Let ρ be a congruence on the free monoid F^1 and W be a set of words in F^1 such that the empty word forms a singleton ρ -class, W is a union of ρ -classes and W is closed under taking subwords. Since W is a union of ρ -classes, the ideal $I(W) = F^1 \setminus W$ is also a union of ρ -classes if it is not empty. Let φ_ρ denote the homomorphism

corresponding to ρ . Since W is closed under taking subwords, $\varphi_\rho(I(W))$ is an ideal of the quotient monoid F^1/ρ . We define $S_\rho(W)$ as the Rees quotient of F^1/ρ over $\varphi_\rho(I(W))$.

The following lemma gives us a connection between monoids of the form $S_\rho(W)$ and ρ -terms for monoid varieties.

Lemma 2.6 ([12, Lemma 7.1]). *Let ρ be a congruence on the free monoid F^1 and W be a set of words in F^1 such that the empty word forms a singleton ρ -class, W is a union of ρ -classes and W is closed under taking subwords. A monoid variety \mathbf{V} contains $S_\rho(W)$ if and only if every word in W is a ρ -term for \mathbf{V} .*

3. THE VARIETY \mathbf{O} AND ITS SUBVARIETIES

The goal of this section is to prove the following

Proposition 3.1. *The variety \mathbf{O} is hereditarily finitely based.*

We need the several auxiliary results. The following statement is evident. We will use it below without references.

Lemma 3.1. *The identities (1.1) and*

$$(3.1) \quad xtyzxy \approx xtyzyx$$

hold in the variety \mathbf{O} . □

Lemma 3.2. *Let $\mathbf{w} = \mathbf{v}_1 a \mathbf{v}_2 a \mathbf{v}_3$ where $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 are possibly empty words. Suppose that $a \in \text{con}(\mathbf{v}_1)$ and $\text{con}(\mathbf{v}_2) \subseteq \text{mul}(\mathbf{w})$. Then \mathbf{O} satisfies the identity $\mathbf{w} \approx \mathbf{v}_1 \mathbf{v}_2 a \mathbf{v}_3$.*

Proof. Let $\mathbf{v}_2 = x_1 x_2 \cdots x_n$ where the letters x_1, x_2, \dots, x_n are not necessarily different. We will use induction by n .

Induction base. Suppose that $n = 0$. Here the identity

$$\mathbf{w} = \mathbf{v}_1 a^2 \mathbf{v}_3 \stackrel{(1.1)}{\approx} \mathbf{v}_1 a \mathbf{v}_3$$

holds in \mathbf{O} and we are done.

Induction step. Let now $n > 0$. If $x_n \in \text{con}(\mathbf{v}_1 x_1 x_2 \cdots x_{n-1})$ then \mathbf{O} satisfies the identities

$$\begin{aligned} \mathbf{v}_1 \mathbf{v}_2 a \mathbf{v}_3 &= \mathbf{v}_1 x_1 x_2 \cdots x_n a \mathbf{v}_3 \\ &\approx \mathbf{v}_1 x_1 x_2 \cdots x_{n-1} a x_n \mathbf{v}_3 && \text{by (3.1)} \\ &\approx \mathbf{v}_1 a x_1 x_2 \cdots x_{n-1} a x_n \mathbf{v}_3 && \text{by the induction assumption} \\ &\approx \mathbf{v}_1 a x_1 x_2 \cdots x_n a \mathbf{v}_3 && \text{by (3.1)} \\ &= \mathbf{w}. \end{aligned}$$

If $x_n \notin \text{con}(\mathbf{v}_1 x_1 x_2 \cdots x_{n-1})$ then $x_n \in \text{con}(\mathbf{v}_3)$ because $x_n \in \text{mul}(\mathbf{w})$. Then the identities

$$\begin{aligned} \mathbf{v}_1 \mathbf{v}_2 a \mathbf{v}_3 &= \mathbf{v}_1 x_1 x_2 \cdots x_n a \mathbf{v}_3 \\ &\approx \mathbf{v}_1 x_1 x_2 \cdots x_{n-1} a x_n a \mathbf{v}_3 && \text{by (1.5)} \\ &\approx \mathbf{v}_1 a x_1 x_2 \cdots x_{n-1} a x_n a \mathbf{v}_3 && \text{by the induction assumption} \\ &\approx \mathbf{v}_1 a x_1 x_2 \cdots x_n a \mathbf{v}_3 && \text{by (1.5)} \\ &= \mathbf{w} \end{aligned}$$

hold in \mathbf{O} . □

Lemma 3.3. *Let \mathbf{v}_1 and \mathbf{v}_2 be words. If $\text{con}(\mathbf{v}_2) = \{x_1, x_2, \dots, x_n\} \subseteq \text{con}(\mathbf{v}_1)$ then the identity $\mathbf{v}_1\mathbf{v}_2 \approx \mathbf{v}_1x_1x_2 \cdots x_n$ holds in \mathbf{O} .*

Proof. The identities

$$\mathbf{v}_1\mathbf{v}_2 \stackrel{(3.1)}{\approx} \mathbf{v}_1x_1^{\text{occ}_{x_1}(\mathbf{v}_2)}x_2^{\text{occ}_{x_2}(\mathbf{v}_2)} \cdots x_n^{\text{occ}_{x_n}(\mathbf{v}_2)} \stackrel{(1.1)}{\approx} \mathbf{v}_1x_1x_2 \cdots x_n$$

hold in \mathbf{O} . □

Let \mathbf{u} and \mathbf{v} be words with decompositions (2.7) and (2.8) respectively. The identity $\mathbf{u} \approx \mathbf{v}$ is *not well-balanced at x* if $\text{occ}_x(\mathbf{u}_i) \neq \text{occ}_x(\mathbf{v}_i)$ for some i . For brevity, put $\mathbf{O}\{\Sigma\} = \mathbf{O} \wedge \text{var}\{\Sigma\}$ for any identity system Σ .

Lemma 3.4. *Let $\mathbf{u} \approx \mathbf{v}$ be an identity that holds in $\mathbf{F} \vee \mathbf{E}$. Then either*

$$\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u}' \approx \mathbf{v}'\} \text{ or } \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u}' \approx \mathbf{v}'\}, \quad (1.4)$$

for some well-balanced identity $\mathbf{u}' \approx \mathbf{v}'$.

Proof. If $\mathbf{u} \approx \mathbf{v}$ is well-balanced then we are done. Suppose that $\mathbf{u} \approx \mathbf{v}$ is not well-balanced. By induction, we may assume that this identity is not well-balanced at precisely one letter x . In view of Lemma 2.4, the claims (2.4)–(2.6) are true. Let (2.7) be the decomposition of \mathbf{u} . Then the decomposition of \mathbf{v} has the form (2.8) by Lemma 2.5. By induction, we may assume that there is k such that $\text{occ}_x(\mathbf{u}_k) \neq \text{occ}_x(\mathbf{v}_k)$ but $\text{occ}_x(\mathbf{u}_i) = \text{occ}_x(\mathbf{v}_i)$ for any $i \neq k$. We may assume without any loss that $\text{occ}_x(\mathbf{u}_k) < \text{occ}_x(\mathbf{v}_k)$. Put

$$\mathbf{u}' = \prod_{i=0}^{k-1} (\mathbf{u}_i t_{i+1}), \quad \mathbf{u}'' = \prod_{i=k+1}^m (t_i \mathbf{u}_i), \quad \mathbf{v}' = \prod_{i=0}^{k-1} (\mathbf{v}_i t_{i+1}), \quad \mathbf{v}'' = \prod_{i=k+1}^m (t_i \mathbf{v}_i).$$

Suppose that $x \in \text{con}(\mathbf{u}_k)$. It follows that $\text{occ}_x(\mathbf{v}_k) \geq 2$. Therefore, the first and the second occurrences of x in \mathbf{v} lie in the subword $\mathbf{v}'\mathbf{v}_k$. In view of the claims (2.4) and (2.5), the first and the second occurrences of x in \mathbf{u} lie in the subword $\mathbf{u}'\mathbf{u}_k$. Then there are subwords \mathbf{u}'_k and \mathbf{u}''_k of \mathbf{u}_k such that $\mathbf{u}_k = \mathbf{u}'_k x \mathbf{u}''_k$ and $x \in \text{con}(\mathbf{u}'_k)$. Put

$$\mathbf{p} = \mathbf{u}'_k x^{\text{occ}_x(\mathbf{v}_k) - \text{occ}_x(\mathbf{u}_k) + 1} \mathbf{u}''_k.$$

The identity $\mathbf{u} \stackrel{(1.1)}{\approx} \mathbf{u}'\mathbf{p}\mathbf{u}''$ holds in \mathbf{O} , whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u}'\mathbf{p}\mathbf{u}'' \approx \mathbf{v}\}$. It remains to note that the identity $\mathbf{u}'\mathbf{p}\mathbf{u}'' \approx \mathbf{v}$ is well-balanced.

Suppose now that $x \notin \text{con}(\mathbf{u}_k)$. In view of [8, Lemma 5.1], $\mathbf{Q} \not\subseteq \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$. This fact and [8, Lemma 5.3] imply that $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies the identity

$$(3.2) \quad x^2 y z x^2 \approx x^2 y x z x^2.$$

Then the identities

$$x y z z t x \stackrel{(1.1)}{\approx} x y x^2 z t x^2 \stackrel{(3.2)}{\approx} x y x^2 z x t x^2 \stackrel{(1.1)}{\approx} x y x z x t x$$

hold in the variety $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$. We see that this variety satisfies the identity (1.4). The second occurrence of x in \mathbf{v} does not lie in the block \mathbf{v}_k by the claim (2.5). If $t_j = h_2(\mathbf{v}, x)$ for some $j > k$ then we obtain a contradiction with the claim (2.4) and the fact that $x \in \text{con}(\mathbf{v}_k) \setminus \text{con}(\mathbf{u}_k)$. So, the second occurrence of x in \mathbf{v} lies in the subword \mathbf{v}' . Taking into account the claims (2.4) and (2.5) again, we get that the first and the second occurrences of x in \mathbf{u} lie in the subword \mathbf{u}' .

Further, the fact that $x \in \text{con}(\mathbf{v}_k) \setminus \text{con}(\mathbf{u}_k)$ and the claim (2.6) imply that the latest occurrences of x in \mathbf{u} and \mathbf{v} lie in the subwords \mathbf{u}'' and \mathbf{v}'' respectively. Put $\mathbf{q} = \mathbf{u}_k x^{\text{occ}_x(\mathbf{v}_k)}$. Then the identity $\mathbf{u} \stackrel{(1.4)}{\approx} \mathbf{u}'\mathbf{q}\mathbf{u}''$ holds in the variety $\mathbf{O}\{(1.4)\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u}'\mathbf{q}\mathbf{u}'' \approx \mathbf{v}, (1.4)\}$. It remains to note that the identity $\mathbf{u}'\mathbf{q}\mathbf{u}'' \approx \mathbf{v}$ is well-balanced. \square

The identity $\mathbf{u} \approx \mathbf{v}$ is said to be 1-*invertible* if $\mathbf{u} = \mathbf{w}'xy\mathbf{w}''$ and $\mathbf{v} = \mathbf{w}'yx\mathbf{w}''$ for some possibly empty words \mathbf{w}' , \mathbf{w}'' and some letters $x, y \in \text{con}(\mathbf{w}'\mathbf{w}'')$. Let now $n > 1$. The identity $\mathbf{u} \approx \mathbf{v}$ is said to be n -*invertible* if there exists a sequence of words

$$\mathbf{u} = \mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n = \mathbf{v}$$

such that the identity $\mathbf{w}_j \approx \mathbf{w}_{j+1}$ is 1-*invertible* for any $j \in \{0, 1, \dots, n-1\}$ and n is the least number with such a property. For convenience, the trivial identity is called 0-*invertible*.

Lemma 3.5. *Let $\mathbf{u} \approx \mathbf{v}$ be a well-balanced identity. Then the variety $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ can be defined by the identities (1.2) and (1.5) together with some of the following identities: (1.3),*

$$(3.3) \quad yx^2txy \approx xytxy,$$

$$(3.4) \quad x^2ytxy \approx xytxy,$$

$$\alpha_n : xy \prod_{i=1}^{n+1} (t_i \mathbf{e}_i) \approx yx \prod_{i=1}^{n+1} (t_i \mathbf{e}_i),$$

$$\beta_n : yx^2 \prod_{i=2}^{n+1} (t_i \mathbf{e}_i) \approx xyx \prod_{i=1}^n (t_i \mathbf{e}_i),$$

$$\gamma_n : x^2y \prod_{i=1}^{n+1} (t_i \mathbf{e}_i) \approx xyx \prod_{i=1}^{n+1} (t_i \mathbf{e}_i),$$

$$\gamma'_n : x^2y \prod_{i=2}^{n+1} (t_i \mathbf{e}_i) \approx xyx \prod_{i=2}^{n+1} (t_i \mathbf{e}_i),$$

where $n \in \mathbb{N}$ and

$$\mathbf{e}_i = \begin{cases} x & \text{if } i \text{ is odd,} \\ y & \text{if } i \text{ is even.} \end{cases}$$

Proof. Put

$$\Phi = \{(1.3), (3.3), (3.4), \alpha_n, \beta_n, \gamma_n, \gamma'_n \mid n \in \mathbb{N}\}.$$

Since the identity $\mathbf{u} \approx \mathbf{v}$ is well-balanced, it is n -invertible for some $n \geq 0$. We will use induction by n .

Induction base. Suppose that $n = 0$. Here $\mathbf{u} = \mathbf{v}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\emptyset\}$.

Induction step. Let now $n > 0$. Let (2.7) be the decomposition of \mathbf{u} . Then the decomposition of \mathbf{v} has the form (2.8) because $\mathbf{u} \approx \mathbf{v}$ is well-balanced. There is $0 \leq i \leq m$ such that $\mathbf{u}_i \neq \mathbf{v}_i$. Let \mathbf{p} be the greatest common prefix of \mathbf{u}_i and \mathbf{v}_i . Suppose that $\mathbf{u}_i = \mathbf{p}x\mathbf{u}'_i$ for some letter x and some word \mathbf{u}'_i . Since $\mathbf{u} \approx \mathbf{v}$

is well-balanced, there are words \mathbf{a}, \mathbf{b} and a letter y such that $\mathbf{v}_i = \mathbf{p} \mathbf{a} y x \mathbf{b}$ and $x \notin \text{con}(\mathbf{a}y)$. We note also that $y \in \text{con}(\mathbf{u}'_i)$ because $\mathbf{u} \approx \mathbf{v}$ is well-balanced. Put

$$\mathbf{v}' = \prod_{j=0}^{i-1} (\mathbf{v}_j t_{j+1}) \mathbf{p}, \quad \mathbf{v}'' = \prod_{j=i+1}^m (t_j \mathbf{v}_j) \quad \text{and} \quad \mathbf{w} = \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}''.$$

We are going to verify that

$$\text{either } \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\} \text{ or } \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \sigma\}$$

for some $\sigma \in \Phi$.

Suppose that $x, y \in \text{con}(\mathbf{v}' \mathbf{a} \mathbf{b})$. If $x, y \in \text{con}(\mathbf{b})$ then the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}'' \\ &\approx \mathbf{v}' \mathbf{a} x^2 y^2 \mathbf{b} \mathbf{v}'' && \text{by Lemma 3.2} \\ &\approx \mathbf{v}' \mathbf{a} y^2 x^2 \mathbf{b} \mathbf{v}'' && \text{by (1.2)} \\ &\approx \mathbf{v}' \mathbf{a} y x \mathbf{b} \mathbf{v}''' && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in \mathbf{O} , whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\}$. So, we may assume that either $x \notin \text{con}(\mathbf{b})$ or $y \notin \text{con}(\mathbf{b})$. By symmetry, it suffices to consider the case $y \notin \text{con}(\mathbf{b})$. Then $y \in \text{con}(\mathbf{v}' \mathbf{a})$. If $x \in \text{con}(\mathbf{b})$ then the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}'' \\ &\approx \mathbf{v}' \mathbf{a} x^2 y \mathbf{b} \mathbf{v}'' && \text{by Lemma 3.2} \\ &\approx \mathbf{v}' \mathbf{a} x^2 y^2 \mathbf{b} \mathbf{v}'' && \text{by (1.1)} \\ &\approx \mathbf{v}' \mathbf{a} y^2 x^2 \mathbf{b} \mathbf{v}'' && \text{by (1.2)} \\ &\approx \mathbf{v}' \mathbf{a} y x^2 \mathbf{b} \mathbf{v}'' && \text{by (1.1)} \\ &\approx \mathbf{v}' \mathbf{a} y x \mathbf{b} \mathbf{v}''' && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in \mathbf{O} . Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\}$. Finally, if $x \notin \text{con}(\mathbf{b})$ then $x \in \text{con}(\mathbf{v}' \mathbf{a})$ and the identities

$$\mathbf{w} = \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}'' \stackrel{(3.1)}{\approx} \mathbf{v}' \mathbf{a} y x \mathbf{b} \mathbf{v}'' = \mathbf{v}$$

hold in \mathbf{O} , whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\}$ again. Thus, it remains to consider the case when either $x \notin \text{con}(\mathbf{v}' \mathbf{a} \mathbf{b})$ or $y \notin \text{con}(\mathbf{v}' \mathbf{a} \mathbf{b})$. By symmetry, we may assume without loss of generality that $y \notin \text{con}(\mathbf{v}' \mathbf{a} \mathbf{b})$. Then $y \in \text{con}(\mathbf{v}'')$ because $y \in \text{mul}(\mathbf{v})$.

Suppose that $x \in \text{con}(\mathbf{v}' \mathbf{a}) \cap \text{con}(\mathbf{b})$. Then the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}'' \\ &\approx \mathbf{v}' \mathbf{a} y \mathbf{b} \mathbf{v}'' && \text{by Lemma 3.2} \\ &\approx \mathbf{v}' \mathbf{a} y x \mathbf{b} \mathbf{v}''' && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in \mathbf{O} , whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\}$. So, we may assume that either $x \notin \text{con}(\mathbf{v}'\mathbf{a})$ or $x \notin \text{con}(\mathbf{b})$. Put

$$\mathbf{w}_j = t_{i+1} \prod_{s=i+2}^j (\mathbf{v}_{s-1} t_s) \quad \text{and} \quad \mathbf{w}'_j = \prod_{s=j+1}^m (t_s \mathbf{v}_s)$$

for any $j > i$.

Case 1: $x \notin \text{con}(\mathbf{v}'\mathbf{a}\mathbf{b})$. Then $x \in \text{con}(\mathbf{v}'')$ because $x \in \text{mul}(\mathbf{v})$.

Subcase 1.1: $x, y \in \text{con}(\mathbf{v}_k)$ for some $k > i$. Then, since $\mathbf{u} \approx \mathbf{v}$ is well-balanced, $\mathbf{u}(x, y, t_{i+1}) = xy t_{i+1} \mathbf{p}$ and $\mathbf{v}(x, y, t_{i+1}) = yx t_{i+1} \mathbf{q}$ where $\text{con}(\mathbf{p}) = \text{con}(\mathbf{q}) = \{x, y\}$. According to Lemma 3.3,

$$\mathbf{O}\{\mathbf{u}(x, y, t_{i+1}) \approx \mathbf{v}(x, y, t_{i+1})\} = \mathbf{O}\{(1.3)\}.$$

Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies (1.3). On the other hand, the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}'\mathbf{a}xy\mathbf{b}\mathbf{w}_k\mathbf{v}_k\mathbf{w}'_k \\ &\approx \mathbf{v}'\mathbf{a}xy\mathbf{b}\mathbf{w}_kxy\mathbf{v}_k\mathbf{w}'_k && \text{by Lemma 3.2} \\ &\approx \mathbf{v}'\mathbf{a}yx\mathbf{b}\mathbf{w}_kxy\mathbf{v}_k\mathbf{w}'_k && \text{by (1.3)} \\ &\approx \mathbf{v}'\mathbf{a}yx\mathbf{b}\mathbf{w}_k\mathbf{v}_k\mathbf{w}'_k && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in $\mathbf{O}\{(1.3)\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, (1.3)\}$.

Subcase 1.2: $|\text{con}(\mathbf{u}_j) \cap \{x, y\}| \leq 1$ for any $j > i$. Then we may assume without any loss that there exists a subsequence $k_1, k_2, \dots, k_r = m+1$ of $i+1, i+2, \dots, m+1$ such that

$$(3.5) \quad \text{con}(\mathbf{u}_{i+1}\mathbf{u}_{i+2} \cdots \mathbf{u}_{k_1-1}) \cap \{x, y\} = \emptyset;$$

$$(3.6) \quad \text{if } s \text{ is odd and } s \leq r-1 \text{ then } \text{con}(\mathbf{u}_{k_s}\mathbf{u}_{k_s+1} \cdots \mathbf{u}_{k_{s+1}-1}) \cap \{x, y\} = \{x\};$$

$$(3.7) \quad \text{if } s \text{ is even and } s \leq r-1 \text{ then } \text{con}(\mathbf{u}_{k_s}\mathbf{u}_{k_s+1} \cdots \mathbf{u}_{k_{s+1}-1}) \cap \{x, y\} = \{y\}.$$

Clearly, $r > 2$ because the letters x and y are multiple in \mathbf{u} and \mathbf{v} . Then, since $\mathbf{u} \approx \mathbf{v}$ is well-balanced,

$$\begin{aligned} \mathbf{u}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &= xy \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j), \\ \mathbf{v}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &= yx \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \end{aligned}$$

where

$$(3.8) \quad \mathbf{f}_j = \begin{cases} x^{a_j} & \text{if } j \text{ is odd,} \\ y^{a_j} & \text{if } j \text{ is even} \end{cases}$$

for some $a_j \in \mathbb{N}$ and $j = 1, 2, \dots, r-1$. In view of Lemma 3.3,

$$\mathbf{O}\{\mathbf{u}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) \approx \mathbf{v}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}})\} = \mathbf{O}\{\alpha_{r-2}\}.$$

Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies the identity α_{r-2} . On the other hand, the identities

$$\mathbf{w} = \mathbf{v}'\mathbf{a}xy\mathbf{b}\mathbf{v}'' \stackrel{\alpha_{r-2}}{\approx} \mathbf{v}'\mathbf{a}yx\mathbf{b}\mathbf{v}'' = \mathbf{v}$$

hold in $\mathbf{O}\{\alpha_{r-2}\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \alpha_{r-2}\}$.

Case 2: $x \notin \text{con}(\mathbf{v}'\mathbf{a})$ but $x \in \text{con}(\mathbf{b})$.

Subcase 2.1: $x, y \in \text{con}(\mathbf{v}_k)$ for some $k > i$. Since $\mathbf{u} \approx \mathbf{v}$ is well-balanced, substituting xt_{i+1} for t_{i+1} and 1 for all letters occurring in $\mathbf{u} \approx \mathbf{v}$ except x, y, t_{i+1} , we obtain the identity

$$x^c y x^d t_{i+1} \mathbf{p} \approx y x^e t_{i+1} \mathbf{q}$$

where $c, d \geq 1$, $e \geq 2$ and $\text{con}(\mathbf{p}) = \text{con}(\mathbf{q}) = \{x, y\}$. Lemma 3.2 implies that

$$\mathbf{O}\{x^c y x^d t_{i+1} \mathbf{p} \approx y x^e t_{i+1} \mathbf{q}\} = \mathbf{O}\{x y x t_{i+1} \mathbf{p} \approx y x^2 t_{i+1} \mathbf{q}\}.$$

According to Lemma 3.3,

$$\mathbf{O}\{x y x t_{i+1} \mathbf{p} \approx y x^2 t_{i+1} \mathbf{q}\} = \mathbf{O}\{(3.3)\}.$$

Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies (3.3). On the other hand, the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}'\mathbf{a} x y \mathbf{b} \mathbf{w}_k \mathbf{v}_k \mathbf{w}'_k \\ &\approx \mathbf{v}'\mathbf{a} x y x \mathbf{b} \mathbf{w}_k x y \mathbf{v}_k \mathbf{w}'_k && \text{by Lemma 3.2} \\ &\approx \mathbf{v}'\mathbf{a} y x^2 \mathbf{b} \mathbf{w}_k x y \mathbf{v}_k \mathbf{w}'_k && \text{by (3.3)} \\ &\approx \mathbf{v}'\mathbf{a} y x \mathbf{b} \mathbf{w}_k \mathbf{v}_k \mathbf{w}'_k && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in $\mathbf{O}\{(3.3)\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, (3.3)\}$.

Subcase 2.2: $|\text{con}(\mathbf{u}_j) \cap \{x, y\}| \leq 1$ for any $j > i$. Then there exists a subsequence $k_1, k_2, \dots, k_r = m + 1$ of $i + 1, i + 2, \dots, m + 1$ such that

$$\begin{aligned} &\text{con}(\mathbf{u}_{i+1} \mathbf{u}_{i+2} \cdots \mathbf{u}_{k_1-1}) \cap \{x, y\} \subseteq \{x\}; \\ (3.9) \quad &\text{if } s \text{ is even and } s \leq r-1 \text{ then } \text{con}(\mathbf{u}_{k_s} \mathbf{u}_{k_s+1} \cdots \mathbf{u}_{k_{s+1}-1}) \cap \{x, y\} = \{x\}; \\ (3.10) \quad &\text{if } s \text{ is odd and } s \leq r-1 \text{ then } \text{con}(\mathbf{u}_{k_s} \mathbf{u}_{k_s+1} \cdots \mathbf{u}_{k_{s+1}-1}) \cap \{x, y\} = \{y\}. \end{aligned}$$

Clearly, $r > 1$ because y is multiple in \mathbf{u} and \mathbf{v} . Since $\mathbf{u} \approx \mathbf{v}$ is well-balanced, substituting xt_{k_1} for t_{k_1} and 1 for all letters occurring in $\mathbf{u} \approx \mathbf{v}$ except $x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}$, we obtain the identity

$$(3.11) \quad x^c y x^d \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \approx y x^e \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j),$$

where $c, d \geq 1$, $e \geq 2$ and

$$(3.12) \quad \mathbf{f}_j = \begin{cases} y^{a_j} & \text{if } j \text{ is odd,} \\ x^{a_j} & \text{if } j \text{ is even} \end{cases}$$

for some $a_j \in \mathbb{N}$ and $j = 1, 2, \dots, r-1$. Lemma 3.2 implies that

$$\mathbf{O}\{(3.11)\} = \mathbf{O}\{x y x \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \approx y x^2 \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j)\}.$$

In view of Lemma 3.3,

$$\mathbf{O}\{x y x \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \approx y x^2 \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j)\} = \mathbf{O}\{\beta_{r-1}\}.$$

Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies β_{r-1} . On the other hand, the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}'\mathbf{a}xy\mathbf{b}\mathbf{v}'' \\ &\approx \mathbf{v}'\mathbf{a}xyx\mathbf{b}\mathbf{v}'' && \text{by Lemma 3.2} \\ &\approx \mathbf{v}'\mathbf{a}yx^2\mathbf{b}\mathbf{v}''' && \text{by the identity } \beta_{r-1} \\ &\approx \mathbf{v}'\mathbf{a}yx\mathbf{b}\mathbf{v}''' && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in $\mathbf{O}\{\beta_{r-1}\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \beta_{r-1}\}$.

Case 3: $x \in \text{con}(\mathbf{v}'\mathbf{a})$ but $x \notin \text{con}(\mathbf{b})$. Then $x \notin \text{con}(\mathbf{u}'_i)$ because $x \notin \text{con}(\mathbf{a})$ and $\text{con}(\mathbf{u}_i) = \text{con}(\mathbf{v}_i)$.

Subcase 3.1: $x, y \in \text{con}(\mathbf{v}_k)$ for some $k > i$. Then, since $\mathbf{u} \approx \mathbf{v}$ is well-balanced,

$$\mathbf{u}(x, y, t_{i+1}) = x^{c+1}yt_{i+1}\mathbf{p} \text{ and } \mathbf{v}(x, y, t_{i+1}) = x^c yxt_{i+1}\mathbf{q}$$

where $c \in \mathbb{N}$ and $\text{con}(\mathbf{p}) = \text{con}(\mathbf{q}) = \{x, y\}$. The identities

$$\mathbf{u}(x, y, t_{i+1}) \stackrel{(1.1)}{\approx} x^2yt_{i+1}\mathbf{p} \text{ and } \mathbf{v}(x, y, t_{i+1}) \stackrel{(1.5)}{\approx} xyxt_{i+1}\mathbf{q}$$

hold in \mathbf{O} . According to Lemma 3.3,

$$\mathbf{O}\{x^2yt_{i+1}\mathbf{p} \approx xyxt_{i+1}\mathbf{q}\} = \mathbf{O}\{(3.4)\}.$$

Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies (3.4). On the other hand, the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}'\mathbf{a}xy\mathbf{b}\mathbf{w}_k\mathbf{v}_k\mathbf{w}'_k \\ &\approx \mathbf{v}'\mathbf{a}x^2y\mathbf{b}\mathbf{w}_kxy\mathbf{v}_k\mathbf{w}'_k && \text{by Lemma 3.2} \\ &\approx \mathbf{v}'\mathbf{a}xyx\mathbf{b}\mathbf{w}_kxy\mathbf{v}_k\mathbf{w}'_k && \text{by (3.4)} \\ &\approx \mathbf{v}'\mathbf{a}yx\mathbf{b}\mathbf{w}_k\mathbf{v}_k\mathbf{w}'_k && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in $\mathbf{O}\{(3.4)\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, (3.4)\}$.

Subcase 3.2: $|\text{con}(\mathbf{u}_j) \cap \{x, y\}| \leq 1$ for any $j > i$. Then there exists a subsequence $k_1, k_2, \dots, k_r = m+1$ of $i+1, i+2, \dots, m+1$ such that the claim (3.5) is true and either the claims (3.6) and (3.7) are true or the claims (3.9) and (3.10) are true. Then

$$\begin{aligned} \mathbf{u}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &= x^{c+1}y \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j), \\ \mathbf{v}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &= x^c yx \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \end{aligned}$$

where $c \in \mathbb{N}$ and either (3.8) or (3.12) holds. The identities

$$\begin{aligned} \mathbf{u}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &\stackrel{(1.1)}{\approx} x^2y \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j), \\ \mathbf{v}(x, y, t_{k_1}, t_{k_2}, \dots, t_{k_{r-1}}) &\stackrel{(1.5)}{\approx} xyx \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \end{aligned}$$

hold in \mathbf{O} . Since the letter y is multiple in \mathbf{u} and \mathbf{v} , we have that $r > 2$ whenever (3.8) holds and $r > 1$ otherwise. In view of Lemma 3.3,

$$\mathbf{O}\left\{x^2y \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j) \approx xyx \prod_{j=1}^{r-1} (t_{k_j} \mathbf{f}_j)\right\} = \mathbf{O}\{\gamma\}$$

where $\gamma = \gamma_{r-2}$ whenever (3.8) holds and $\gamma = \gamma'_{r-1}$ whenever (3.12) holds. Therefore, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\}$ satisfies γ . On the other hand, the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{v}' \mathbf{a} x y \mathbf{b} \mathbf{v}'' \\ &\approx \mathbf{v}' \mathbf{a} x^2 y \mathbf{b} \mathbf{v}'' && \text{by (1.1)} \\ &\approx \mathbf{v}' \mathbf{a} x y x \mathbf{b} \mathbf{v}''' && \text{by the identity } \gamma \\ &\approx \mathbf{v}' \mathbf{a} y x \mathbf{b} \mathbf{v}''' && \text{by Lemma 3.2} \\ &= \mathbf{v} \end{aligned}$$

hold in $\mathbf{O}\{\gamma\}$, whence $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \gamma\}$.

So, we have proved that

$$\text{either } \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}\} \text{ or } \mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \sigma\}$$

for some $\sigma \in \Phi$. The identity $\mathbf{u} \approx \mathbf{w}$ is $(n-1)$ -invertible. By the induction assumption, $\mathbf{O}\{\mathbf{u} \approx \mathbf{w}\} = \mathbf{O}\{\Sigma\}$ for some $\Sigma \subseteq \Phi$. It follows that $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\Sigma, \sigma\}$ whenever $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\mathbf{u} \approx \mathbf{w}, \sigma\}$ and $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{O}\{\Sigma\}$ otherwise. Lemma 3.5 is proved. \square

Proof of Proposition 3.1. Let \mathbf{V} be a subvariety of \mathbf{O} . If $\mathbf{E} \not\subseteq \mathbf{V}$ then \mathbf{V} is finitely based by Lemmas 2.2 and 2.3(i). If $\mathbf{F} \not\subseteq \mathbf{V}$ then Lemmas 2.2 and 2.3(ii) imply that \mathbf{V} is finitely based again. So, we may assume that $\mathbf{FV}\mathbf{E} \subseteq \mathbf{V}$. In view of Lemma 3.4, the variety \mathbf{V} can be given within the variety \mathbf{O} by a set of well-balanced identities Σ together with the identity (1.4). According to Lemma 3.5, $\mathbf{O}\{\Sigma\} = \mathbf{O}\{\Psi\}$ for some $\Psi \subseteq \Phi$, where Φ has the same sense as in the proof of Lemma 3.5. Evidently, the identities $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$ and γ'_{n+1} follow from the identities $\alpha_n, \beta_n, \gamma_n$ and γ'_n respectively for any $n \in \mathbb{N}$. Therefore, the monoid variety given by the identity system Ψ is finitely based. Then \mathbf{V} is finitely based too. \square

4. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we need one auxiliary result.

Lemma 4.1. *Let \mathbf{V} be a monoid variety that contains the variety $\mathbf{FV}\mathbf{E}$ and satisfies the identity (1.1). If \mathbf{V} does not contain \mathbf{J} then \mathbf{V} satisfies (1.5).*

Proof. The words \mathbf{p} and \mathbf{q} are of the same type if \mathbf{p} can be obtained from \mathbf{q} by changing the individual exponents of letters and the second occurrence of a is next to the first occurrence of a in \mathbf{p} if and only if the second occurrence of a is next to the first occurrence of a in \mathbf{q} for any letter a . For example, the words $xy^4xz^3x^5y$ and $xy^3x^4z^2x^2y^2$ are of the same type. A word \mathbf{w} is called *reduced* if $a \notin \text{con}(\mathbf{p})$ whenever $\mathbf{w} = \mathbf{p}a^2\mathbf{q}$ for any $a \in \text{con}(\mathbf{w})$. It is shown in [13, Section 3] that for any word \mathbf{w} there is a unique reduced word $r(\mathbf{w})$ such that $\text{sim}(\mathbf{w}) = \text{sim}(r(\mathbf{w}))$ and the words \mathbf{w} and $r(\mathbf{w})$ are of the same type. This implies that the relation τ given by $\mathbf{u} \tau \mathbf{v}$ if and only if $r(\mathbf{u}) = r(\mathbf{v})$ is a congruence on F^1 . Let $W \leq$ denote the set of all subwords of a set of words W .

In view of [13, Theorem 5.1(x)], the variety \mathbf{J} is generated by the monoid $S_\tau(\{xzyx^kty^\ell \mid k, \ell \in \mathbb{N}\}^\leq)$. Then a word \mathbf{u} in $\{xzyx^kty^\ell \mid k, \ell \in \mathbb{N}\}^\leq$ is not a τ -term for \mathbf{V} by Lemma 2.6. This means that \mathbf{V} satisfies an identity $\mathbf{u} \approx \mathbf{v}$ with $(\mathbf{u}, \mathbf{v}) \notin \tau$. Since $\mathbf{F} \vee \mathbf{E} \subseteq \mathbf{V}$, Theorem 5.1(vii) and Fact 5.3(ii) of [13] imply that the word \mathbf{u} contains a block with two distinct letters. Then $\mathbf{u} \in \{xzyx^kty^\ell, zyx^mty^\ell, yx^mty^\ell \mid k, \ell \in \mathbb{N}, m \geq 2\}$.

If $\mathbf{u} = yx^mty^\ell$ then $\mathbf{v} = x^p yx^q ty^r$ for some $p, r \in \mathbb{N}$ and $q \geq 0$ by Lemma 2.4. It follows that \mathbf{V} satisfies $xzyx^mty^\ell \approx xz\mathbf{v}$, whence the word $xzyx^mty^\ell$ is not a τ -term for \mathbf{V} . If $\mathbf{u} = zyx^mty^\ell$ then $\mathbf{v} = zx^p yx^q ty^r$ for some $p, r \in \mathbb{N}$ and $q \geq 0$ by Lemma 2.4. It follows that \mathbf{V} satisfies $xzyx^mty^\ell \approx x\mathbf{v}$, whence the word $xzyx^mty^\ell$ is not a τ -term for \mathbf{V} again. So, we may assume that $\mathbf{u} = xzyx^kty^\ell$. Then $\mathbf{v} = tx^p yx^q ty^r$ for some $p, r \in \mathbb{N}$ and $q \geq 0$ by Lemma 2.4. If $q \geq 1$ then \mathbf{V} satisfies the identities

$$xzyxty \stackrel{(1.1)}{\approx} xzyx^kty^\ell \approx tx^p yx^q ty^r \stackrel{(1.1)}{\approx} txxyxty,$$

and we are done. If $q = 0$ then the identities

$$xzyxty \stackrel{(1.1)}{\approx} xzyx^{k+1}ty^\ell \approx tx^p yxty^r \stackrel{(1.1)}{\approx} txxyxty,$$

hold in \mathbf{V} , and we are done again. \square

Proof of Theorem 1.1. Let \mathbf{V} be a limit variety within the class \mathbf{A}_{com} . If $S(xy x) \in \mathbf{V}$ then $\mathbf{V} \in \{\mathbf{L}, \mathbf{M}\}$ by [7, Theorem 3.2]. Suppose now that $S(xy x) \notin \mathbf{V}$. In view of Lemma 2.1, \mathbf{V} satisfies one of the identities (1.1) or (2.1). By symmetry, we may assume without any loss that \mathbf{V} satisfies (1.1). It is well known and can be easily verified that every variety from \mathbf{A}_{com} satisfies the identity $x^n y^n \approx y^n x^n$ for some $n \in \mathbb{N}$. This identity together with (1.1) implies (1.2). According to Lemmas 2.2 and 2.3, $\mathbf{F} \vee \mathbf{E} \subseteq \mathbf{V}$. Suppose that $\mathbf{V} \neq \mathbf{J}$. Then Lemma 4.1 implies that \mathbf{V} satisfies the identity (1.5). Thus, $\mathbf{V} \subseteq \mathbf{O}$. We obtain a contradiction with Proposition 3.1.

Theorem 1.1 is proved. \square

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