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Asymptotics and Hille-Type Results for Dynamic Equations of Third Order with Deviating Arguments

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Abstract: The aim of this paper is to deduce the asymptotic and Hille-type criteria of the dynamic equations of third order on time scales. Some of the presented results concern the sufficient condition for the oscillation of all solutions of third-order dynamical equations. Additionally, compared with the related contributions reported in the literature, the Hille-type oscillation criterion which is derived is superior for dynamic equations of third order. The symmetry plays a positive and influential role in determining the appropriate type of study for the qualitative behavior of solutions to dynamic equations. Some examples of Euler-type equations are included to demonstrate the finding.

Keywords: asymptotic behavior; Hille-type oscillation criteria; Euler-type equation; time scales; dynamic equations

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1. Introduction

The growing interest in oscillatory properties of solutions to dynamic equations on time scales has resulted from their large applications in the engineering and natural sciences. In this paper, we are concerned with the asymptotic and Hille-type criteria of the linear functional dynamic equation of third order

$$\left\{p_2(\xi)\left[p_1(\xi)z^{\Delta}(\xi)\right]^{\Delta}\right\}^{\Delta}+a(\xi)z(\phi(\xi))=0 \tag{1}$$

on an above-unbounded time scale $\mathbb{T}$, where $a\in C_{\mathrm{rd}}([\xi_0,\infty)_{\mathbb{T}},\mathbb{R})$ is non-negative and does not vanish eventually, where C_{rd} is the space of right-dense continuous functions; $p_i\in C_{\mathrm{rd}}([\xi_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $i=1,2$, satisfy

$$\int_{\xi_0}^{\infty}\frac{\Delta s}{p_i(s)}=\infty,\ i=1,2, \tag{2}$$

and $\phi\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ is strictly increasing function such that $\lim_{\xi\to\infty}\phi(\xi)=\infty$. As a notational convenience, we let

$$z^{[i]}(\xi):=p_i(\xi)[z^{[i-1]}(\xi)]^{\Delta},\ i=1,2,3,\quad \text{with } z^{[0]}(\xi)=z(\xi),\ p_3=1,$$

$$H_i(\xi,\tau):=\int_{\tau}^{\xi}\frac{H_{i-1}(s,\tau)}{p_{i-1}(s)}\Delta s,\ i=1,2,\quad \text{with } H_0(\xi,\tau):=\frac{1}{p_2(\xi)},\ p_0=1,$$

and

$$G_i(\xi,\tau):=\int_{\tau}^{\xi}\frac{G_{i-1}(s,\tau)}{p_{i-1}(s)}\Delta s,\ i=1,2,\quad \text{with } G_0(\xi,\tau):=\frac{1}{p_1(\xi)},\ p_0=1.$$

By a solution of Equation (1) we mean a nontrivial real–valued function $z \in C^1_{rd}[T_z, \infty)_{\mathbb{T}}$ for some $T_z \geq \xi_0$ for a positive constant $\xi_0 \in \mathbb{T}$ such that $z^{[1]}(\xi)$, $z^{[2]}(\xi) \in C^1_{rd}[T_z, \infty)_{\mathbb{T}}$ and $z(\xi)$ satisfies Equation (1) on $[T_z, \infty)_{\mathbb{T}}$, for an excellent introduction to the calculus on time scales, see [1–4]. The solutions vanishing in some neighbourhood of infinity will be excluded from the consideration. A solution z of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The symmetry of the dynamic equations in terms of nonoscillatory solutions plays an essential and fundamental role in deciding the right way to study the oscillatory behavior of solutions to these equations. In the following, we introduce some oscillation criteria for differential equations that will be connected to our oscillation results for (1) on time scales and explain the significant contributions of this paper. Fite [5] deliberated the oscillatory behavior of the linear differential equation of second order

$$z''(\xi) + a(\xi)z(\xi) = 0, \tag{3}$$

and saw that if

$$\int_{\xi_0}^{\infty} a(s)\mathrm{d}s = \infty, \tag{4}$$

then every solution of Equation (3) is oscillatory. Hille [6] ameliorated the condition (4) and showed that if

$$\liminf_{\xi \to \infty} \xi \int_{\xi}^{\infty} a(s)\mathrm{d}s > \frac{1}{4}, \tag{5}$$

then every solution of (3) is oscillatory. Wong [7] improved the Hille-type condition (5) for the differential equation

$$z''(\xi) + a(\xi)z(\phi(\xi)) = 0, \tag{6}$$

where $\phi(\xi) \geq k\xi$ with $0 < k < 1$, and proved that if

$$\liminf_{\xi \to \infty} \xi \int_{\xi}^{\infty} a(s)\mathrm{d}s > \frac{1}{4k}, \tag{7}$$

then every solution of (6) is oscillatory. Erbe [8] generalized the condition (7) and established that if

$$\liminf_{\xi \to \infty} \xi \int_{\xi}^{\infty} a(s)\frac{\phi(s)}{s}\mathrm{d}s > \frac{1}{4}, \tag{8}$$

then every solution of (6) is oscillatory where $\phi(\xi) \leq \xi$.

Regarding oscillation criteria for dynamic equations that will be related to our main results, Erbe et al. [9] studied Hille type oscillation criterion for the dynamic equation of third order

$$z^{\Delta\Delta\Delta}(\xi) + a(\xi)z(\xi) = 0. \tag{9}$$

We list the main results of [9] as follows:

Theorem 1 ([9]). *Every solution of Equation* (9) *is either oscillatory or tends to zero eventually provided that*

$$\int_{\xi_0}^{\infty} \int_{v}^{\infty} \int_{u}^{\infty} a(s)\Delta s \Delta u \Delta v = \infty, \tag{10}$$

and

$$\liminf_{\xi \to \infty} \xi \int_{\xi}^{\infty} \frac{h_2(s, \xi_0)}{\sigma(s)} a(s)\Delta s > \frac{1}{4}; \tag{11}$$

where $h_2(\xi, s)$ is the Taylor monomial of degree 2*, see ([2] Section 1.6).*

Agarwal et al. [10] suggested some Hille type oscillation criterion to the delay dynamic Equation (1), where $\phi(\xi) \leq \xi$ on $[\xi_0, \infty)_{\mathbb{T}}$ and under the assumptions (2) and

$$\int_{\xi_0}^{\infty} \frac{1}{p_1(v)} \int_{v}^{\infty} \frac{1}{p_2(u)} \int_{u}^{\infty} a(s)\, \Delta s \Delta u \Delta v = \infty. \tag{12}$$

One of these results in [10] reads as follows.

Theorem 2 ([10])**.** *Every solution of Equation* (1) *is either oscillatory or tends to zero eventually if* (2) *and* (12) *hold, and*

$$\liminf_{\xi \to \infty} H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_2)}{H_1(\sigma(s), \xi_1)} a(s) \Delta s > \frac{1}{4}, \tag{13}$$

for $[\xi_2, \infty)_{\mathbb{T}} \subseteq (\xi_1, \infty)_{\mathbb{T}} \subseteq (\xi_0, \infty)_{\mathbb{T}}$.

We note that the results in [10] included the results which were established in [9]. We refer the reader to the papers [11–28], and the references cited therein.

It should be noted that research in this paper was strongly motivated by the contributions of Hille [6]. The objective of this paper is to infer asymptotics and improved Hille-type oscillation criteria for (1) in the cases where $\phi(\xi) \leq \xi$ and $\phi(\xi) \geq \xi$.

Below, all functional inequalities presented in this paper would be supposed to hold eventually, that is, they are fulfilled for all sufficiently large ξ.

2. Main Results

We start this section with the following preliminary lemmas, which will perform an essential role in proving the main results. The proof of first two lemmas follow immediately from the canonical form ((2) holds) of (1); thus, we omit the details.

Lemma 1. *If* $z(\xi)$ *is a nonoscillatory solution of the Equation* (1)*, then* $z^{[i]}(\xi)$, $i = 0, 1, 2, 3$, *are eventually of one sign.*

Lemma 2. *Let* (2) *hold. If* $z(\xi)$ *is a nonoscillatory solution of the Equation* (1)*, then* $z(\xi)\, z^{[2]}(\xi)$ *is eventually positive.*

Lemma 3 ([11])**.** *(Lemma 2.1) Assume that*

(B) *either*

$$\int_{\xi_0}^{\infty} \frac{1}{p_2(u)} \int_{u}^{\infty} a(s)\, \Delta s\, \Delta u = \infty;$$

or

$$\int_{\xi_0}^{\infty} \frac{1}{p_1(v)} \int_{v}^{\infty} \frac{1}{p_2(u)} \int_{u}^{\infty} a(s)\, \Delta s\, \Delta u \Delta v = \infty.$$

If $z(\xi)$ *is an eventually nonoscillatory solution of the Equation* (1) *and corresponding* $z(\xi)$ *satisfies* $z^{[i-1]}(\xi)\, z^{[i]}(\xi) < 0$, $i = 1, 2$, *eventually, then* $z(\xi)$ *tends to zero eventually.*

Lemma 4. *If* $z(\xi)$ *is a nonoscillatory solution of the Equation* (1) *and corresponding* $z(\xi)$ *satisfies* $z^{[i-1]}(\xi)\, z^{[i]}(\xi) > 0$, $i = 1, 2$, *eventually, then* $z(\xi)$ *satisfies the following for sufficiently large* $\xi \in (\xi_0, \infty)_{\mathbb{T}}$,

$$\left(\frac{\left| z^{[1]}(\xi) \right|}{H_1(\xi, \xi_0)} \right)^{\Delta} < 0, \tag{14}$$

and

$$\frac{z(\xi)}{z^{[1]}(\xi)} \geq \frac{H_2(\xi, \xi_0)}{H_1(\xi, \xi_0)}, \tag{15}$$

and there is a $k \in (0,1)$ *such that*

$$k\frac{z(\xi)}{z^{[1]}(\xi)} \leq G_1(\xi,\xi_0). \tag{16}$$

Proof. Without loss of generality, let

$$z^{[i]}(\xi) > 0,\ i = 0,1,2 \text{ and } z(\phi(\xi)) > 0 \text{ on } [\xi_0,\infty)_{\mathbb{T}}.$$

By using the fact that $\left(z^{[2]}(\xi)\right)^{\Delta} < 0$ on $[\xi_0,\infty)_{\mathbb{T}}$. Then for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z^{[1]}(\xi) &\geq z^{[1]}(\xi) - z^{[1]}(\xi_0) = \int_{\xi_0}^{\xi} z^{[2]}(s)H_0(s,\xi_0)\,\Delta s \\ &\geq z^{[2]}(\xi)\int_{\xi_0}^{\xi} H_0(s,\xi_0)\,\Delta s \\ &= z^{[2]}(\xi)H_1(\xi,\xi_0). \end{aligned} \tag{17}$$

Hence, we conclude that for $\xi \in (\xi_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned} \left(\frac{z^{[1]}(\xi)}{H_1(\xi,\xi_0)}\right)^{\Delta} &= \frac{\left[z^{[1]}(\xi)\right]^{\Delta}H_1(\xi,\xi_0) - H_0(\xi,\xi_0)z^{[1]}(\xi)}{H_1(\xi,\xi_0)H_1(\sigma(\xi),\xi_0)} \\ &= \frac{H_0(\xi,\xi_0)\left\{z^{[2]}(\xi)H_1(\xi,\xi_0) - z^{[1]}(\xi)\right\}}{H_1(\xi,\xi_0)H_1(\sigma(\xi),\xi_0)} < 0. \end{aligned}$$

Thus (14) holds for $\xi \in (\xi_0,\infty)_{\mathbb{T}}$. Since $\dfrac{z^{[1]}(\xi)}{H_1(\xi,\xi_0)}$ is strictly decreasing on $(\xi_0,\infty)_{\mathbb{T}}$, we have for $\xi \in (\xi_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z(\xi) &\geq z(\xi) - z(\xi_0) = \int_{\xi_0}^{\xi} \frac{z^{[1]}(s)}{p_1(s)}\Delta s \\ &\geq \frac{z^{[1]}(\xi)}{H_1(\xi,\xi_0)}\int_{\xi_0}^{\xi} \frac{H_1(s,\xi_0)}{p_1(s)}\Delta s \\ &= \frac{z^{[1]}(\xi)}{H_1(\xi,\xi_0)}H_2(\xi,\xi_0). \end{aligned} \tag{18}$$

Thus (15) holds for $\xi \in (\xi_0,\infty)_{\mathbb{T}}$. Additionally, since $z(\xi)$ and $z^{[1]}(\xi)$ are strictly increasing on $[\xi_0,\infty)_{\mathbb{T}}$, then there is a $k \in (0,1)$ such that

$$\begin{aligned} kz(\xi) &\leq z(\xi) - z(\xi_0) = \int_{\xi_0}^{\xi} z^{[1]}(s)G_0(s,\xi_0)\Delta s \\ &\leq z^{[1]}(\xi)\int_{\xi_0}^{\xi} G_0(s,\xi_0)\Delta s \\ &= z^{[1]}(\xi)G_1(\xi,\xi_0). \end{aligned}$$

The proof is complete. □

3. Convergence of Nonoscillatory Solutions of Equation (1)

First, we present a Fite–Wintner type for (1). The proof is similar to that in [29] (Theorem 2.1), and hence is deleted.

Theorem 3. *Let* (2) *hold. If*

$$\int_{\xi_0}^{\infty} a(s)\,\Delta s = \infty, \tag{19}$$

then all nonoscillatory solutions of the Equation (1) *tend to zero eventually.*

From Theorem 3, we assume without loss of generality that

$$\int_{\xi_0}^{\infty} \varphi(s)a(s)\,\Delta s < \infty$$

for any function $\varphi(t) \leq 1$. Otherwise, we have that (19) holds due to $\varphi(t) \leq 1$.

Theorem 4. *Let $\phi(\xi) \leq \xi$ and* (2) *hold. If, for sufficiently large $T \in [\xi_0, \infty)_{\mathbb{T}}$,*

$$\liminf_{\xi\to\infty} H_1(\xi,\xi_0)\int_{\xi}^{\infty} \frac{H_2(\phi(s),T)}{H_1(s,T)} a(s)\Delta s > \frac{1}{4}, \tag{20}$$

then all nonoscillatory solutions of the Equation (1) *tend to a finite limit eventually.*

Proof. Suppose Equation (1) has a nonoscillatory solution $z(\xi)$. Then without loss of generality, let $z(\xi) > 0$ and $z(\phi(\xi)) > 0$ for $\xi \in [\xi_0, \infty)_{\mathbb{T}}$. By Lemmas 1 and 2 we have

$$z^{[2]}(\xi) > 0 \text{ and } z^{[3]}(\xi) < 0,$$

eventually and $z^{[1]}(\xi)$ is eventually of one sign. We consider the following two cases:

(I) $z^{[1]}(\xi)$ is positive eventually. Thus there is $\xi_1 \in [\xi_0, \infty)_{\mathbb{T}}$ such that

$$z^{[i]}(\xi) > 0,\ i = 1, 2, \text{ and } z^{[3]}(\xi) < 0 \qquad \text{for } \xi \geq \xi_1.$$

Define

$$x(\xi) := \frac{z^{[2]}(\xi)}{z^{[1]}(\xi)}. \tag{21}$$

Using the product and quotient rules, we obtain

$$\begin{aligned} x^{\Delta}(\xi) &= \frac{1}{z^{[1]}(\xi)}\left(z^{[2]}(\xi)\right)^{\Delta} + \left(\frac{1}{z^{[1]}(\xi)}\right)^{\Delta} z^{[2]}(\sigma(\xi)) \\ &= \frac{\left(z^{[2]}(\xi)\right)^{\Delta}}{z^{[1]}(\xi)} - \frac{\left(z^{[1]}(\xi)\right)^{\Delta} z^{[2]}(\sigma(\xi))}{z^{[1]}(\xi)z^{[1]}\sigma(\xi)}. \end{aligned}$$

From (1) and the definition of $x(\xi)$, we see that for $\xi \geq \xi_1$,

$$\begin{aligned} x^{\Delta}(\xi) &= -a(\xi)\frac{z(\phi(\xi))}{z^{[1]}(\xi)} - \frac{\left(z^{[1]}(\xi)\right)^{\Delta}}{z^{[1]}(\xi)} x(\sigma(\xi)) \\ &= -a(\xi)\frac{z(\phi(\xi))}{z^{[1]}(\xi)} - \frac{1}{p_2(\xi)} x(\xi)x(\sigma(\xi)). \end{aligned}$$

It follows from (15) and using the fact that $\dfrac{z^{[1]}(\xi)}{H_1(\xi,\xi_1)}$ is strictly decreasing that there exists a $\xi_2 \in (\xi_1, \infty)_{\mathbb{T}}$ such that for $\xi \in [\xi_2, \infty)_{\mathbb{T}}$,

$$z(\phi(\xi)) \geq z^{[1]}(\phi(\xi))\frac{H_2(\phi(\xi),\xi_1)}{H_1(\phi(\xi),\xi_1)} \geq z^{[1]}(\xi)\frac{H_2(\phi(\xi),\xi_1)}{H_1(\xi,\xi_1)}. \tag{22}$$

Hence, we conclude that for every $\xi \in [\xi_2, \infty)_{\mathbb{T}}$,

$$x^{\Delta}(\xi) \leq -\frac{H_2(\phi(\xi), \xi_1)}{H_1(\xi, \xi_1)} a(\xi) - \frac{1}{p_2(\xi)} x(\xi) x(\sigma(\xi)). \tag{23}$$

Integrating (23) from $\xi \geq \xi_2$ to $v \in [\xi, \infty)_{\mathbb{T}}$ and using the fact that $x > 0$, we have

$$-x(\xi) \leq x(v) - x(\xi) \leq -\int_{\xi}^{v} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s)\, \Delta s - \int_{\xi}^{v} \frac{1}{p_2(s)} x(s) x(\sigma(s)) \Delta s.$$

Taking $v \to \infty$ we get

$$-x(\xi) \leq -\int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s)\, \Delta s - \int_{\xi}^{\infty} \frac{1}{p_2(s)} x(s) x(\sigma(s)) \Delta s. \tag{24}$$

Multiplying both sides of (24) by $H_1(\xi, \xi_0)$, we obtain for $\xi \in [\xi_2, \infty)_{\mathbb{T}}$,

$$\begin{aligned} H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s) \Delta s \quad &\leq \quad H_1(\xi, \xi_0) x(\xi) \\ &\quad - H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{1}{p_2(s)} x(s) x(\sigma(s)) \Delta s. \end{aligned}$$

Now, for any $\varepsilon > 0$, there exists a $\xi_3 \in [\xi_2, \infty)_{\mathbb{T}}$ such that, for $\xi \in [\xi_3, \infty)_{\mathbb{T}}$,

$$H_1(\xi, \xi_1) x(\xi) \geq h_* - \varepsilon, \tag{25}$$

where

$$h_* := \liminf_{\xi \to \infty} H_1(\xi, \xi_1) x(\xi), \quad 0 \leq h_* \leq 1$$

due to (17) and (21). It follows from (25) that

$$\begin{aligned} & H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s) \Delta s \\ \leq \quad & H_1(\xi, \xi_0) x(\xi) - (h_* - \varepsilon)^2 H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{1/p_2(s)}{H_1(s, \xi_0) H_1(\sigma(s), \xi_0)} \Delta s. \end{aligned} \tag{26}$$

By the quotient rule, we have

$$\left(\frac{-1}{H_1(s, \xi_0)} \right)^{\Delta} = \frac{1/p_2(s)}{H_1(s, \xi_0) H_1(\sigma(s), \xi_0)}. \tag{27}$$

Using (27) in (26) and then by (2), we achieve that

$$\begin{aligned} & H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s) \Delta s \\ \leq \quad & H_1(\xi, \xi_0) x(\xi) - (h_* - \varepsilon)^2 H_1(\xi, \xi_0) \int_{\xi}^{\infty} \left(\frac{-1}{H_1(s, \xi_0)} \right)^{\Delta} \Delta s \\ = \quad & H_1(\xi, \xi_0) x(\xi) - (h_* - \varepsilon)^2. \end{aligned}$$

Applying the lim inf on both sides of this inequality as $\xi \to \infty$, we arrive at

$$\liminf_{\xi \to \infty} H_1(\xi, \xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s), \xi_1)}{H_1(s, \xi_1)} a(s) \Delta s \leq h_* - (h_* - \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we achieve at

$$\liminf_{\xi\to\infty} H_1(\xi,\xi_0)\int_{\xi}^{\infty}\frac{H_2(\phi(s),\xi_1)}{H_1(s,\xi_1)}a(s)\Delta s \le h_* - h_*^2 \le \frac{1}{4},$$

which is a contradiction with (20).

(II) $z^{[1]}(\xi)$ is negative eventually. One can easily see that z tends to a finite limit eventually. The proof is completed. □

Theorem 5. *Let $\phi(\xi) \ge \xi$ and (2) hold. If, for sufficiently large $T \in [\xi_0,\infty)_{\mathbb{T}}$,*

$$\liminf_{\xi\to\infty} H_1(\xi,\xi_0)\int_{\xi}^{\infty}\frac{G_1(\phi(s),T)}{G_1(s,T)}\frac{H_2(s,T)}{H_1(s,T)}a(s)\Delta s > \frac{1}{4}, \tag{28}$$

then all nonoscillatory solutions of the Equation (1) tend to a finite limit eventually.

Proof. Suppose Equation (1) has a nonoscillatory solution $z(\xi)$. Then without loss of generality, let $z(\xi) > 0$ and $z(\phi(\xi)) > 0$ for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$. Using Lemmas 1 and 2 we have

$$z^{[2]}(\xi) > 0 \text{ and } z^{[3]}(\xi) < 0,$$

eventually and $z^{[1]}(\xi)$ is eventually of one sign. We consider the following two cases:

(I) $z^{[1]}(\xi)$ is positive eventually. Thus there is $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that

$$z^{[i]}(\xi) > 0,\ i = 1,2, \text{ and } z^{[3]}(\xi) < 0 \qquad \text{for } \xi \ge \xi_1.$$

By the same way as in the proof of Theorem 4 we have for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$

$$x^{\Delta}(\xi) = -a(\xi)\frac{z(\phi(\xi))}{z^{[1]}(\xi)} - \frac{1}{p_2(\xi)}x(\xi)x(\sigma(\xi)).$$

It follows from the fact that $z^{[1]}$ is strictly increasing that

$$\begin{aligned} z(\phi(\xi)) - z(\xi) &= \int_{\xi}^{\phi(\xi)} z^{[1]}(s)G_0(s,\xi_1)\Delta s \\ &\ge z^{[1]}(\xi)\int_{\xi}^{\phi(\xi)} G_0(s,\xi_1)\Delta s \\ &= z^{[1]}(\xi)(G_1(\phi(\xi),\xi_1) - G_1(\xi,\xi_1)). \end{aligned}$$

That is,

$$z(\phi(\xi)) \ge z(\xi) + z^{[1]}(\xi)(G_1(\phi(\xi),\xi_1) - G_1(\xi,\xi_1))$$

which yield by (16) and for $k \in (0,1)$ there exists a $\xi_k \in [\xi_1,\infty)_{\mathbb{T}}$ such that for $\xi \in [\xi_k,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z(\phi(\xi)) &\ge z(\xi)\left(1 + \frac{k}{G_1(\xi,\xi_1)}(G_1(\phi(\xi),\xi_1) - G_1(\xi,\xi_1))\right) \\ &\ge k\, z(\xi)\frac{G_1(\phi(\xi),\xi_1)}{G_1(\xi,\xi_1)}, \end{aligned}$$

and by (15), we have that for $\xi \in [\xi_k,\infty)_{\mathbb{T}}$,

$$z(\phi(\xi)) \ge k\, z(\xi)\frac{G_1(\phi(\xi),\xi_1)}{G_1(\xi,\xi_1)} \ge k\, z^{[1]}(\xi)\frac{G_1(\phi(\xi),\xi_1)}{G_1(\xi,\xi_1)}\frac{H_2(\xi,\xi_1)}{H_1(\xi,\xi_1)},$$

and so
$$\frac{z(\phi(\xi))}{z^{[1]}(\xi)} \geq k \, \frac{G_1(\phi(\xi),\xi_1)}{G_1(\xi,\xi_1)} \frac{H_2(\xi,\xi_1)}{H_1(\xi,\xi_1)}.$$

Hence, we conclude that, for every $\xi \in [\xi_k,\infty)_{\mathbb{T}}$,
$$x^{\Delta}(\xi) \leq -k \, \frac{G_1(\phi(\xi),\xi_1)}{G_1(\xi,\xi_1)} \frac{H_2(\xi,\xi_1)}{H_1(\xi,\xi_1)} a(\xi) - \frac{x(\xi)x(\sigma(\xi))}{p_2(\xi)}.$$

The rest of the proof is identical to that the proof of Theorem 4 and hence is omitted. □

Theorem 6. *Let $\phi(\xi) \geq \xi$ and (2) hold. If, for sufficiently large $T \in [\xi_0,\infty)_{\mathbb{T}}$,*
$$\liminf_{\xi\to\infty} H_1(\xi,\xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s),T)}{H_1(\phi(s),T)} a(s)\Delta s > \frac{1}{4}, \tag{29}$$
then all nonoscillatory solutions of the Equation (1) tend to a finite limit eventually.

Proof. Suppose Equation (1) has a nonoscillatory solution $z(\xi)$. Then without loss of generality, let $z(\xi) > 0$ and $z(\phi(\xi)) > 0$ for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$. Using Lemmas 1 and 2 we have
$$z^{[2]}(\xi) > 0 \text{ and } z^{[3]}(\xi) < 0,$$
eventually and $z^{[1]}(\xi)$ is eventually of one sign. We consider the following two cases:

(I) $z^{[1]}(\xi)$ is positive eventually. Thus there is $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that
$$z^{[i]}(\xi) > 0, \; i = 1,2, \text{ and } z^{[3]}(\xi) < 0 \qquad \text{for } \xi \geq \xi_1.$$

By the same way as in the proof of Theorem 4 we have for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$
$$x^{\Delta}(\xi) = -a(\xi)\frac{z(\phi(\xi))}{z^{[1]}(\xi)} - \frac{1}{p_2(\xi)} x(\xi)x(\sigma(\xi)).$$

From (15) and using the fact that $z^{[1]}(\xi)$ is strictly increasing that there exists a $\xi_2 \in (\xi_1,\infty)_{\mathbb{T}}$ such that for $\xi \in [\xi_2,\infty)_{\mathbb{T}}$,
$$z(\phi(\xi)) \geq z^{[1]}(\phi(\xi))\frac{H_2(\phi(\xi),\xi_1)}{H_1(\phi(\xi),\xi_1)} \geq z^{[1]}(\xi)\frac{H_2(\phi(\xi),\xi_1)}{H_1(\phi(\xi),\xi_1)}.$$

Therefore,
$$x^{\Delta}(\xi) \leq -\frac{H_2(\phi(\xi),\xi_1)}{H_1(\phi(\xi),\xi_1)} a(\xi) - \frac{1}{p_2(\xi)} x(\xi)x(\sigma(\xi)).$$

The rest of the proof is identical to that the proof of Theorem 4 and hence is deleted. □

Theorem 7. *Let $\phi(\xi) \leq \xi$, (2) and (B) hold. If (20), then all nonoscillatory solutions of the Equation (1) tend to zero eventually.*

Theorem 8. *Let $\phi(\xi) \geq \xi$, (2) and (B) hold. If (28), then all nonoscillatory solutions of the Equation (1) tend to zero eventually.*

Theorem 9. *Let $\phi(\xi) \geq \xi$, (2) and (B) hold. If (29), then all nonoscillatory solutions of the Equation (1) tend to zero eventually.*

Example 1. *Consider the third-order Euler type dynamic equation*

$$\left(\frac{1}{2\xi}\left(\frac{1}{4\xi}z^{\Delta}\right)^{\Delta}\right)^{\Delta} + \frac{\beta}{\xi^5}\, z(\alpha\xi) = 0,\ \xi \in [1,\infty), \tag{30}$$

where $\alpha,\ \beta > 0$ is a constant. It is clear that conditions (2) hold. Now

$$\begin{aligned}&\liminf_{\xi\to\infty} H_1(\xi,\xi_0)\int_{\xi}^{\infty}\frac{H_2(\phi(s),T)}{H_1(s,T)}a(s)\Delta s\\ \geq\ &\beta\liminf_{\xi\to\infty}\left(\xi^2-1\right)\int_{\xi}^{\infty}\left(\frac{\alpha^4}{s^3}-\frac{2\alpha^2}{s^5}+\frac{1}{s^7}\right)\mathrm{d}s = \frac{\beta\alpha^4}{2}\end{aligned}$$

and

$$\begin{aligned}&\liminf_{\xi\to\infty} H_1(\xi,\xi_0)\int_{\xi}^{\infty}\frac{G_1(\phi(s),T)}{G_1(s,T)}\frac{H_2(s,T)}{H_1(s,T)}a(s)\Delta s\\ \geq\ &\beta\liminf_{\xi\to\infty}\left(\xi^2-1\right)\int_{\xi}^{\infty}\left(\frac{1}{s^3}-\frac{2}{s^5}+\frac{1}{s^7}\right)\mathrm{d}s = \frac{\beta}{2}\end{aligned}$$

For $\alpha \in (0,1]$, an application of Theorem 4 implies that all nonoscillatory solutions of the Equation (30) converge if $\beta > \frac{1}{2\alpha^4}$. Additionally, it is easy to prove that

$$\int_{\xi_0}^{\infty}\frac{1}{p_1(v)}\int_{v}^{\infty}\frac{1}{p_2(u)}\int_{u}^{\infty}a(s)\,\Delta s\,\Delta u\Delta v = 2\beta\int_{1}^{\infty}v\int_{v}^{\infty}\frac{1}{u^3}\,\mathrm{d}u\,\mathrm{d}v = \infty.$$

Therefore, by Theorem 7, all nonoscillatory solutions of the Equation (30) converge to zero.

For $\alpha \in [1,\infty)$, Theorem 5 implies that all nonoscillatory solutions of the Equation (30) converge if $\beta > \frac{1}{2}$ and by Theorem 8, all nonoscillatory solutions of the Equation (30) converge to zero.

4. Oscillatory Solutions of Equation (1)

In the following, we give some sufficient conditions for Hille type oscillation of Equation (1). These results solve a problem posed by [10] (Remark 3.3) when $\phi(\xi) \leq \xi$ for $\xi \geq \xi_0 \in \mathbb{T}$.

Theorem 10. *Let $\phi(\xi) \leq \xi$ and (2) hold. If (20) holds and either*

$$\limsup_{\xi\to\infty}\int_{\phi(\xi)}^{\xi} H_2(\phi(s),\phi(\xi))a(s)\,\Delta s > 1, \tag{31}$$

or

$$\limsup_{\xi\to\infty}\int_{\phi(\xi)}^{\xi}\frac{1}{p_1(v)}\int_{v}^{\xi}\frac{1}{p_2(u)}\int_{u}^{\xi}a(s)\,\Delta s\,\Delta u\,\Delta v > 1, \tag{32}$$

then all solutions of the Equation (1) are oscillatory.

Proof. Suppose Equation (1) has a nonoscillatory solution $z(\xi)$. Then without loss of generality, let $z(\xi) > 0$ and $z(\phi(\xi)) > 0$ for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$. By Lemmas 1 and 2 we have

$$z^{[2]}(\xi) > 0 \text{ and } z^{[3]}(\xi) < 0,$$

eventually and $z^{[1]}(\xi)$ is eventually of one sign. We consider the following two cases:

(I) $z^{[1]}(\xi)$ is positive eventually. The proof is identical to that in the proof of Theorem 4, Case (I).

(II) $z^{[1]}(\xi)$ is negative eventually. Thus there is $\xi_1 \in [\xi_0, \infty)_{\mathbb{T}}$ such that

$$z^{[1]}(\xi) < 0,\ z^{[2]}(\xi) > 0,\ \text{and}\ z^{[3]}(\xi) < 0 \qquad \text{for } \xi \geq \xi_1.$$

Let (31) hold. By using the fact that $z^{[2]}(\xi)$ is strictly decreasing on $[\xi_1, \infty)_{\mathbb{T}}$, then for $\phi(\xi) \geq s$ and $s, \xi \in [\xi_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} -z^{[1]}(s) &\geq z^{[1]}(v) - z^{[1]}(u) = \int_u^v z^{[2]}(\tau) H_0(\tau, u)\, \Delta s \\ &\geq z^{[2]}(v) \int_u^v H_0(s, u)\, \Delta s \\ &= z^{[2]}(v) H_1(v, u). \end{aligned} \tag{33}$$

Hence,

$$-z^{\Delta}(u) \geq z^{[2]}(v) \frac{H_1(v, u)}{p_1(u)}$$

Integrating the last inequality again from u to v $(\geq u)$ in u, we get

$$z(u) \geq z(u) - z(v) \geq z^{[2]}(v) \int_u^v \frac{H_1(v, s)}{p_1(s)}\, \Delta s = z^{[2]}(v) H_2(u, v). \tag{34}$$

Integrating (1) from $\phi(\xi)$ to ξ and using (33) with $u = \phi(s)$ and $v = \phi(\xi)$, we obtain

$$\begin{aligned} z^{[2]}(\phi(\xi)) &\geq z^{[2]}(\phi(\xi)) - z^{[2]}(\xi) = \int_{\phi(\xi)}^{\xi} a(s) z(\phi(s))\, \Delta s \\ &\geq z^{[2]}(\phi(\xi)) \int_{\phi(\xi)}^{\xi} a(s) H_2(\phi(s), \phi(\xi))\, \Delta s \end{aligned}$$

Dividing the above inequality by $z^{[2]}(\phi(\xi))$ and taking the lim sup on both sides of the resulting inequality as $\xi \to \infty$, we get a contradiction with (31).

Let (32) hold. Integrating (1) from u to ξ and using the facts that $z^{\Delta} < 0$ and $\phi^{\Delta} > 0$, we obtain

$$z^{[2]}(u) \geq z^{[2]}(u) - z^{[2]}(\xi) = \int_u^{\xi} a(s) z(\phi(s))\, \Delta s \geq z(\phi(\xi)) \int_u^{\xi} a(s)\, \Delta s.$$

Hence,

$$\left(z^{[1]}(u)\right)^{\Delta} \geq \frac{z(\phi(\xi))}{p_2(u)} \int_u^{\xi} a(s)\, \Delta s.$$

Integrating above inequality from v to ξ , we arrive at

$$-z^{[1]}(v) \geq z^{[1]}(\xi) - z^{[1]}(v) \geq z(\phi(\xi)) \int_v^{\xi} \frac{1}{p_2(u)} \int_u^{\xi} a(s)\, \Delta s\, \Delta u.$$

That is,

$$-z^{\Delta}(v) \geq \frac{z(\phi(\xi))}{p_1(v)} \int_v^{\xi} \frac{1}{p_2(u)} \int_u^{\xi} a(s)\, \Delta s\, \Delta u.$$

Integrating again from $\phi(\xi)$ to ξ , we achieve

$$z(\phi(\xi)) \geq -z(\xi) + z(\phi(\xi)) \geq z(\phi(\xi)) \int_{\phi(\xi)}^{\xi} \frac{1}{p_1(v)} \int_v^{\xi} \frac{1}{p_2(u)} \int_u^{\xi} a(s)\, \Delta s\, \Delta u\, \Delta v$$

Dividing the above inequality by $z(\phi(\xi))$ and taking the lim sup on both sides of the resulting inequality as $\xi \to \infty$, we get a contradiction with (32). The proof is completed. □

Example 2. *Consider the third-order Euler type delay dynamic Equation (30) with $\phi(\xi) \le \xi$. Now*

$$\begin{aligned}\limsup_{\xi\to\infty} \int_{\phi(\xi)}^{\xi} a(s) H_2(\phi(s),\phi(\xi))\,\Delta s &= \beta\alpha^4 \limsup_{\xi\to\infty} \int_{\alpha\xi}^{\xi} \left(\frac{1}{s} + \frac{\xi^4}{s^5} - \frac{2\xi^2}{s^3}\right) \mathrm{d}s \\ &= \frac{\beta}{4}\left(4\alpha^4 \ln\left(\frac{1}{\alpha}\right) + 3\alpha^4 - 4\alpha^2 + 1\right)\end{aligned}$$

and

$$\begin{aligned}&\limsup_{\xi\to\infty} \int_{\phi(\xi)}^{\xi} \frac{1}{p_1(v)} \int_v^{\xi} \frac{1}{p_2(u)} \int_u^{\xi} a(s)\,\Delta s\,\Delta u\,\Delta v \\ =\;& \beta \limsup_{\xi\to\infty} \int_{\alpha\xi}^{\xi} \left(\frac{1}{v} + \frac{v^3}{\xi^4} - \frac{2v}{\xi^2}\right) \mathrm{d}v \\ =\;& \frac{\beta}{4}\left(4\ln\left(\frac{1}{\alpha}\right) - \alpha^4 + 4\alpha^2 - 3\right).\end{aligned}$$

So, by Theorem 10, all solutions of the Equation (30) are oscillatory if $\beta > \frac{1}{2\alpha^4}$ and either

$$\beta > \frac{4}{4\alpha^4 \ln\left(\frac{1}{\alpha}\right) + 3\alpha^4 - 4\alpha^2 + 1}$$

or

$$\beta > \frac{4}{4\ln\left(\frac{1}{\alpha}\right) - \alpha^4 + 4\alpha^2 - 3}.$$

5. Discussion and Conclusions

1. The results here have been offered for Equation (1) on an unbounded above arbitrary time scale; therefore, they can be correct to various kinds of time scales, e.g., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, etc., see [2].
2. We notice that the results on the asymptotic behavior of solutions are viable to dynamic Equation (1) for both cases $\phi(\xi) \le \xi$ and $\phi(\xi) \ge \xi$ while the oscillation criteria are viable to dynamic Equation (1) when $\phi(\xi) \le \xi$. Thus, as will be known, it is the delay in equations that can ensure oscillations.
3. The reported results have solved a problem posed by [10] (Remark 3.3) that is attentive with studying the sufficient conditions which ensure that all solutions of third-order dynamic equations oscillate, see Theorem 10.
4. Hille-type criteria for dynamic Equation (1) have been derived and the results in this paper is a considerable improvement contrasted to the results in the cited papers. In particular, our criteria ameliorate those reported in [9,10]; see the following details:

(i) If $p_1(\xi) = p_2(\xi) = 1$, and $\phi(\xi) = \xi$, then condition (20) reduces to

$$\liminf_{\xi\to\infty} \xi \int_{\xi}^{\infty} \frac{h_2(s,\xi_0)}{s} a(s)\Delta s > \frac{1}{4}$$

By virtue of

$$\xi \int_{\xi}^{\infty} \frac{h_2(s,\xi_0)}{s} a(s)\Delta s \ge \xi \int_{\xi}^{\infty} \frac{h_2(s,\xi_0)}{\sigma(s)} a(s)\Delta s$$

Theorem 7 improves Theorem 1 (condition (20) improves (11)).

(ii) If $\phi(\xi) \le \xi$. Since

$$H_1(\xi,\xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s),T)}{H_1(s,T)} a(s)\Delta s \ge H_1(\xi,\xi_0) \int_{\xi}^{\infty} \frac{H_2(\phi(s),T)}{H_1(\sigma(s),T)} a(s)\Delta s$$

Theorem 7 improves Theorem 2 (condition (20) improves (13)).

5. It would be of interest to extend the sharp criterion that the solutions of third-order Euler differential equation $x'''(\xi) + \frac{\beta}{\xi^3}x(\xi) = 0$ are oscillatory when $\beta > \frac{2}{3\sqrt{3}}$ to a third-order dynamic equation, see [30].

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