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Participation in fraudulent elections

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Abstract I analyze a costly voting model of elections in which the incumbent can stuff the ballot box to investigate how electoral fraud affects the decisions of voters to participate. I find that two stable equilibria may exist: an abstention equilibrium, where none of the voters vote and the incumbent always wins, and a more efficient coordination equilibrium, where a substantial share of a challenger's supporters vote and the candidate preferred by the majority is likely to win. I further show that because the higher capability of the incumbent to stuff a ballot box discourages the participation of his own supporters and creates participation incentives for the challenger's supporters, higher fraud does not always benefit the incumbent, even when costless. The model may help to explain two empirical observations related to fraudulent elections: a positive relationship between fraud and the margin of victory and a negative relationship between fraud and voter turnout.

1 Introduction

Voters' participation in elections has been widely studied by economists. Though approaches to modeling and analyzing the behavior of voters differ, they all share one common feature: it is assumed that elections are well-functioning mechanisms for converting public preferences into social choice, and that the outcomes of elections are fully determined by votes cast for the candidates. However, in reality various manip-

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ulations have become an integral part of all kinds of electoral competition. Though in recent years scholars have started paying closer attention to elections that lack integrity, the literature primarily focuses on the behavior of politicians in fraudulent contexts (see, for example, Gehlbach et al. 2015, for an extensive review), while the behavior of voters has not garnered much attention. Moreover, when studying electoral fraud, scholars focus almost exclusively on large state-level elections, though manipulations of elections with smaller numbers of voters, in committees, organizations and small communities also occur. In this paper I theoretically analyze how the presence of fraud affects voters' participation decisions and thus social welfare, and discuss how the findings may be used to increase understanding of electoral manipulations and their consequences in both small and large elections.

Indeed, if voters anticipate that elections will be tainted by fraud, their decisions must be different. In fact, there is a substantial body of empirical evidence suggesting that voters behave differently in fraudulent elections than they do in 'clean' elections. For instance, it is well-established that voters are less likely to participate in elections when they expect fraud. This observation has been verified by, for example, McCann and Dominguez (1998), Hiskey and Bowler (2005), Birch (2010), Simpser (2012) and Carreras and Irepoglu (2013). Though it is well-known that voters reach different decisions in fraudulent elections than in clean elections, the mechanisms which lead to these differences have not been investigated in any depth. In this paper I explicitly study the mechanism of making participation decision in the presence of fraud.

The literature generally considers fraud to be more than just a means of getting extra votes, and often models it as political pressure or violence which directly affects the utility of voters or candidates (Chaturvedi 2005; Collier and Vicente 2012), or as a tool to manipulate voters' beliefs about candidates' popularity or strength (Egorov and Sonin 2014; Gehlbach and Simpser 2015). On the contrary, in this paper I suggest thinking about fraud solely as an instrument for affecting electoral outcomes. Such an approach allows me to focus on the effect of fraud on voters' behavior from a purely pivotal perspective, while the other effects can be simply added, if necessary, to the pivotal effects studied in this work.

I model electoral fraud as ballot stuffing, assuming that if a voter does not participate in elections, his unused ballot may be transformed into a vote for the incumbent. Indeed there is a wide range of technologies for rigging elections, and ballot stuffing is just one of them.[1] From the modeling perspective, most of the variety of fraud technologies which directly influence the reported vote shares of the candidates may be divided into several groups according to the underlying mechanism of lending an advantage to the incumbent. Techniques from the first group directly or indirectly transform unused ballots into additional votes for the incumbent: ballot stuffing, multiple voting, and vote buying, for example. Though some of these methods like multiple voting and vote buying do not literally convert unused ballot papers into votes, in contrast to pure ballot stuffing, the number of extra votes that the incumbent may get through them is limited

[1] See, for example, Lehoucq (2003) for the description of electoral irregularities observed in various elections.

by the number of unused ballots, since reported turnout cannot exceed 100 %.[2] The second group utilizes technologies that transform votes cast for the challenger into votes for the incumbent. This category includes rigging the software for electronic voting machines or designing the ballot so that it consistently leads voters to vote for another candidate than they prefer.[3] The third group of technologies consists of techniques that reduce the number of votes cast for the challenger, such as invalidation and destruction of ballots for wrong candidates, and intimidation of the challenger's supporters. There are also techniques that lend an advantage to the incumbent by increasing the number of registered voters and shaping the electorate through, for example, manipulation of demography or disenfranchisement.

Though direct evidence is hard to find, the methods of the first group of techniques (ballot stuffing, vote buying, multiple voting, etc) are likely to be more widespread (especially in the cases of elections with small number of voters) and to account for a larger share of fraudulent activities than the methods from the other groups, which are much less cost effective and far more limited in adding to the incumbent's advantage. Modeling fraud technology as adding extra votes for the incumbent at the expense of voters who do not participate is, therefore, the most natural way to proceed. In the rest of the paper I thus refer to the fraud technology used in my model as ballot stuffing, though it also accounts for multiple voting, vote buying, and all other fraud techniques that increase the number of votes for the incumbent using, directly or indirectly, the actual ballots of voters who abstained from voting.

The main findings of the paper are the following:

1. Two stable equilibria may exist: an equilibrium with abstention, where none of the voters with positive voting costs vote and the incumbent wins regardless of ex-ante candidates' support, and a coordination equilibrium where a substantial share of the challenger's supporters votes, and the candidate who is ex-ante preferred by the majority is most likely to win. Abstention equilibrium exists only when the incumbent's capability to commit fraud is sufficiently large.
2. Coordination equilibrium is likely to deliver higher welfare than abstention equilibrium. Participation in coordination equilibrium is below an efficient level, since voters do not take into account externalities they produce on other voters when making participation decisions.
3. Higher capability of the incumbent to stuff the ballot box discourages the participation of the incumbent's supporters, requires stronger coordination among the challenger's supporters, and leads to higher participation of the supporters of the challenger, conditional on coordination being achieved.
4. Higher fraud capability does not always benefit the incumbent, even when costless.

The rest of the paper is organized as follows. In the next section I describe a pivotal private value model of costly voting, where voters decide whether to participate

[2] Though there are some documented cases when reported turnout slightly exceeded 100 %, such as in several districts of the Chechnya and Dagestan Republics in the 2008 presidential and 2011 parliamentary Russian elections, these cases are exceptionally rare and are more likely to have occurred due to bureaucratic mistakes rather than extreme fraud.

[3] The infamous Florida butterfly ballot in the US 2000 Presidential elections is an example.

in elections or abstain by comparing their individual specific voting costs with the expected benefit, which involves a probability to cast a decisive vote, i.e. to be pivotal. I then analyze the case where the incumbent can stuff a ballot box perfectly. I show that in addition to the abstention equilibrium, where none of the voters vote, the incumbent stuffs 100 % of votes, and wins with probability one, a relatively more efficient stable coordination equilibrium exists, where a substantial share of the challenger's supporters vote, the number of stuffed ballots is relatively low, and the candidate ex-ante supported by the majority is likely to win. After characterizing properties of the equilibria, I focus on welfare and show that in a coordination equilibrium voters' participation is generally inefficiently low, since the voters ignore the externalities they produce on other voters when making voting decisions. I then generalize the model by allowing fraud to be imperfect. Instead of assuming that the vote of a non-participant is stuffed in favor of the incumbent with certainty, I allow the incumbent to steal a non-participant's vote with some probability, which can be thought of as the incumbent's fraud capability. This generalization allows me to analyze the whole range of elections, from clean to totally fraudulent, though it makes the analytical solution extremely challenging to obtain. I explore how changes in this probability affect properties of the equilibria, and then study the choice of the incumbent if he is free to choose his fraud capability. In the final section, I discuss how the findings may be applied to both small and large elections, and how the model can shed light on several puzzling observations such as a positive relationship between fraud and victory margins and a negative effect of fraud on turnout.

2 Model

Participation in fraudulent elections is analyzed within a pivotal voting framework. Elections are modeled in a way similar to a large body of pivotal costly voting literature, where voters are assumed to make participation decisions based on the probability that their votes can alter the outcome of elections. Costly private value voting models of a similar type have been widely studied by, for example, Palfrey and Rosenthal (1983, 1985), Ledyard (1984), Borgers (2004), and more recently by Krasa and Polborn (2009) as well as Taylor and Yildirim (2010).

2.1 Setup

There are N voters ($N \geq 2$) and two candidates to vote for, the incumbent (A) and the challenger (B). Voters have preferences for candidates but the exact distribution of support for candidates is unknown. Instead, a voter may support the challenger with commonly known probability $\beta \in (0, 1)$ or the incumbent with probability $1 - \beta$. Each voter has an individual specific voting cost c_i drawn from a commonly known distribution over interval $[c_{min}, c_{max}]$ where $0 \leq c_{min} < c_{max} \leq 1$, independently of his type and other voters. The distribution is assumed to be continuous, single-peaked, with positive density over (c_{min}, c_{max}), and differentiable cdf F. If a voter's preferred candidate wins, the voter gains utility 1 if he did not vote, and $1 - c_i$ otherwise. If his favored candidate loses, the voter gains utility 0 if he abstained, and $-c_i$ if he voted. I refer to a voter as "A-type" or "the incumbent's supporter" if his preferred candidate is the incumbent, and as "B-type", "the challenger's supporter" or "opposition supporter" otherwise.

Elections are run under majority rule and, without loss of generality, a tie is resolved in favor of the incumbent. Elections are fraudulent: the incumbent is able to commit fraud through ballot stuffing, meaning that if a voter abstains, his unused ballot may be counted in favor of the incumbent with certain exogenously given probability $\alpha \in [0, 1]$. Thus, α can be thought of as the incumbent's fraud capability. Note that if $\alpha = 0$, my model becomes very similar to the models by Krasa and Polborn (2009) and Borgers (2004). In Borgers (2004) the probability that a voter supports certain candidate is 0.5, while in Krasa and Polborn (2009) this probability is arbitrary. Thus, Borgers' model is a special case of the model by Krasa and Polborn. Since I also allow this probability to take any value, my model can be thought of as a generalization of the models by Krasa and Polborn (2009) as well as by Borgers (2004).[4]

2.2 Analysis

I first analyze a simplified model, assuming $\alpha = 1$, wherein the incumbent stuffs the ballot box perfectly: if a voter abstains, his vote is counted in favor of the incumbent with certainty. Though this assumption might seem too strict, it allows for analytical characterization of the properties of equilibria, and understanding of the intuition behind the electoral game. In Sect. 3 I relax the assumption on perfect fraud and analyze the model with arbitrary α.

The analysis of voters' behavior in elections with perfect fraud begins from the observation that, conditional on voting, a voter's weakly dominant strategy is to vote for his preferred candidate; thus the analysis focuses on participation decisions only. Further note that none of the incumbent's supporters have incentives to vote as long as the costs of voting are non-negative. This is because an A-type voter's vote will be counted in favor of the incumbent regardless of whether the voter participates or abstains. Thus, I restrict my attention to the voting behavior of the challenger's supporters. First, note that a B-type voter i decides to vote if and only if his expected benefit exceeds his participation cost:

$$\Pi(p) > c_i. \tag{1}$$

$\Pi(p)$ is the voter's probability of being pivotal given that a randomly chosen B-type voter votes with probability p, and, at the same time, expected benefit because the voter's benefit from electing the challenger is 1. In this paper I focus on within-group symmetric equilibria where all the voters of the same type adopt the same voting strategy.

To build the pivotal probability function $\Pi(p)$, let V be a number of individuals other than i who choose to vote. Since, only B-type voters vote and their exact number is unknown, the probability that V takes a particular value v is then

4 Both Borgers (2004) and Krasa and Polborn (2009) also assume that a tie is resolved with a toss of a coin, while I assume that a tie is resolved in favor of the incumbent, but this difference is purely technical and does not crucially affect any result.

$$
\begin{aligned}
Pr(V=v) &= \sum_{i=v}^{N-1} \binom{N-1}{i} \beta^i (1-\beta)^{(N-i-1)} \binom{i}{v} p^v (1-p)^{i-v} \\
&= \binom{N-1}{v} (\beta p)^v \sum_{i=0}^{N-v-1} \binom{N-v-1}{i} (\beta - \beta p)^i (1-\beta)^{(N-v-i-1)} \\
&= \binom{N-1}{v} (\beta p)^v (1-\beta p)^{(N-v-1)}. \qquad (2)
\end{aligned}
$$

Without loss of generality, assume N is even, since all further calculations can be straightforwardly adjusted for the case when N is odd. A B-type voter is pivotal when, in case of his abstention, the numbers of votes for the incumbent and challenger are equal. Then, the voter is pivotal if and only if the number of B-type participants is exactly $N/2$. From (2) the probability of being pivotal is

$$
\Pi(p) = Pr(V = N/2) = \binom{N-1}{N/2} (\beta p)^{N/2} (1-\beta p)^{N/2-1}. \qquad (3)
$$

For $p \in [0, 1]$, this function is non-negative, equals zero when $p = 0$, and achieves maximum at $p = \min\{1, \frac{N}{2\beta(N-1)}\}$.

Given pivotal probability function $\Pi(p)$, it is now possible to characterize equilibrium. I search for a symmetric equilibrium where all B-type voters adopt the same voting strategy. Specifically, there must be a common threshold value c^* such that a B-type voter i votes if $c_i \le c^*$ and abstains otherwise. Thus, c^* should satisfy:

$$
\Pi(F(c^*)) \ge c^*, \qquad (4)
$$

with equality when $c^* < c_{max}$.

For further analysis the condition can be rewritten as:

$$
\Pi(F(c^*)) \ge F^{-1}(F(c^*)). \qquad (5)
$$

Note that $F(c)$ is the expected share of B-type voters with voting costs below c, i.e. those who participate in elections. Thus, $F(c^*)$ is the expected equilibrium turnout of B-type voters, and $\beta F(c^*)$ is the overall expected turnout in the elections. For further analysis I will use short notation $F(c) = p$. Then, one can construct a graph in $(p, \Pi(p))$ space.

The fact that the cost distribution is single-peaked guarantees (except the degenerated cases, where some parts of the inverse cdf coincide with the pivotal probability function, and which I ignore for simplicity) that there could be up to two points (which may potentially coincide under certain conditions) that satisfy Eq. (5) and thus constitute an equilibrium. I denote the arguments of these intersections and the corresponding equilibria as p^t and p^*. Furthermore, if none of the voters participates (i.e. $p = 0$), the pivotal probability of each voter is 0, and thus the net expected benefit from participation is negative. Hence, $p = 0$ is also an equilibrium. I denote it as

p^0. Equilibrium p^0 is an equilibrium with abstention, while in equilibria p^t and p^* a strictly positive share of B-type voters participates.

Consider equilibrium p^t and suppose there is arbitrary small perturbation in the equilibrium actions of the voters: they vote whenever their costs are below some $\bar{c} = c^t + \epsilon$, where c^t is the cost threshold which corresponds to p^t and $\epsilon > 0$, and abstain otherwise. Anticipating this, every voter will expect to obtain benefit $\Pi(\bar{c}) > \bar{c}$, and thus the cost threshold will shift even further from $\bar{c}$. Continuing the iteration, one will end up with cost threshold c^*, i.e. in equilibrium p^*. Similarly, when $\epsilon < 0$, perturbation will lead to zero participation, i.e. equilibrium p^0. Contrariwise, such perturbations will not result in this type of divergence in case of equilibria p^0 and p^*. Hence, p^0 and p^* are stable equilibria, while p^t is not. As a result, p^t constitutes a participation threshold value, which needs to be enforced in order to achieve stable coordination equilibrium p^*. From now on I focus on stable equilibria only. Finally, note that if $\Pi(1) > c_{max}$ all the B-type voters have incentive to vote, which corresponds to $p^* = 1$.

Further note that equilibrium p^0 always exists, while existence of p^t and p^* depends on the model's parameters. Necessary and sufficient conditions for the existence of coordination equilibrium p^* can be formulated as follows:

$$\exists p \in (0, 1] : \Pi(p) - F^{-1}(p) \geq 0. \tag{6}$$

Then, $p^* = \max\{p > 0 | \Pi(p) - F^{-1}(p) \geq 0\}$ and $p^t = \min\{p > 0 | \Pi(p) - F^{-1}(p) \geq 0\} \leq \frac{N}{2\beta(N-1)}$.

In terms of exogenous parameters, the following condition is sufficient for the existence of coordination equilibrium:

$$\Pi\left(\frac{N}{2\beta(N-1)}\right) > F^{-1}\left(\frac{N}{2\beta(N-1)}\right), \tag{7}$$

for $\beta > \frac{N}{2(N-1)}$, i.e. when the challenger's supporters are ex-ante in majority. In this case condition (7) implies $p^t < \frac{N}{2\beta(N-1)} < p^*$. The latter condition, in its turn, implies that participation of B-type voters is higher than $\frac{1}{2\beta}$, and thus overall turnout is expected to be higher than 50 %. When $\beta \leq \frac{N}{2(N-1)}$, i.e. when the challenger is ex-ante supported by either minority or just slight majority, condition (7) takes the form of $\Pi(1) > c_{max}$, which implies full participation among B-type voters, i.e. $p^* = 1$. In either case, if sufficient condition (7) holds, more than 50 % of the challenger's supporters participate in coordination equilibrium.

Conditions under which coordination equilibrium is more likely to exist as well as the properties of the equilibrium are summarized in the following proposition.

Proposition 1 *Coordination Equilibrium*

1. *If coordination equilibrium exists for some N_0, β_0, and cost distribution F_0, then it exists for (a) any $N < N_0$; (b) any $\beta > \beta_0$; (c) any F which is first-order stochastically dominated by F_0.*

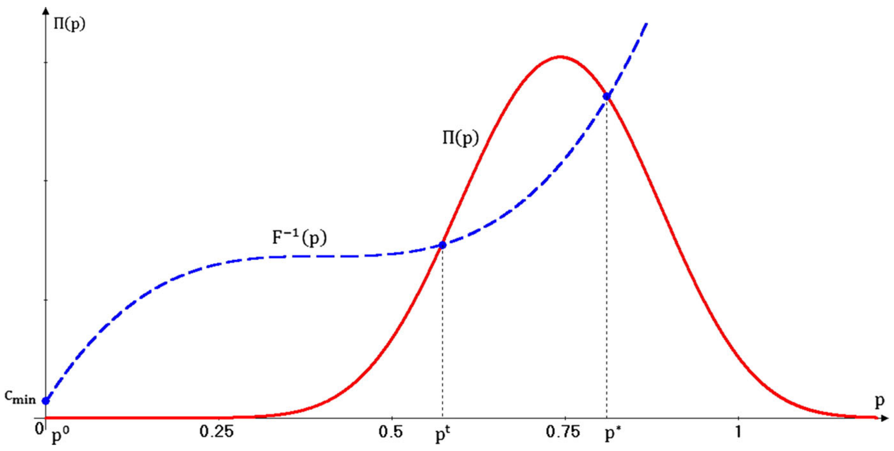

Fig. 1 Equilibrium

2. *If sufficient condition (7) holds, then* (*a*) *equilibrium participation* p^* *is decreasing and threshold value* p^t *is increasing in* N*;* (*b*) p^* *and* p^t *are decreasing in* β*;* (*c*) *If some cost distributions* G *and* F *are such that* G *is first-order stochastically dominated by* F*, then* p^* *is higher and* p^t *is lower for* G.

Proof See the Appendix. □

It is easy to understand Proposition 1 using the graph in Fig. 1. There are three exogenous components in the model: total number of voters N, parameter that determines ex-ante distribution of preferences for candidates across voters β, and cost distribution F. Lower N increases function $\Pi(p)$ for all p (See Lemma 2 of the Appendix 2 for the rigorous proof), since with a lower number of voters every individual is more likely to be pivotal. Thus, with a lower number of voters there is greater likelihood to have coordination equilibrium, and, the lower N, the higher p^* and the lower p^t. An increase in β implies that the pivotal probability function achieves its maximum at a lower participation level while the maximum probability itself stays the same. Graphically, an increase in β results in a horizontal shrink of $\Pi(p)$ function and thus guarantees existence of coordination equilibrium and moves both intersection points between Π and F^{-1} to lower values. Finally, if one cost distribution is first-order stochastically dominated by another, then the graph of the first inverse distribution is lower: lower costs of voting straightforwardly result in stronger participation incentives.

The model predicts the existence of two stable equilibria: abstention equilibrium and coordination equilibrium. In abstention equilibrium, nobody votes and the incumbent steals all the votes through ballot stuffing, winning with a 100 % victory margin. Coordination equilibrium is characterized by a strictly positive participation[5] among the B-type voters and thus by ex-ante lower fraud than abstention equilibrium. Finally,

[5] Throughout the paper, the measures of turnout and fraud are in ex-ante terms since the actual number of participants and thus the actual number of stuffed ballots are random variables, whose exact realizations depend on the realization of candidates' support and voting costs.

note that in both equilibria, the official reported turnout is always 100 % since all the ballots of absentees are converted in favor of the incumbent.

Properties of the abstention equilibrium, including zero turnout and 100% victory margin, may seem too extreme. A slight modification of the model would generate a less extreme result but keep the logic and all the established properties unchanged. Starting from Riker and Ordeshook (1968), the voting literature often argues that voters' participation in elections is driven not solely by their likelihood to be pivotal, but also by the utility they derive from voting, which can be, for example, thought of as a utility from fulfilling a civic duty. Technically, such a modification would mean that a voter now compares his voting costs with the expected benefit plus some utility from voting. This would result in a downward shift of the inverse cost distribution on Fig. 1, straightforwardly implying that, instead of zero participation, the abstention equilibrium is characterized by a strictly positive share of B-type and A-type voters who participate (in expectation). Note that if there is a sufficient number of vote lovers (those who have negative net voting costs) the abstention equilibrium may disappear: there are enough B-type voters with negative costs to create participation incentives for voters with positive costs and, thus, to induce coordination equilibrium. A similar effect on equilibria may be generated by introducing a mass point at zero cost and assuming that voters with zero cost participate. If a voter would have zero cost with some probability $\rho > 0$ and with probability $1 - \rho$ cost would be drawn from distribution F, pivotal probability function $\Pi(0)$ will become strictly positive and thus full abstention will not be an equilibrium. Further, sufficiently large ρ may create enough participation incentives for voters with strictly positive cost and thus induce coordination equilibrium in the same way as negative costs do. Since this is the only property of the model which is substantially affected by allowing voting costs to be negative or to have a mass point at 0, and all the propositions stated above and below stay valid, hereafter I analyze the model with non-negative costs without mass points to avoid unnecessary complications. Participation in abstention equilibrium should be then thought of as "low" rather than exactly 0, and fraud magnitude should be thought of as "large" rather than 100 %.

Another important characteristic of the equilibria is the probability of the candidates' victory. Since in abstention equilibrium, the incumbent stuffs 100 % of ballots, he wins with certainty. The likelihood of his winning in coordination equilibrium depends on the actual turnout of B-type voters. Note that the turnout of B-type voters is a random variable. This is because, first, the exact number of B-type voters is ex-ante unknown, and, second, individual voting costs are independent random draws, with unknown exact realization, and thus the exact number of B-types with costs below some particular threshold is also unknown ex-ante. Given some voting coordination equilibrium rule c^*, turnout would follow a binomial distribution with parameters N and $\beta F(c^*)$. Thus, the probability that the incumbent defeats the challenger is the probability that no more than $N/2$ votes are cast for the challenger:

$$w_A = \sum_{i=0}^{N/2} \binom{N}{i} (\beta F(c^*))^i (1 - \beta F(c^*))^{N-i}. \tag{8}$$

Similarly, the probability that the challenger wins is

$$w_B = \sum_{i=N/2+1}^{N} \binom{N}{i} (\beta F(c^*))^i (1 - \beta F(c^*))^{N-i} = 1 - w_A. \tag{9}$$

Note that w_B is increasing in participation of the challenger's supporters[6]. Further note that when sufficient condition (7) holds, $\beta F(c^*) > \frac{N}{2(N-1)} > 0.5$ whenever $\beta > \frac{N}{2(N-1)}$, and $\beta F(c^*) \leq \frac{N}{2(N-1)}$ whenever $\beta \leq \frac{N}{2(N-1)}$. Then, properties of binomial distribution imply that in the former case $w_A < 0.5$ and in the latter case $w_A \geq 0.5$, i.e. in coordination equilibrium the candidate who is ex-ante preferred by the majority is most likely to win. Given the winning probabilities of the candidates the welfare properties of the equilibria may be investigated.

2.3 Welfare

When voting is costly, participation implies a tradeoff between the quality of the aggregation of voters' preferences and participation costs. Higher participation decreases the probability of electing the wrong candidate (preferred by the minority), but at the same time implies higher total costs borne by society. In this section I explore welfare properties of equilibria in my model of fraudulent elections. First, note that there are two stable equilibria in this model. To see which of the two equilibria, coordination or abstention, is more desirable from a social point of view, consider social welfare as a function of some strategy $\tilde{c}$ adopted by all B-type voters. The expected utility of an A-type voter is $1 - v_B$, where v_B is the probability that the challenger will receive enough votes from other $N - 1$ voters to win:

$$v_B = \sum_{i=N/2+1}^{N-1} \binom{N-1}{i} (\beta F(\tilde{c}))^i (1 - \beta F(\tilde{c}))^{N-i-1}. \tag{10}$$

Note that v_B is also the probability that the challenger wins, conditional on abstention of a B-type. Denoting the probability that the challenger wins if a B-type voter participates as u_B, the expected utility of a B-type voter can be then expressed as follows:

$$\int_0^{\tilde{c}} (u_B - c)\, \mathrm{d}F(c) + \int_{\tilde{c}}^{c_{max}} v_B\, \mathrm{d}F(c). \tag{11}$$

Given that the voter participates, the challenger will win if out of the other $N - 1$ voters at least $N/2$ voters participate. Thus,

[6] w_A is in fact a cdf of binomial distribution with parameters $(N, \beta F(c^*))$ evaluated at $N/2$ and, as any binomial distribution, is decreasing in the probability of success in each Bernoulli trial $\beta F(c^*)$. Intuitively, w_A is the probability of no more than $N/2$ success in N Bernoulli trials, and it is smaller whenever success in each trial is more likely to occur. Since $w_B = 1 - w_A$, w_B is increasing in $F(c^*)$.

$$u_B = \sum_{i=N/2}^{N-1} \binom{N-1}{i} (\beta F(\tilde{c}))^i (1 - \beta F(\tilde{c}))^{N-i-1} = v_B + \Pi(F(\tilde{c})). \qquad (12)$$

Thus, $\Pi(F(\tilde{c}))$ is the marginal contribution of a participating B-type voter to the ex-ante probability that the challenger wins. Then, the total expected utility of all the voters can be expressed as follows:

$$W = N\left((1-\beta)(1-v_B) + \beta v_B + \beta \int_0^{\tilde{c}} (\Pi(F(\tilde{c})) - c)\, \mathrm{d}F(c)\right). \qquad (13)$$

The latter expression can be re-written as

$$W = N\left((2\beta - 1)v_B + \beta\Pi(F(\tilde{c}))F(\tilde{c}) - \beta \int_0^{\tilde{c}} c\, \mathrm{d}F(c)\right). \qquad (14)$$

Having defined the welfare function, it is possible to compare the efficiency of abstention equilibrium ($\tilde{c} = 0$) and coordination equilibrium ($\tilde{c} = c^*$). First, recall that in this model the voters' utility does not directly depend on the cleanness of the elections, and thus the fact that coordination equilibrium results in lower fraud than does abstention equilibrium, which is characterized by 100 % ballot stuffing, is irrelevant for the welfare comparison. The only two features that affect voters' welfare are participation costs and the candidates' probabilities of winning.

The abstention equilibrium implies that the incumbent is elected at no cost. Recall that the exact numbers of the supporters of the incumbent and challenger are random variables. If it turns out ex-post that the majority of voters supported the challenger, the abstention equilibrium will result in substantial welfare loss from electing the wrong candidate. Contrariwise, the coordination equilibrium implies that in such a case there will be high probability that the challenger will win. In fact, coordination equilibrium guarantees, in contrast to abstention equilibrium, that whatever the true distribution of voters' preferences for candidates would be, the preferred candidate will be most likely to win. However, this guarantee comes with participation costs, and whether the welfare gain from it outweighs these costs depends on the model parameters. Intuitively, the larger the probability that the majority will support the challenger and the larger the expected number of the challenger's supporters, the larger the gain from the "insurance" provided by the coordination equilibrium. Hence, the higher β, the more likely coordination equilibrium welfare dominates abstention equilibrium. This result is formalized in the following proposition.

Proposition 2 *For any $N > 2$, and any cost distribution F such that sufficient condition* (7) *holds, there exists β_0 such that for any $\beta > \beta_0$ coordination equilibrium yields higher expected welfare than does abstention equilibrium.*

Proof See the Appendix. □

Further, coordination equilibrium is generally still not socially efficient. Specifically, participation in coordination equilibrium is always below the participation level that would maximize social welfare, except in cases when coordination equilibrium is characterized by participation of all the challenger's supporters. This result is formalized in the following proposition.

Proposition 3 *Let c^e be the efficient voting rule, i.e. the one that maximizes expected social welfare. Then, $c^* \leq c^e$ with inequality for $c^* < c_{max}$.*

Proof See the Appendix. □

This result suggests that participation of B-type voters in coordination equilibrium is always below the efficient level, unless all the B-types participate. The result is driven by two types of externalities produced by B-type participants. First, every vote increases the challenger's winning probability. Second, every vote changes pivotal probability, hence the marginal contribution to the probability that the challenger will win, hence the utility of each B-type participant. When the challenger is ex-ante preferred by the majority, an increase in participation would increase the winning probability of the challenger and thus total welfare. If sufficient condition (7) holds, such an increase would decrease the pivotal probability and thus the utility of each participant. It turns out that the first effect always dominates the second effect, and thus equilibrium participation is below the efficient level. When sufficient condition (7) does not hold but coordination equilibrium still exists, an increase in participation increases the pivotal probability, both effects have the same sign, and thus equilibrium participation is again inefficiently low. When the majority prefers the incumbent, a change in participation of B-types would have a stronger effect on pivotal probabilities than on the candidates' winning probabilities. Hence, if condition (7) holds, the equilibrium outcome, which is full participation among B-types, is efficient, while if it does not, the equilibrium participation is again below the efficient level.

The established welfare result differs from the results typically obtained in literature on costly voting in clean elections. For instance, in the model of Borgers (2004) as well as that of Chakravarty et al. (2010), a vote cast by a voter produces only a negative externality on all other voters by decreasing their pivotal probabilities and thus, in equilibrium, participation is higher than the welfare-maximizing level. In Ghosal and Lockwood (2009), voters' preferences combine both private and common values (i.e. voters prefer different candidates but also possess heterogeneous information on the common state of the world) and if voters vote according to their private preferences, equilibrium participation is inefficiently high, whereas if they vote according to their private information, participation appears to be below the efficient level. The mechanism behind the welfare result in my model is close to that of Krasa and Polborn (2009), where in addition to an externality that affects voters' pivotal probabilities, there is also an externality that affects the candidates' winning probabilities. While in their model, equilibrium participation is typically, though not always, less than the efficient level, I demonstrate that a similar mechanism in a fraudulent context always leads to inefficiently low participation.

3 General model

3.1 Analysis

Consider a more general model where α is arbitrary. Suppose that all other B-type voters adopt voting strategy c_B, i.e. a B-type voter votes if his voting costs are below c_B and abstains otherwise. Similarly, suppose A-type voters adopt strategy c_A. Then, the probability that a randomly picked voter votes is $F(c_B)$ and $F(c_A)$ for B-types and A-types respectively.

Denote $P_i^j(k) = \binom{j}{i} k^i (1-k)^j$ for shorter notation. Then, the probability that there are a A-types among other $N-1$ voters is $P_a^{N-1}(1-\beta)$. The probability that l of them participate in elections is $P_l^a(F(c_A))$. The probability that m out of another $N-a-1$ B-type voters participate is $P_m^{N-a-1}(F(c_B))$.

A B-type voter is pivotal in two cases. First, if the number of stolen votes is such that the number of votes for each candidate is equal, and second, if the challenger leads by one vote and the voter's ballot is stolen if he abstains. If x votes are stolen, the incumbent gets $x + l$ and the challenger gets m votes. Thus, given a, l and m, a B-type voter is pivotal if and only if $x = m - l$ or $x = m - l - 1$. The probability of this event is $P_{m-l}^{N-l-m-1}(\alpha) + \alpha P_{m-l-1}^{N-l-m-1}(\alpha)$.

The probability that a out of $N - 1$ voters support the incumbent, l out of these a A-types participate, m out of $N - a - 1$ B-types participate, and $m - l$ or $m - l + 1$ votes out of $N - l - m - 1$ non-participants' votes are stolen is then:

$$P_a^{N-1}(1-\beta) P_l^a\,(F(c_A))\, P_m^{N-a-1}(F(c_B)) \left(P_{m-l}^{N-l-m-1}(\alpha) + \alpha P_{m-l-1}^{N-l-m-1}(\alpha)\right). \tag{15}$$

Denote this probability as $P_B(a, l, m, c_A, c_B)$. Finally, the probability that a B-type voter is pivotal is a function of voting strategies c_A and c_B adopted by all the A-type and all the B-type voters respectively:

$$\Pi_B(c_A, c_B) = \sum_{a=0}^{N-1} \sum_{l=0}^{a} \sum_{m=l-1}^{N-a-1} P_B(a, l, m, c_A, c_B). \tag{16}$$

Similarly, one can construct a pivotal probability function for an A-type voter:

$$\Pi_A(c_A, c_B) = (1-\alpha) \sum_{a=0}^{N-1} \sum_{l=0}^{a} \sum_{m=l-1}^{N-a-1} P_A(a, l, m, c_A, c_B), \tag{17}$$

where

$$P_A(a, l, m, c_A, c_B) = P_a^{N-1}(1-\beta) P_l^a(F(c_A)) P_m^{N-a-1}(F(c_B)) P_{m-l-1}^{N-l-m-1}(\alpha).$$

Note, that there is a $(1-\alpha)$ term in the pivotal probability function for the incumbent's supporters: an A-type voter will be pivotal only if his vote is not stolen in case of abstention; otherwise his participation decision will not change the outcome.

Symmetric equilibrium is characterized by a pair (c_A, c_B) such that all A-type voters with costs below c_A and all B-type voters with costs below c_B participate, and the others abstain. Equilibrium values of c_A and c_B are the solution for the following system of equations:

$$\begin{aligned} \Pi_A(c_A, c_B) &\geq c_A, \\ \Pi_B(c_A, c_B) &\geq c_B, \end{aligned} \tag{18}$$

with equalities when $c_A < c_{max}$ and $c_B < c_{max}$ respectively.

If one defines a function $L : [c_{min}, c_{max}]^2 \to [c_{min}, c_{max}]^2$ as:

$$L(c_A, c_B) = (max\{M_A, c_{min}\}, max\{M_B, c_{min}\}), \tag{19}$$

where $M_A = min\{\Pi_A(c_A, c_B), c_{max}\}$ and $M_B = min\{\Pi_B(c_A, c_B), c_{max}\}$, then Brouwer's fixed point theorem will imply the existence of equilibrium.

As in the case of the perfect fraud model, equilibrium voting rules (c_A, c_B) are one-to-one matched to the pair $(F(c_A), F(c_B))$ which is equilibrium participation of the incumbent's and the challenger's supporters respectively. Figure 2 demonstrates an equilibrium for the following values of parameters: $N = 25, \beta = 0.7, \alpha = 0.3$, costs are distributed uniformly over the interval $[0.01, 0.1]$. The red (darkest) surface is the net expected benefit of a B-type voter as a function of $F(c_A), F(c_B)$: $\Pi_B(F(c_A), F(c_B)) - F^{-1}(F(c_B))$. Similarly, the blue surface is the net expected benefit of an A-type voter: $\Pi_A(F(c_A), F(c_B)) - F^{-1}(F(c_A))$. The yellow (lightest) surface is a zero-plane. Thus, a point where all three surfaces intersect would be an equilibrium.

There is one such point on the figure above: $(0.27, 0.63)$. However, there are two more equilibria here. First $(0, 0)$ is an equilibrium (stable) because for both types of voters net expected benefits are negative at this point, and thus no voter has any incentive to vote. Second, $(0, 0.05)$ is an equilibrium (unstable) since the net expected

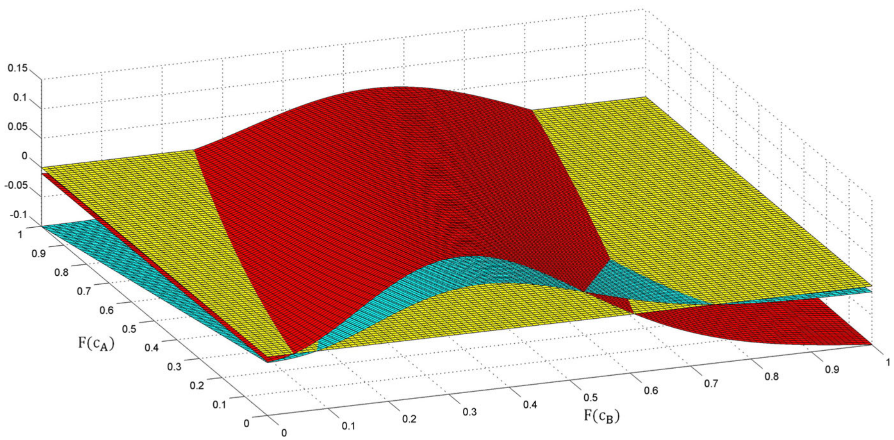

Fig. 2 Equilibrium in the generalized model

benefit of A-type voters is negative, so they do not vote, while B-types' net expected benefit is exactly 0 at this point.

Obtaining a closed-form solution and even characterizing equilibria for the generalized model is quite challenging. However, numerical simulations provide a number of consistent observations about the equilibria and their properties. As in the case of perfect fraud model, I focus on stable coordination equilibrium. The first question is how changes in fraud capability α affect equilibrium. Consider a simple numerical example with just two voters.

Example 1 $N = 2$, costs are distributed uniformly over $[0, k]$, where $0 < k < 1$.

An A-type voter is pivotal only if his vote would not be stolen in case of abstention, another voter appears to be a B-type, and decides to participate. The probability of this event is $(1-\alpha)\beta F(c_B)$. A B-type voter may be pivotal in three cases. First, if his vote would be stolen in case of abstention, another voter is also a B-type and participates. Second, if another voter is B-type, abstains and his unused ballot is stolen. Third, if another voter is A-type, abstains and his vote is not stolen. Thus, equilibrium is given by the following system of equations:

$$(1-\alpha)\beta F(c_B) = c_A,$$
$$\alpha\beta F(c_B) + (1-\alpha)\beta(1-F(c_B)) + (1-\alpha)(1-\beta)(1-F(c_A)) = c_B.$$

To guarantee that coordination equilibrium exists for any α, assume that $\beta > k$, i.e. costs are reasonable in comparison to benefit from electing the preferred candidate (which is 1) or/and there is ex-ante substantial share of B-type voters. If this is not the case, then, for example, for the values of α close to 1, the benefit function of B-type voters is always below the cost function, and thus coordination equilibrium does not exist.

The system has a unique solution, which constitutes coordination equilibrium as long as both c_A^* and c_B^* are within $[0, k]$ range:

$$c_B^* = \frac{(1-\alpha)k^2}{k^2 + (1-\alpha)^2\beta(1-\beta) - k\beta(2\alpha-1)}, \quad c_A^* = \frac{(1-\alpha)\beta}{k}c_B^*.$$

It can be checked that participation threshold for B-type voters c_B^*, and thus B-types' participation rate, is increasing in α while c_A^* is decreasing in α, whenever $\beta > k$ and both c_A^* and c_B^* are within $[0, k]$.

In the example above, fraud capability α leads to a decrease in the equilibrium participation of A-type voters and to an increase in the equilibrium participation of B-type voters, conditional on coordination. Simulations of the model for a higher number of voters provide similar results: an increase in α decreases the participation incentives of the incumbent's supporters and increases the participation incentives of the challenger's supporters. To understand how equilibrium participation rates of each group of voters change in response to an increase in α consider Fig. 3, which displays two-dimensional representation of the equilibrium in the generalized model with the following values of the parameters: $N = 25$, $\beta = 0.7$, costs are distributed

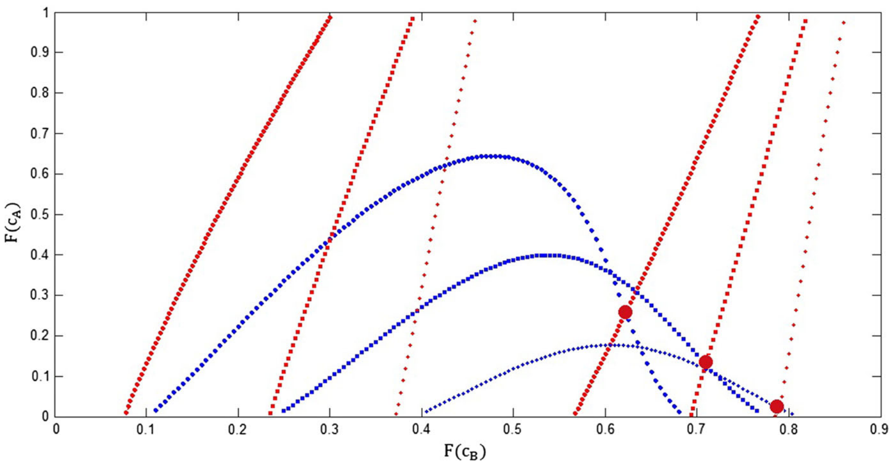

Fig. 3 Equilibrium for different values of α

uniformly over the interval [0.01, 0.1]. In Fig. 3 the blue (dark) curve is the solution $c_A(c_B)$ of the first equation of system (18) and a red (light) curve is the solution of the second equation. The graph presents the solutions for three distinct values of α: 0.3, 0.5 and 0.7. Equilibrium is determined by the intersection of red and blue curves of the same type which correspond to the same value of α (the blue curve with the highest maximum corresponds to $\alpha = 0.3$, the blue curve with the lowest maximum correspond to $\alpha = 0.7$). For each pair of curves, the intersection with lower value of $F(c_B)$ determines unstable equilibrium, while the intersection with higher $F(c_B)$ intersection determines stable equilibrium.

This result is robust to different choices of the model's parameters. Figure 4 shows how changes in α affect participation of the incumbent's supporters (blue curve) and the challenger's supporters (red curve) for $N = 25$, $\beta = 0.7$, costs distributed uniformly over the interval [0.01, 0.1].

Figure 4 shows that the participation incentives of A-type voters decrease with higher α. Intuitively, the higher the incumbent's fraud capability, the more likely the incumbent will steal an A-type's vote if the voter abstains and avoids cost, the less incentive to participate the voter has. The effect of an increase in α on coordination equilibrium participation of the B-types is the opposite. Higher fraud capability implies that higher participation among B-types is needed to maintain sufficiently high pivotal probabilities. Also note that though higher α leads to higher participation of the challenger's supporters in coordination equilibrium, at the same time it requires more people to coordinate in order to achieve this equilibrium.

Further note that when elections are perfectly fraudulent, there are two stable equilibria: abstention equilibrium and coordination equilibrium. In clean elections, as shown by Borgers (2004) and Krasa and Polborn (2009), there is unique equilibrium. Thus, it must be the case that sufficiently large fraud capability leads to the emergence of a bad abstention equilibrium. This observation is formalized in Proposition 4.

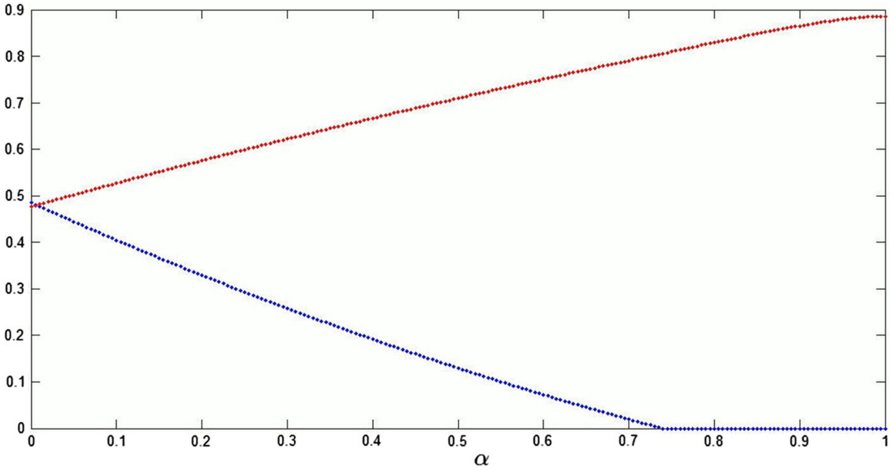

Fig. 4 Coordination equilibrium participation of A-supporters (*downward sloping*) and B-supporters (*upward sloping*) as a function of fraud capability

Proposition 4 *For given values of N and β, and cost distribution F, there is a unique value $\alpha_0 < 1$ such that for any $\alpha \geq \alpha_0$ abstention is an equilibrium, and for any $\alpha < \alpha_0$ it is not.*

Proof See Appendix. □

The proposition is very intuitive: when fraud capability is low, full abstention cannot be an equilibrium because a single voter has a good chance of influencing the outcome of elections by deviating and participating. But when fraud capability is high, implying that there is a high probability that the vote of a non-participant will be stolen, participating when all the others abstain is unlikely to be profitable as there is a high probability that a sufficient number of votes will be stuffed in favor of the incumbent, and thus deviation from abstention will not change the outcome of elections.

3.2 Incumbent's problem

Suppose now that α is no longer exogenous, but rather the incumbent is free to choose it. Anticipating the equilibrium response of the voters to any level of fraud capability α, c_A^* and c_B^*, a rational incumbent would choose an α that maximizes his expected net benefit $Bw_A - c_f(\alpha)$, where B is some benefit from remaining in office, $c_f(\alpha)$ is the cost of fraud, and w_A is the probability of winning elections:

$$w_A = 1 - \sum_{a=0}^{N} \sum_{l=0}^{a} \sum_{m=l+1}^{N-a} \sum_{x=0}^{m-l-1} P_a^N (1-\beta) P_l^a (F(c_A^*)) P_m^{N-a} (F(c_B^*)) P_x^{N-l-m} (\alpha). \tag{20}$$

Assume for simplicity that fraud is costless, i.e. $c_f(\alpha) = 0$ for all $\alpha \in [0, 1]$. It might seem at first glance that in this case the incumbent should choose the maximum possible fraud capability, which is $\alpha = 1$. However an increase in fraud capability has several effects. First, higher α, other things being equal, implies a higher number of stuffed ballots and thus a higher probability that the incumbent wins. But changes in α also affect the participation of voters. Specifically, as stated above, a higher α has a deterrent effect on the participation of the incumbent's supporters and a stimulating effect on participation of the challenger's supporters (conditional on the fact that coordination equilibrium is achieved), which together lead to a decrease in the probability that the incumbent will win. Thus, the resulting effect of an increase in α on the probability that the incumbent wins depends on which of the two effects dominates. To illustrate this intuition, consider further the example from the previous section.

Example 2 $N = 2$, costs are distributed uniformly over $[0, k]$, where $0 < k < \beta$. Equilibrium is given by the following system of equations:

$$(1-\alpha)\beta F(c_B) = c_A,$$
$$\alpha\beta F(c_B) + (1-\alpha)\beta(1 - F(c_B)) + (1-\alpha)(1-\beta)(1 - F(c_A)) = c_B.$$

When elections are perfectly fraudulent, i.e. $\alpha = 1$, A-type voters always abstain, and B-type voters either always participate (coordination equilibrium) or always abstain (abstention equilibrium). If B-types always participate, the probability that the incumbent wins elections is $w_A^F = 1 - \beta^2$.

When elections are clean, i.e. $\alpha = 0$, equilibrium participation is given by the following cost thresholds:

$$c_A^* = \frac{\beta k}{k^2 + \beta(1-\beta) - k\beta}, \quad c_B^* = \frac{k^2}{k^2 + \beta(1-\beta) - k\beta},$$

whenever both c_A^* and c_B^* are within $[0, k]$.

Then, the probability that the incumbent wins clean elections is

$$w_A^C = \beta^2(1 - F(c_B^*))^2 + (1-\beta)^2 + 2\beta(1-\beta)\left(F(c_A^*)(1 - F(c_A^*)(1 - F(c_B^*))\right).$$

Whenever both c_A^* and c_B^* are within $[0, k]$ range, $w_A^C \geq w_A^F$, implying that if the incumbent can choose whether to hold clean elections or stuff the ballot box perfectly, clean elections would be preferable if the challenger's supporters coordinate well, even if fraud is costless.

For arbitrary $\alpha \in [0, 1]$ the probability that the incumbent wins elections can be expressed as follows:

$$w^A(\alpha) = 1 - (\beta^2((F(c_B^*)^2 + 2(1-\alpha)F(c_B^*)(1 - F(c_B^*))$$
$$+2(1-\alpha)F(c_B^*)(1 - F(c_B^*)) + 2\beta(1-\beta)F(c_B^*)(1 - F(c_A^*)),$$

where

$$c_B^* = \frac{(1-\alpha)k^2}{k^2 + (1-\alpha)^2\beta(1-\beta) - k\beta(2\alpha - 1)}, \quad c_A^* = \frac{(1-\alpha)\beta}{k}c_B^*.$$

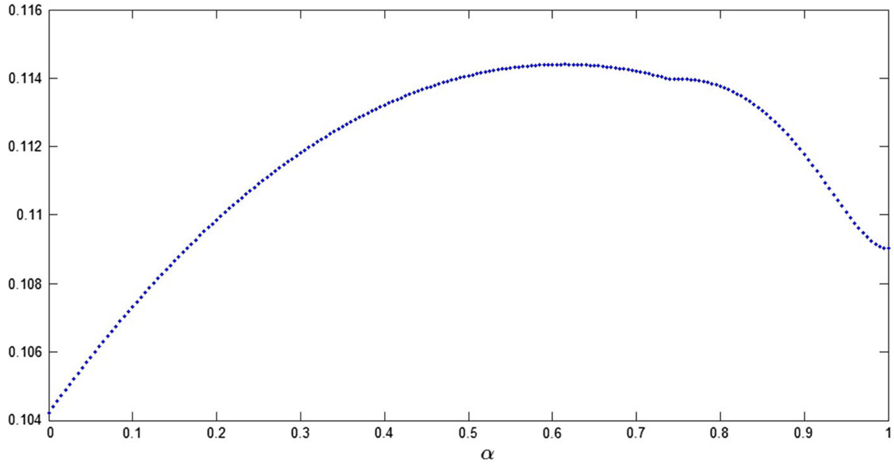

Fig. 5 The incumbent's winning probability as a function of fraud capability

If the incumbent was free to chose any level of fraud at no cost, he would choose the level of α which maximizes $w_A(\alpha)$. The important result is that this level would rarely be 1. For example, for $\beta = 0.75$ and $k = 0.5$ the optimal level of fraud is $\alpha^* = 0.21$, while for $\beta = 0.5$ and $k = 0.25$ it is $\alpha^* = 0.16$.

Simulations of the model for a higher number of voters provide similar results: the level of fraud capability α which maximizes the probability that the incumbent wins elections where the challenger's supporters coordinate is almost always less then 1. Figure 5 shows the incumbent's winning probability in coordination equilibrium as a function of α for $N = 25$, $\beta = 0.7$, and voting costs distributed uniformly over the interval $[0.01, 0.1]$. From Fig. 5 it is clear that that maximum feasible fraud level is not necessarily optimal.

Further, recall Proposition 4 which states that a threshold value exists, such that for all α above this value equilibrium with full abstention exists, and for all α below this value it does not exist. From the incumbent's point of view, abstention equilibrium is the first best as it guarantees the incumbent's victory with certainty. For the parameters used to construct Fig. 5 the threshold equals $\alpha_0 = 0.22$.

Further, the winning probability achieves its maximum for some α^*. Note that a higher value of α implies that the participation rate of the challenger's supporters in coordination equilibrium is higher, which in turn means that solving the collective action problem is more difficult and thus achieving coordination equilibrium is less likely. Ultimately, when choosing fraud level, the incumbent faces a triple tradeoff: costs of fraud, likelihood of coordination among the challenger's supporters, and winning probability given that coordination is achieved. If the incumbent is strong enough to deter coordination of the challenger's supporters, he would probably choose a low level of fraud close to α_0, while a weak incumbent would prefer a high level fraud close to α^*, which makes coordination harder and provides relatively high chances of winning even if this coordination is achieved.

The fact that higher fraud does not necessarily increase the incumbent's winning probability has another important implication: zero fraud is not necessarily socially optimal. As it might seem at first glance that the incumbent should choose maximum possible fraud capability when it is costless, it might seem that clean elections should always be welfare dominant over fraudulent elections. However, it has been demonstrated that the winning chances of an incumbent who is ex-ante supported by the minority may be hurt by higher α. Hence, higher α may be socially preferred since it can increase the winning probability of the challenger, who is the candidate preferred by the majority. Indeed, there may also be extra participation costs as well costs of fraud, which may or may not be outweighed by the gain from increased winning chances of a challenger. For instance, in the example with two voters and costless fraud analyzed above, high fraud (the minimal level which ensures full participation among B-types) is typically socially optimal. Furthermore, when there are ex-ante more A-type voters (i.e. $\beta < 0.5$), the abstention equilibrium, which exists for sufficiently large α, may easily welfare dominate clean election equilibrium since it would guarantee the ex-ante preferred candidate will win at no participation cost.

When fraud capacity α is considered as an endogenous variable which is subject to the incumbent's choice, then given the timing of the model, one may argue that there is a commitment problem if α is easily adjustable. Since, first, the incumbent sets α, then voters make their participation decisions, and only then fraud is realized, the incumbent would always benefit by increasing α from the announced level once voters have made their decisions. If this is the case, then the discussion on optimal α should be thought of as a comparative static exercise with respect to fraud capability rather than the rational incumbent's choice. However, in reality, targeted fraud level is unlikely to be such an easily adjustable variable as it is the result of a comprehensive rigging process which takes place not only on election day, but begins also long before. In this case, though the incumbent does not commit to maintaining the announced level of α, his ability to adjust it at the last moment is very limited.

4 Applications and discussion

Pivotal voting models are typically used for studying elections in organizations, committees, and other similar settings where the number of potential voters is relatively small. Thus, since the model of fraudulent elections is developed within a traditional pivotal framework, it should be first considered as a tool for understanding fraud in small elections. Even though voting fraud in committees or organizations may not seem as widespread as in large elections, there are a number of settings beyond purely fraudulent small elections which can be analyzed within the developed framework.

One important class of such settings would be voting in organizations with several hierarchy levels and an agency problem. Consider an organization where an owner or higher level manager is interested in informed decision making and wants to have the opinion of employees on some issue. If voting is organized by a lower level manager, the latter may then interpret employees' abstention from voting in a way that fits his personal preferences, while participation removes this possibility. An example of such a setting would be a university where a department head organizes voting among faculty members and then uses the results in his communication with the dean. Though the setup considered in the paper differs from this setting, my model can be useful for analysis of such a voting game. For instance, it would suggest that allowing the voting organizer to interpret abstentions according to his interests may be an optimal policy from the perspective of the higher level manager since this may encourage participation of those voters who have different preferences than the lower level manager and thus may help him or her to make more informed decisions. Moreover, it would suggest that it may be optimal for the higher level manager to even run "voting over status quo" instead of the more usual voluntary elections. In such a voting arrangement, an alternative may win only if a sufficient number of its supporters cast votes, and an abstention is effectively a vote for the status quo, which is equivalent to perfectly fraudulent elections analyzed in Sect. 2 of the paper.

Furthermore, the developed model provides a framework for comprehensive analysis of "voting over status quo". To my knowledge, the only paper which explicitly considers such a voting procedure is the early version of Borgers (2004). In his model, however, the main focus of the analysis is on the relative efficiency of voting over status quo versus voluntary majority voting and random decision making procedures, while I focus on equilibria and their properties.

Though voters' participation decisions in large elections are unlikely to be exclusively driven by pivotal perspectives, there is substantial evidence suggesting that even in large elections voters care about being pivotal to a certain extent. For instance, Cox and Munger (1989), Shachar and Nalebuff (1999), Fauvelle-Aymar and Francois (2006), Landry et al. (2010), Simonovits (2012) and De Paola and Scoppa (2014) analyze data from various large elections and show that voter turnout significantly increases when results are expected to be close, even after controlling for mobilization efforts of competing parties. Thus, the model developed in this paper can also contribute to understanding voters' behavior in large fraudulent elections. Specifically, it generates predictions which may be at least partially explanatory of several puzzling empirical observations about fraudulent elections.

First, a number of survey-based studies have shown that perceived electoral integrity has a strong positive effect on the propensity to vote. This result was established by, for example, McCann and Dominguez (1998) as well as Hiskey and Bowler (2005) for Mexico, by Carreras and Irepoglu (2013) for other Latin American countries, by Birch (2010) for a set of almost 30 countries including both new and established democracies, and by Alvarez et al. (2008) who show that confidence in electoral process and in its transparency are positively correlated with US citizens' intension to participate in elections. Since these studies use survey data, this result is about the relationship between electoral integrity and true turnout rather than reported turnout. In contrast to the survey-based research, Simpser (2012) explores Mexican electoral data to assess the relationship between voters' participation incentives and fraud. Using variation in fraud and turnout across Mexican states, and explicitly distinguishing between reported and true turnout, he finds that electoral manipulations discourage voter participation.

The negative relationship between fraud perception and turnout is generally explained by the low incentives of the electorate to participate in costly voting if elections lack competitiveness. Low incentives are assumed to be the result of either direct disutility from participating in corrupt elections (e.g. Simpser 2008) or from low likelihood of a vote to be pivotal, which in turn comes from a lack of competition (e.g. Birch 2010). Though the idea that electoral fraud affects a voter's probability to be pivotal, the mechanism behind this relationship has not been explored in any depth. Indeed, the relationship is not monotonic: fraud can both decrease and increase the competitiveness of elections: it may give a corrupt candidate an overwhelming advantage, but it can also create a chance for an incumbent with low support to make the race competitive. The framework developed in this paper may provide more consistent explanation for the relationship between fraud and turnout: the observed correlation may be not the result of a direct negative causal effect of fraud on participation, but rather an equilibrium outcome of an electoral game. The key point is that the incumbent needs to maintain low participation to guarantee he will win, while the only way to ensure this is to commit extensive fraud: if opposition supporters fail to coordinate, then their participation incentives are low, given extensive fraud commitment. But extensive fraud is possible only when a lot of opposition supporters abstain, since it uses residual turnout to transform unused ballots into votes. This situation corresponds to the abstention equilibrium. Alternatively, if opposition supporters coordinate, then there is not much opportunity for fraud, and the extent of fraud is relatively moderate, which in turn provides sufficient incentives for voters to coordinate. This corresponds to coordination equilibrium. Together, abstention and coordination equilibria generate a negative fraud-turnout relationship, implying that low (high) turnout is not just a consequence of high (low) fraud, but rather low (high) turnout and high (low) fraud are simultaneously determined equilibrium outcomes.

Similar logic lies behind the potential explanation for the second empirical observation about fraudulent elections, which relates integrity, victory margin, and fraud excessiveness, established by, for example, Simpser (2008). Analyzing several datasets, he finds that in elections whose integrity is widely questioned, a high victory margin is observed far more frequently than in presumably clean elections. Moreover, in about 40 % of elections considered as fraudulent the victory margin exceeds 40 %, implying that corrupt politicians often commit excessive fraud. Simpser (2008) explains such excessiveness using a two-period voting game where a high victory margin in the first period discourages the participation of opposition supporters in the second period. My model suggests an alternative and simpler explanation: abstention equilibrium is characterized by extensive fraud and large victory margin, which is far beyond the level needed to guarantee the incumbent's victory. In contrast, coordination equilibrium implies that elections are relatively clean ex-post and the winning candidate has a reasonable victory margin. Again, the fact that relatively clean elections correspond to a reasonable victory margin, while fraudulent elections are associated with an extremely large margin, comes not from the causal effect of fraud on victory margin, but arises as an equilibrium outcome.

5 Conclusion

In this paper I explore the mechanism through which electoral fraud affects the decisions of voters to participate in elections. Using a pivotal voting model of elections with costly participation, where the incumbent can stuff the ballot box, I show that fraud expectations are crucial to the voters' participation decisions, affecting them in a non-trivial way. Unlike the cases of clean elections which are usually characterized by a unique equilibrium, when voters expect fraud two stable equilibria may exist: an abstention equilibrium, where the incumbent is likely to win, and a more yet not fully efficient coordination equilibrium, where a substantial share of a challenger's supporters vote, and the candidate who is ex-ante preferred by the majority is most likely to win. I then show that higher capability of the incumbent to stuff a ballot box discourages the participation of his own supporters and creates coordination incentives for supporters of a challenger. Since the effect of fraud on the participation incentives of the voters is non-linear, higher fraud does not always benefit the incumbent in terms of his winning probability, even when it is costless. I demonstrate that if the incumbent can freely chose the degree of ballot stuffing at no cost and credibly commit to maintain it, perfectly fraudulent elections would rarely be his choice, and sometimes he even prefers to hold clean elections.

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Conflict of interest The author declares that he has no conflict of interest other than double affiliation with Ural Federal University (Yekaterinburg, Russia) and CERGE-EI (Prague, Czech Republic).

Appendix 1

Proof of Proposition 1 1. Suppose coordination equilibrium exists for some N_0, β_0, and F_0.

a) Suppose, one decreases population size to $N_1 = N_0 - t$, where $t \geq 2$ is an even integer. Denote:

$$\Pi_0(p) = \binom{N_0 - 1}{N_0/2}(\beta_0 p)^{N_0/2}(1 - \beta_0 p)^{N_0/2-1},$$
$$\Pi_1(p) = \binom{N_0 - t - 1}{N_0/2 - t/2}(\beta_0 p)^{N_0/2-t/2}(1 - \beta_0 p)^{N_0/2-t/2-1}.$$

Lemma 1 *$\Pi_0(p) < \Pi_1(p)$ for all p.*

Proof See Appendix 2. □

Lemma 1 immediately implies the result: if coordination equilibrium exists for N_0 then $\exists \bar{p} \in (0, 1]$ such that ${F_0}^{-1}(\bar{p}) \leq \Pi_0(\bar{p}) < \Pi_1(\bar{p})$ and thus, equilibrium exists for N_1.

b) Assume by contrast, that equilibrium does not exist for some $\beta_1 > \beta_0$. Denote:

$$\Pi_0(p) = \binom{N_0 - 1}{N_0/2}(\beta_0 p)^{N_0/2}(1 - \beta_0 p)^{N_0/2-1},$$
$$\Pi_1(p) = \binom{N_0 - 1}{N_0/2}(\beta_1 p)^{N_0/2}(1 - \beta_1 p)^{N_0/2-1}.$$

Note that $\Pi_1(p) = \Pi_0\left(\frac{\beta_1}{\beta_0}p\right)$, and, since $\beta_1 > \beta_0$, $\Pi_1(p) \geq \Pi_0(p)$ for all $0 < p \leq \frac{N_0}{2\beta_1(N_0-1)}$. Further note that

$$\max_p \Pi_1(p) = \max_p \Pi_0(p) = \binom{N_0-1}{N_0/2}\left(\frac{N_0}{2(N_0-1)}\right)^{N_0/2}\left(1-\frac{N_0}{2(N_0-1)}\right)^{N_0/2-1},$$

and denote the latter value as Π_{max}. If there is no coordination equilibrium for β_1, then $\Pi_0 \leq \Pi_1(p) < F_0^{-1}(p)$ for all $0 < p \leq \frac{N_0}{2\beta_1(N_0-1)}$. Since $F_0^{-1}(p)$ is increasing function, for all $\frac{N_0}{2\beta_1(N_0-1)} < p \leq 1$ it must be $\Pi_0(p) \leq \Pi_{max} < F_0^{-1}(p)$. Hence, for any $0 < p \leq 1$, $\Pi_0(p) < F_0^{-1}(p)$, implying that coordination equilibrium does not exist for β_0 either, which contradicts the initial assumption.

c) If coordination equilibrium exists for some cost distribution F_0, then $\exists \bar{p} \in (0, 1]$ such that $F_0{}^{-1}(\bar{p}) \leq \Pi(\bar{p})$. Any F which is first order stochastically dominated by F_0 satisfies $F(c) \geq F_0(c)$ for all c. Hence, $F^{-1}(p) \leq F_0{}^{-1}(p)$ for all $p \in [0, 1]$, including $\bar{p}$. From $F^{-1}(\bar{p}) \leq F_0{}^{-1}(\bar{p}) \leq \Pi(\bar{p})$ existence of equilibrium for cost distribution F follows immediately.

2. Suppose, condition (7) holds for some N_0, β_0, and F_0, and let p^* and p^t be the equilibrium participation level and the enforcement threshold under these parameters.

a) Suppose, one decreases population size to $N_1 = N_0 - t$, and let $\tilde{p}$ and $\tilde{p}^t$ be the equilibrium participation level and the threshold under N_1. Denote:

$$\Pi_0(p) = \binom{N_0-1}{N_0/2}(\beta_0 p)^{N_0/2}(1-\beta_0 p)^{N_0/2-1},$$
$$\Pi_1(p) = \binom{N_0-t-1}{N_0/2-t/2}(\beta_0 p)^{N_0/2-t/2}(1-\beta_0 p)^{N_0/2-t/2-1}.$$

According to Lemma 1, $\Pi_0(p) < \Pi_1(p)$ for all p. Because p^* and $\tilde{p}$ are intersections of increasing function $F_0^{-1}(p)$ with $\Pi_0(p)$ and $\Pi_1(p)$ respectively in their decreasing parts, $p^* < \tilde{p}$. Similarly, since p^t and $\tilde{p}^t$ are intersections of $F_0^{-1}(p)$ with $\Pi_0(p)$ and $\Pi_1(p)$ respectively in their increasing parts, $p^t > \tilde{p}^t$. Thus, equilibrium participation is decreasing and participation threshold is increasing in population size.

(b) Let $\tilde{p}$ is the equilibrium participation level and $\tilde{p}^t$ is the enforcement threshold under $\beta_1 > \beta_0$. Then, p^* is an argument of an intersection point between decreasing part of $\Pi_0(p) = \binom{N_0-1}{N_0/2}(\beta_0 p)^{N_0/2}(1-\beta_0 p)^{N_0/2-1}$, , i.e. for $p > \frac{1}{2\beta_0}$, and increasing function $F_0^{-1}(p)$. Likewise, $\tilde{p}$ is an argument of an intersection point between decreasing part of $\Pi_1(p) = \binom{N_0-1}{N_0/2}(\beta_1 p)^{N_0/2}(1-\beta_1 p)^{N_0/2-1}$, i.e. for $p > \frac{1}{2\beta_1}$, and the same $F_0^{-1}(p)$. Thus, to prove that $p^* > \tilde{p}$ it is sufficient to show that $\Pi_0(p) > \Pi_1(p)$ for $p > \frac{1}{\beta_0}$. The latter inequality follows from the fact that $\Pi_1(p) = \Pi_0(\frac{\beta_1}{\beta_0}p)$, i.e. $\Pi_1(p)$ is a horizontal stretch (in fact, compression, since $\beta_1 > \beta_0$) of $\Pi_0(p)$.

Similarly, p^t is an argument of an intersection point between non-decreasing function $F_0^{-1}(p)$ and increasing part of $\Pi_0(p)$, i.e. for $p < \frac{1}{2\beta_0}$. Likewise, $\tilde{p}^t$ is an argument of an intersection point between the same $F^{-1}(p)$ and increasing part of $\Pi_1(p)$, i.e. for $p < \frac{1}{2\beta_1}$. Since $\Pi_0(p) < \Pi_1(p)$ for $p < \frac{1}{2\beta_1}$, it follows that $p^t > \tilde{p}^t$.

(c) Let $\tilde{p}$ is the equilibrium participation level and $\tilde{p}^t$ is the enforcement threshold under some cost distribution G which is first-order stochastically dominated by F_0, and assume by contrast that $p^* \geq \tilde{p}$. Then $F_0^{-1}(p^*) \geq F_0^{-1}(\tilde{p}) \geq G^{-1}(\tilde{p})$. Because p^* satisfies $\Pi(p^*) = F_0^{-1}(p^*)$ and $\tilde{p}$ satisfies $\Pi(\tilde{p}) = G^{-1}(\tilde{p})$ it must be that $\Pi(p^*) \geq \Pi(\tilde{p})$. Recall that Π is a decreasing function for any $p \geq \frac{N_0/2}{\beta(N_0-1)}$ and any equilibrium participation $p \geq \frac{N_0/2}{\beta(N_0-1)}$, implying that $p^* \leq \tilde{p}$, which contradicts the initial assumption. Thus, $p^* \leq \tilde{p}$. Similarly, since p^t and $\tilde{p}^t$ are intersections of the increasing part of $\Pi(p)$ with $F_0^{-1}(p)$ and $G^{-1}(p)$ respectively, and $F_0^{-1}(p) \geq G^{-1}(p)$ for all p, $p^t \geq \tilde{p}^t$. □

Proof of Proposition 2 Welfare as a function of some strategy $\tilde{c}$ is expressed as

$$W = N\left((1-\beta)(1-v_B) + \beta v_B + \beta\Pi(F(\tilde{c}))F(\tilde{c}) - \beta\int_0^{\tilde{c}} c\,\mathrm{d}F(c)\right)$$

In abstention equilibrium, where $\tilde{c} = 0$ and $v_B = 0$, social welfare is then simply $N(1-\beta)$. Consider the difference between welfare in coordination equilibrium and welfare in abstention equilibrium:

$$\Delta W = N\left((2\beta-1)v_B + \beta\Pi(F(c^*))F(c^*) - \beta\int_0^{c^*} c\,\mathrm{d}F(c)\right).$$

Since $\int_0^{c^*} c\,\mathrm{d}F(c) < \Pi(F(c^*))$, $\Delta W > N\left((2\beta-1)v_B - \Pi(F(c^*))(1-\beta F(c^*)\right)$.

Note that the right hand side of the last inequality is non-positive for $\beta < 1/2$, and strictly positive for $\beta = 1$. To see the latter, note that condition (7) implies $F(c^*) > \frac{N}{2\beta(N-1)}$ and re-write the last inequality for $\beta = 1$:

$$\Delta W > N\left(v_B - \Pi(F(c^*))(1-F(c^*)\right) > N\left(v_B - \frac{N/2-1}{N-1}\Pi(F(c^*))\right).$$

Further, using the first element of v_B:

$$\begin{aligned}\Delta W &> N\left(\binom{N-1}{N/2+1}(F(c^*))^{N/2+1}(1-F(c^*))^{N/2-2} - \frac{N/2-1}{N-1}\Pi(F(c^*))\right)\\ &= N\frac{N/2-1}{N-1}\Pi(F(c^*))\left(\frac{N-1}{N/2+1}\frac{F(c^*)}{1-F(c^*)} - 1\right)\\ &> N\frac{N/2-1}{N-1}\Pi(F(c^*))\left(\frac{N-1}{N/2+1}\frac{N}{N-2} - 1\right).\end{aligned}$$

Since the latter expression is strictly positive for $N > 2$, from the Intermediate Value Theorem it immediately follows that for any even $N > 2$ there exists some β_0 such that $\Delta W > 0$ for any $\beta > \beta_0$. □

Proof of Proposition 3 Consider social welfare as a function of some strategy $\tilde{c}$:

$$W = N\left((1-\beta)(1-v_B) + \beta v_B + \beta \Pi(F(\tilde{c}))F(\tilde{c}) - \beta \int_0^{\tilde{c}} c\,\mathrm{d}F(c)\right)$$

It is easy to check that the welfare function $W(F(\tilde{c}))$ is concave in $F(\tilde{c})$. Hence, the first-order condition with respect to $F(\tilde{c})$ defines the efficiency condition:

$$N\left((2\beta-1)\frac{\partial v_B}{\partial F(\tilde{c})} + \beta\Pi(F(\tilde{c})\left(1+\frac{N/2-\beta N F(\tilde{c})+\beta F(\tilde{c})}{1-\beta F(\tilde{c})}\right) - \beta\tilde{c}\right) = 0.$$

Lemma 2 $\frac{\partial v_B}{\partial F(\tilde{c})} = \frac{\beta(N/2-1)}{1-\beta F(\tilde{c})}\Pi(F(\tilde{c}))$ *for all* $\tilde{c}$.

Proof See Appendix 2. □

After applying Lemma 2, dropping constants and re-arranging the terms, the efficiency condition takes the following form:

$$\Pi(F(\tilde{c}))\frac{\beta N(1-F(\tilde{c}))+2(1-\beta)}{1-\beta F(\tilde{c})} = F^{-1}(F(\tilde{c})).$$

Let c^e be the efficient participation rule, i.e. the one that satisfies the efficiency condition above. Recall the equilibrium condition:

$$\Pi(F(c^*)) \geq F^{-1}(F(c^*)),$$

with equality for $c^* < c_{max}$, and note that $\frac{\beta N(1-F(\tilde{c}))+2(1-\beta)}{1-\beta F(\tilde{c})} > 1$ for all $\tilde{c}$. Hence, it immediately follows that $c^* \leq c^e$ with inequality for $c^* < c_{max}$. □

Proof of Proposition 4 If all the voters abstain, an A-type voter is never pivotal, while a B-type voter is pivotal if and only if none of the non-participants' votes is stolen. Thus, the expected benefit function of a B-type voter at point (c_{min}, c_{min}), which corresponds to full abstention, is $\Pi_{min}(\alpha) \equiv \Pi_B(c_{min}, c_{min}) = (1-\alpha)^{N-1}$. Since, $\Pi_{min}(0) = 1 > c_{min}$, $\Pi_{min}(1) = 0 < c_{min}$, and $\Pi_{min}(\alpha)$ is strictly decreasing in α, the Intermediate Value Theorem implies that there exists a unique value of $\alpha = \alpha_0 < 1$ such that $\Pi_B(c_{min}, c_{min}) = c_{min}$; for any $\alpha \geq \alpha_0$ $\Pi_B(c_{min}, c_{min}) < c_{min}$, i.e. deviation from abstention is never profitable for a B-type voter; and for any $\alpha < \alpha_0$ $\Pi_B(c_{min}, c_{min}) > c_{min}$, i.e. deviation is profitable, and thus, full abstention is not an equilibrium. □

Appendix 2

Proof of Lemma 1 Consider two functions:

$$S_0(q) = \binom{N_0 - 1}{N_0/2} q^{N_0/2}(1-q)^{N_0/2-1},$$
$$S_1(q) = \binom{N_1 - 1}{N_1/2} q^{N_1/2}(1-q)^{N_1/2-1},$$

where $N_1 = N_0 - t$, $t \geq 2$, and $t/2$ is integer. Denoting $N_1/2 = x$ and $t/2 = m$ for shorter notation:

$$S_0(q) = \binom{2x + 2m - 1}{x + m} q^{x+m}(1-q)^{x+m-1}.$$
$$S_1(q) = \binom{2x - 1}{x} q^{x}(1-q)^{x-1}.$$

To show that $S_0(q) < S_1(q)$ it is sufficient to show that

$$\binom{2x + 2m - 1}{x + m} q^{x+m}(1-q)^{x+m-1} - \binom{2x - 1}{x} q^{x}(1-q)^{x-1} < 0,$$

or, equivalently, that

$$\frac{\binom{2x+2m-1}{x+m}}{\binom{2x-1}{x}} q^m (1-q)^m < 1,$$

Since function $g(p) = q^m(1-q)^m$ achieves its maximum at $q = 1/2$, to complete the proof it is sufficient to show that

$$\frac{\binom{2x+2m-1}{x+m}}{\binom{2x-1}{x}} \frac{1}{4^m} < 1.$$

The left hand side of the last expression can be rewritten as

$$\begin{aligned}\frac{\binom{2x+2m-1}{x+m}}{\binom{2x-1}{x}} \frac{1}{4^m} &= \frac{2x(2x+1)...(2x+2m-1)}{(x+1)..(x+m)x(x+1)..(x+m-1)} \frac{1}{4^m} \\ &= \frac{2x(2x+1)...(2x+2m-1)}{(2x+2)..(2x+2m)2x(2x+2)..(2x+2m-2)}.\end{aligned}$$

Note that the numerator consists of $2m$ elements in the form of $(2x + i)$, $i = 0..2m - 1$. The denominator contains two groups of m elements each. The first group consists of elements in the form of $2x + 2i$, $i = 1..m$, while the second group consists of elements in the same form $2x + 2i$, $i = 0..m - 1$. Hence, the entire expression can be rewritten as

$$\frac{\prod_{i=0}^{2m-1}(2x+i)}{\prod_{i=1}^{m}(2x+2i)\prod_{i=0}^{m-1}(2x+2i)} = \frac{2x\left(\prod_{i=1}^{m-1}(2x+2i-1)(2x+2i)\right)(2x+2m-1)}{2x\left(\prod_{i=1}^{m-1}(2x+2i)(2x+2i)\right)(2x+2m)}.$$

After cancelling out equal elements, each element in the numerator is smaller than the corresponding element in the denominator. Hence, the entire expression is less than 1:

$$\frac{\left(\prod_{i=1}^{m-1}(2x+2i-1)\right)(2x+2m-1)}{\left(\prod_{i=1}^{m-1}(2x+2i)\right)(2x+2m)} < 1.$$

□

Proof of Lemma 2 Recall the following identity[7]:

$$1 - I_x(a,b) = (1-x)^{a+b-1}\sum_{i=0}^{a-1}\binom{a+b-1}{i}\left(\frac{x}{1-x}\right)^i,$$

where $I_x(a,b)$ is regularized incomplete beta-function. Denote $\beta F(\tilde{c}) = q$ for shorter notation. Then

$$v_B = \sum_{i=N/2+1}^{N-1}\binom{N-1}{i}q^i(1-q)^{N-i},$$

Consider the following regularized incomplete beta-function: $I_q(N/2+1, N/2-1)$. Using the identity above:

$$\begin{aligned} 1 - I_q(N/2+1, N/2-1) &= (1-q)^{N-1}\sum_{i=0}^{N/2}\binom{N-1}{i}\left(\frac{q}{1-q}\right)^i \\ &= \sum_{i=0}^{N/2}\binom{N-1}{i}q^i(1-q)^{N-i-1} = 1 - v_B. \end{aligned}$$

Hence, $v_B = I_q(N/2+1, N/2-1)$. Also recall Chebyshev's integral:

$$\int x^a(1-x)^b\,\mathrm{d}x = B_x(a+1, b+1),$$

where $B_x(a+1,b+1)$ is incomplete beta-function. Thus,

$$B_q(N/2+1, N/2-1) = \int q^{N/2}(1-q)^{N/2-2}\,\mathrm{d}q.$$

[7] See, for example, Pearson, K., 1968. Tables of Incomplete Beta-Function, Second Edition, Cambridge University Press, page 28.

By definition of regularized incomplete beta-function:

$$I_q(N/2+1, N/2-1) = \frac{B_q(N/2+1, N/2-1)}{B(N/2+1, N/2-1)},$$

where $B(N/2+1, N/2-1)$ is beta-function. Since $B(N/2+1, N/2-1) = \frac{(N/2)!(N/2-2)!}{(N-1)!}$:

$$\begin{aligned}
&(N/2-1)\binom{N-1}{N/2}\int q^{N/2}(1-q)^{N/2-2}\,\mathrm{d}q \\
&= \frac{(N-1)!}{(N/2)!(N/2-2)!}B_q(N/2+1, N/2-1) \\
&= \frac{B_q(N/2+1, N/2-1)}{B(N/2+1, N/2-1)} = I_q(N/2+1, N/2-1) = v_B.
\end{aligned}$$

Hence, $v_B = (N/2-1)\binom{N-1}{N/2}\int q^{N/2}(1-q)^{N/2-1}\frac{1}{1-q}\,\mathrm{d}q$. Differentiating both parts of the last identity with respect to $F(\tilde{c})$:

$$\begin{aligned}
\frac{\partial v_B}{\partial F(\tilde{c})} &= \beta\frac{\partial v_B}{\partial q} = \beta(N/2-1)\frac{\partial}{\partial q}\left(\int\binom{N-1}{N/2}q^{N/2}(1-q)^{N/2-1}\frac{1}{1-q}\,\mathrm{d}q\right) \\
&= \frac{\beta(N/2-1)}{1-\beta F(\tilde{c})}\Pi(F(\tilde{c})).
\end{aligned}$$

□

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